Surveying Problems & Solutions

Shepherd

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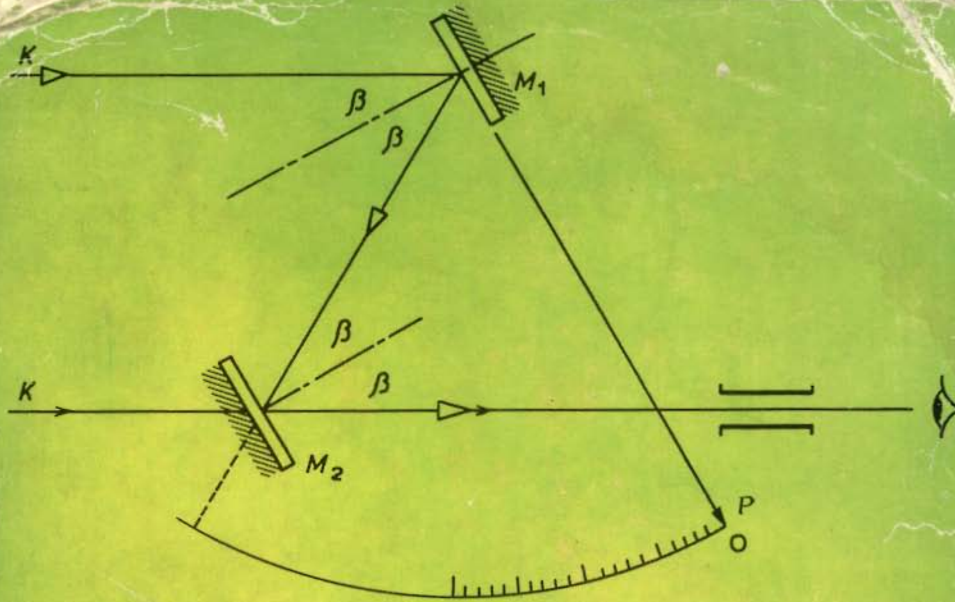

Surveying Problems and Solutions

F. A. Shepherd

This new book gives a presentation concentrating on mathematical problems, an aspect of the subject which usually causes most difficulty.

Summaries of basic theory are followed by worked examples and selected exercises. The book covers three main branches of surveying: measurement, surveying techniques and industrial applications. It is a book concerned mainly with engineering surveying as applied, for example, in the construction and mining industries.

Edward Arnold **80s.** net

Contents

Values in both imperial and metric (S.I.) units are given in the problems

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SURVEYING PROBLEMS & SOLUTIONS

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Surveying Problems and Solutions

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GREEK ALPHABET

Α	α		alpha	Ν	ν	nu
Β	β		beta	Ξ	ξ	xi
Γ	γ		gamma	Ο	o	omicron
Δ	δ		delta	Π	π	pi
Ε	ϵ		epsilon	Ρ	ρ	rho
Ζ	ζ		zeta	Σ	σ	sigma
Η	η		eta	Τ	τ	tau
Θ	θ	ϑ	theta	Υ	υ	upsilon
Ι	ι		iota	Φ	ϕ	phi
Κ	κ		kappa	Χ	χ	chi
Λ	λ		lambda	Ψ	ψ	psi
Μ	μ		mu	Ω	ω	omega

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PREFACE

This book is an attempt to deal with the basic mathematical aspects of 'Engineering Surveying', i.e. surveying applied to construction and mining engineering projects, and to give guidance on practical methods of solving the typical problems posed in practice and, in theory, by the various examining bodies.

The general approach adopted is to give a theoretical analysis of each topic, followed by worked examples and, finally, selected exercises for private study. Little claim is made to new ideas, as the ground covered is elementary and generally well accepted. It is hoped that the mathematics of surveying, which so often causes trouble to beginners, is presented in as clear and readily understood a manner as possible. The main part of the work of the engineering surveyor, civil and mining engineer, and all workers in the construction industry is confined to plane surveying, and this book is similarly restricted.

It is hoped that the order of the chapters provides a natural sequence, viz.:

(a) *Fundamental measurement*
- (i) Linear measurement in the horizontal plane.
- (ii) Angular measurement and its relationship to linear values, i.e. trigonometry.
- (iii) Co-ordinates as a graphical and mathematical tool.

(b) *Fundamental surveying techniques*
- (i) Instrumentation.
- (ii) Linear measurement in the vertical plane, i.e. levelling.
- (iii) Traversing as a control system.
- (iv) Tacheometry as a detail and control system.

(c) *Industrial applications*
- (i) Three-dimensional aspects involving inclined planes.
- (ii) Mensuration.
- (iii) Curve surveying.

Basic trigonometry is included, to provide a fundamental mathematical tool for the surveyor. It is generally found that there is a deficiency in the student's ability to apply numerical values to trigonometrical problems, particularly in the solution of triangles, and it is hoped that the chapter in question shows that more is required than the sine and cosine formulae. Many aspects of surveying, e.g. errors in surveying, curve ranging, etc. require the use of small angles, and the application of radians is suggested. Few numerical problems are posed relating to instrumentation, but it is felt that a knowledge of basic

physical properties affords a more complete understanding of the construction and use of instruments. To facilitate a real grasp of the subject, the effects of errors are analysed in all sections. This may appear too advanced for students who are not familiar with the elementary calculus, but it is hoped that the conclusions derived will be beneficial to all.

With the introduction of the Metric System in the British Isles and elsewhere, its effect on all aspects of surveying is pin-pointed and conversion factors are given. Some examples are duplicated in the proposed units based on the International System (S.I.) and in order to give a 'feel' for the new system, during the difficult transition period, equivalent S.I. values are given in brackets for a few selected examples.

The book is suitable for all students in Universities and Technical Colleges, as well as for supplementary postal tuition, in such courses as Higher National Certificates, Diplomas and Degrees in Surveying, Construction, Architecture, Planning, Estate Management, Civil and Mining Engineering, as well as for professional qualification for the Royal Institution of Chartered Surveyors, the Institution of Civil Engineers, the Incorporated Association of Architects and Surveyors, the Institute of Quantity Surveyors, and the Institute of Building.

ACKNOWLEDGMENTS

I am greatly indebted to the Mining Qualifications Board (Ministry of Power) and the Controller of H.M. Stationery Office, who have given permission for the reproduction of examination questions. My thanks are also due to the Royal Institution of Chartered Surveyors, the Institution of Civil Engineers, to the Senates of the Universities of London and Nottingham, to the East Midlands Educational Union and the Nottingham Regional College of Technology, all of whom have allowed their examination questions to be used.

My special thanks are due to many of my colleagues at Nottingham, but especially to Messrs. J. H. Ball, A.R.I.C.S., A.I.A.S., A.M.I.Min.E., A. Eaton, B.Sc., C.Eng., A.M.I.C.E., A.M.B.I.M., G. M. Lewis, B.Sc., Ph.D., M. B. Pate, M.Sc., A. A. Payne, B.Sc., C. Rayner, B.Sc., A.R.I.C.S., R. Robb, A.R.I.C.S., A.M.I.Min.E., D.B. Shaw, B.Sc., and J. P. Withers, B.Sc., C.Eng., A.M.I.C.E., all of whom have offered advice and help in checking the text

The ultimate responsibility for the accuracy is, of course, my own. I am very conscious that, even with the most careful checking, it is not to be expected that every mistake has been eliminated, and I can only ask readers if they will kindly bring any errors to my notice.

Nottingham
1968

F. A. SHEPHERD

CONVERSION FACTORS (A)

LINEAR	Inches	Feet	Yards	Chains	Furlongs	Miles	Metres
1 Inch (in.)	1	0·083 3	0·027 78				0·025 4
1 Foot (ft)	12	1	0·33				0·304 8
1 Yard (yd)	36	3	1				0·914 4
1 Chain (ch)	792	66	22	1	0·1	0·012 5	20·116 8
1 Furlong	7920	660	220	10	1	0·125	201·168
1 Mile	63 360	5280	1760	80	8	1	1609·34
1 Fathom	72	6	2				1·828 8
1 Cable	8640	720	240				219·456
1 Nautical mile	72 913·32	6076·11	2025·37				1853·18
1 Metre (m)	39·370 078	3·280 84	1·093 61	0·049 709 7	0·004 970 9	0·000 621 3	1
1 Kilometre (km)	39 370·078	3280·84	1093·613 0	49·709 68	4·970 968	0·621 371	1000

The above metric values relate to the International Metre (S.I. metric units) defined as "the length equal to 1 650 763·73 wavelengths in vacuum of the radiation corresponding to the transition between the energy levels $2p_{10}$ and $5d_5$ of the krypton-86 atom"

(*Changing to Metric*, H.M.S.O., 1965)

CONVERSION FACTORS (B)

(Ref.: *Changing to the Metric System*, H.M.S.O., 1967)

Length

1 mile	=	1·609 34 km	1 km	=	0·621 371 mile
1 furlong	=	0·201 168 km			
1 chain	=	20·116 8 m			
1 yd	=	0·914 4 m	1 m	=	1·093 61 yd
1 ft	=	0·304 8 m			
1 in.	=	2·54 cm	1 cm	=	0·393 701 in.
1 fathom	=	1·828 8 m			
1 link	=	0·201 168 m			

Area

1 sq. mile	=	2·589 99 km^2			
1 acre	=	4046·86 m^2	1 km^2	=	247·105 acres
1 rood	=	1011·71 m^2			
1 yd^2	=	0·836 127 m^2	1 m^2	=	1·195 99 yd^2
1 ft^2	=	0·092 903 m^2			
1 in^2	=	6·451 6 cm^2	1 cm^2	=	0·155 00 in^2
1 sq. chain	=	404·686 m^2			

Volume

1 yd^3	=	0·764 555 m^3	1 m^3	=	1·307 95 yd^3
1 ft^3	=	0·028 316 8 m^3	1 m^3	=	35·314 7 ft^3
1 in^3	=	16·387 1 cm^3	1 cm^3	=	0·061 023 7 in^3
1 gal	=	0·004 546 09 m^3			
	=	4·546 09 litre	1 litre	=	0·220 0 gal

Velocity

1 mile/h	=	1·609 34 km/h	1 km/h	=	0·621 371 m.p.h.
1 ft/s	=	0·304 8 m/s	1 m/s	=	3·280 84 ft/s

Acceleration

1 ft/s^2	=	0·304 8 m/s^2	1 m/s^2	=	3·280 84 ft/s^2

Mass

1 ton	=	1016·05 kg			
1 cwt	=	50·802 3 kg			
1 lb	=	0·453 592 37 kg	1 kg	=	2·204 62 lb

Mass per unit length

1 lb/ft	=	1·488 16 kg/m		

Mass per unit area

1 lb/ft^2	=	4·882 43 kg/m^2		

Density

1 ton/yd^3	=	1328·94 kg/m^3		
1 lb/ft^3	=	16·018 5 kg/m^3	1 kg/m^3 =	0·062 428 lb/ft^3
1 lb/gal	=	99·776 3 kg/m^3		
	=	0·099 78 kg/l		

Force

1 lbf	=	4·448 22 N	1 N =	0·224 809 lbf
1 kgf	=	9·806 65 N	1 kgf =	2·204 62 lbf

Force (weight)/unit length

1 lbf/ft	=	14·593 9 N/m		

Pressure

1 lbf/ft^2	=	47·880 3 N/m^2	1 N/m^2 =	0·000 145 038 lbf/in^2
1 lbf/in^2	=	6894·76 N/m^2		
1 kgf/cm^2	=	98·066 5 kN/m^2		
1 kgf/m^2	=	9·806 65 N/m^2		

Standard gravity

32·1740 ft/s^2	=	9·806 65 m/s^2		
N.B. 1 lb	=	0·453 592 kg		
1 lbf	=	0·453 592 × 9·806 65	=	4·448 22 N

1 newton (N) unit of force = that force which applied to a mass of 1 kg gives an acceleration of 1 m/s^2.

CONTENTS

Abbreviations used for Examination Papers

E.M.E.U.	East Midlands Educational Union
I.C.E.	Institution Of Civil Engineers
L.U.	London University B.Sc. (Civil Engineering)
L.U./E	London University B.Sc. (Estate Management)
M.Q.B./S	Mining Qualifications Board (Mining Surveyors)
M.Q.B./M	Mining Qualifications Board (Colliery Managers)
M.Q.B./UM	Mining Qualifications Board (Colliery Undermanagers)
N.R.C.T.	Nottingham Regional College of Technology
N.U.	Nottingham University
R.I.C.S./G	Royal Institution of Chartered Surveyors (General)
R.I.C.S./M	Royal Institution of Chartered Surveyors (Mining)
R.I.C.S./ML	Royal Institution of Chartered Surveyors (Mining/Land)
R.I.C.S./Q	Royal Institution of Chartered Surveyors (Quantity)

1 LINEAR MEASUREMENT

1.1 The Basic Principles of Surveying

Fundamental rule 'Always work from the whole to the part'. This implies 'precise control surveying' as the first consideration, followed by 'subsidiary detail surveying'.

A point C in a plane may be fixed relative to a given line AB in one of the following ways:

1. *Triangulation* Angular measurement from a fixed base line. The length AB is known. The angles α and β are measured.

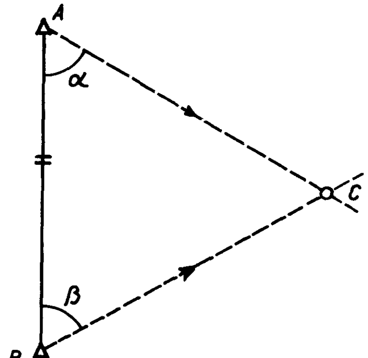


Fig. 1.1 (a)

2. *Trilateration* Linear measurement only. The lengths AC and BC are measured or plotted. The position of C is always fixed provided $AC + BC > AB$.

   Uses: (a) Replacing triangulation with the use of microwave measuring equipment.

   (b) Chain surveying.

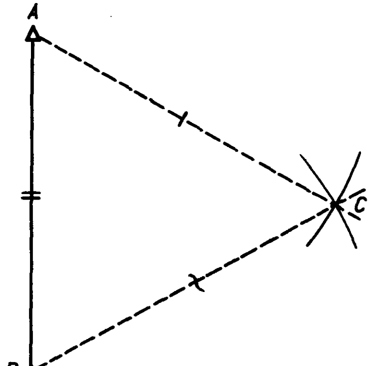


Fig. 1.1 (b)

3. *Polar co-ordinates* Linear and angular measurement.

Uses: (a) Traversing.
(b) Setting out.
(c) Plotting by protractor.

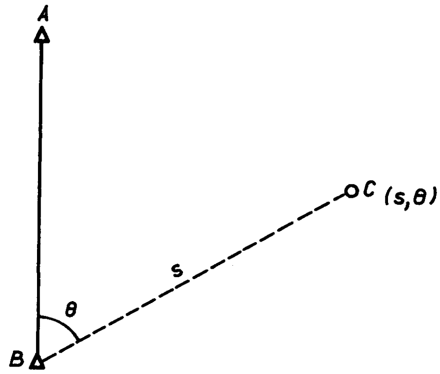


Fig. 1.1 (c)

4. *Rectangular co-ordinates* Linear measurement only at right-angles.

Uses: (a) Offsets.
(b) Setting out.
(c) Plotting.

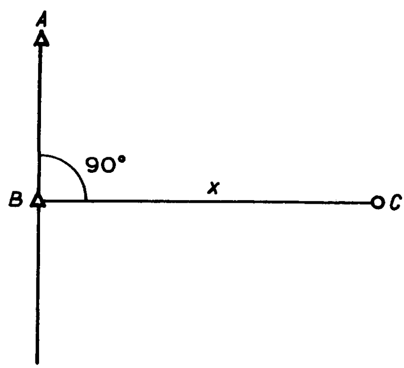


Fig. 1.1 (d)

1.2 General Theory of Measurement

The following points should be noted:

(1) There is no such thing as an exact measurement. All measurements contain some error, the magnitude of the error being dependent on the instruments used and the ability of the observer.

(2) As the true value is never known, the true error is never deter-

mined.

(3) The degree of accuracy, or its precision, can only be quoted as a relative accuracy, i.e. the estimated error is quoted as a fraction of the measured quantity. Thus 100 ft measured with an estimated error of 1 inch represents a relative accuracy of 1/1200. An error of 1 cm in 100 m = 1/10 000.

(4) Where readings are taken on a graduated scale to the nearest subdivision, the maximum error in estimation will be ± ½ division.

(5) Repeated measurement increases the accuracy by $\sqrt{n}$, where n is the number of repetitions. N.B. This cannot be applied indefinitely.

(6) Agreement between repeated measurements does not imply accuracy but only consistency.

1.3 Significant Figures in Measurement and Computation

If a measurement is recorded as 205 ft to the nearest foot, its most probable value is 205 ± 0·5 ft, whilst if measured to the nearest 0·1 ft its most probable value is 205·0 ± 0·05 ft. Thus the smallest recorded digit is subject to a maximum error of half its value.

In computation, figures are rounded off to the required degree of precision, generally by increasing the last significant figure by 1 if the following figure is 5 or more. (An alternative is the rounding off with 5 to the nearest even number.)

Thus 205·613 becomes 205·61 to 2 places,
whilst 205·615 becomes 205·62 to 2 places,
or 205·625 may also be 205·62, giving a less biased value.

It is generally better to work to 1 place of decimals more than is required in the final answer, and to carry out the rounding-off process at the end.

In *multiplication* the number of significant figures depends on the accuracy of the individual components, e.g.,

if $$P = x.y,$$

then $$P + \delta P = (x + \delta x)(y + \delta y)$$
$$= xy + x\delta y + y\delta x + \delta x\delta y$$

Neglecting the last term and substracting P from both sides of the equation,
$$\delta P = x\delta y + y\delta x$$

$\div P$ gives $$\frac{\delta P}{P} = \frac{x\delta y}{xy} + \frac{y\delta x}{xy} = \frac{\delta y}{y} + \frac{\delta x}{x}$$

i.e. $$\delta P = P\left(\frac{\delta y}{y} + \frac{\delta x}{x}\right) \tag{1.1}$$

Thus the relative accuracy of the product is the sum of all the relative accuracies involved in the product.

Example 1.1 A rectangle measures 3·82 in. and 7·64 in. with errors of ± 0·005 in. Express the area to the correct number of significant figures.

$$P = 3{\cdot}82 \times 7{\cdot}64 = 29{\cdot}184\,8 \text{ in}^2$$

relative accuracies

$$\frac{0{\cdot}005}{3{\cdot}82} \simeq \frac{1}{750}$$

$$\frac{0{\cdot}005}{7{\cdot}64} \simeq \frac{1}{1500}$$

$$\delta P = 29\left(\frac{1}{750} + \frac{1}{1500}\right) = \frac{29}{500}$$

$$= \underline{\pm\, 0{\cdot}06}$$

∴ the area should be given as $\underline{29{\cdot}2 \text{ in}^2}$.

As a general rule the number of significant figures in the product should be at least the same as, or preferably have one more significant figure than, the least significant factor.

The area would thus be quoted as $\underline{29{\cdot}18 \text{ in}^2}$

In *division* the same rule applies.

$$Q = \frac{x}{y}$$

$$Q + \delta Q = \frac{x + \delta x}{y + \delta y} = \frac{x}{y} + \frac{\delta x}{y} - \frac{x\delta y}{y^2} + \ldots$$

Subtracting Q from both sides and dividing by Q gives

$$\underline{\delta Q = Q\left(\frac{\delta x}{x} - \frac{\delta y}{y}\right)} \qquad (1.2)$$

Powers

$$R = x^n$$

$$R + \delta R = (x + \delta x)^n$$

$$= x^n + n\delta x + \ldots$$

$$\therefore \quad \frac{\delta R}{R} = \frac{n\delta x}{x^n} \quad \text{i.e. } n \times \text{relative accuracy of single value.}$$

$$\underline{\delta R = n\delta x} \qquad (1.3)$$

Roots This is the opposite relationship

$$R = \sqrt[n]{x} \qquad \therefore \quad R^n = x$$

From the above

$$R^n + n\delta R = x + \delta x$$

$$\therefore \quad n\delta R = \delta x$$

$$\frac{\delta R}{R^n} = \frac{\delta x}{nx}$$

$$\delta R = \frac{1}{n}\delta x \tag{1.4}$$

Example 1.2 If

$$R = (5{\cdot}01 \pm 0{\cdot}005)^2$$
$$5{\cdot}01^2 = 25{\cdot}100\,1$$
$$\delta R = 2 \times 0{\cdot}005 = 0{\cdot}01$$

$\therefore$ R should be given as $\underline{25{\cdot}10}$

Example 1.3 If

$$R = \sqrt{25{\cdot}10 \pm 0{\cdot}01}$$
$$\sqrt{25{\cdot}10} = 5{\cdot}009\,9$$
$$\delta R = \frac{0{\cdot}01}{2} = 0{\cdot}005$$

$\therefore$ R should be given as $\underline{5{\cdot}01}$

Example 1.4 A rectangular building has sides approximately 480 metres and 300 metres. If the area is to be determined to the nearest 10 m² what will be the maximum error permitted in each line, assuming equal precision ratios for each length? To what degree of accuracy should the lines be measured?

$$A = 480 \times 300 = 144\,000\ \text{m}^2$$
$$\delta A = 10\ \text{m}^2$$
$$\therefore \quad \frac{\delta A}{A} = \frac{1}{14\,400} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

but $\dfrac{\delta x}{x} = \dfrac{\delta y}{y} \qquad \therefore \quad \dfrac{\delta x}{x} + \dfrac{\delta y}{y} = \dfrac{2\delta x}{x}$

$$\therefore \quad \frac{\delta x}{x} = \frac{1}{2 \times 14\,400} = \frac{1}{28\,800}$$

i.e. the precision ratio of each line is $\dfrac{1}{28\,800}$

This represents a maximum in 480 m of $\dfrac{480}{28\,800} = 0{\cdot}016\,7$ m

and in 300 m of $\dfrac{300}{28\,800} = 0{\cdot}010\,4$ m

If the number of significant figures in the area is 5, i.e. to the nearest 10 m², then each line also must be measured to at least 5 significant figures, i.e. 480·00 m and 300·00 m.

1.4 Chain Surveying

The chain

There are two types:

(a) Gunter's chain

1 chain* = 100 links = 66 ft

1 link = 0·66 ft = 7·92 in.

Its advantage lies in its relationship to the acre

10 sq chains = 100 000 sq links = 1 acre.

(b) Engineer's chain 100 links = 100 ft

(Metric chain 100 links = 20 m

1 link = 0·2 m)

Basic figures

There are many combinations of chain lines all dependent on the linear dimensions forming trilateration, Fig. 1.2.

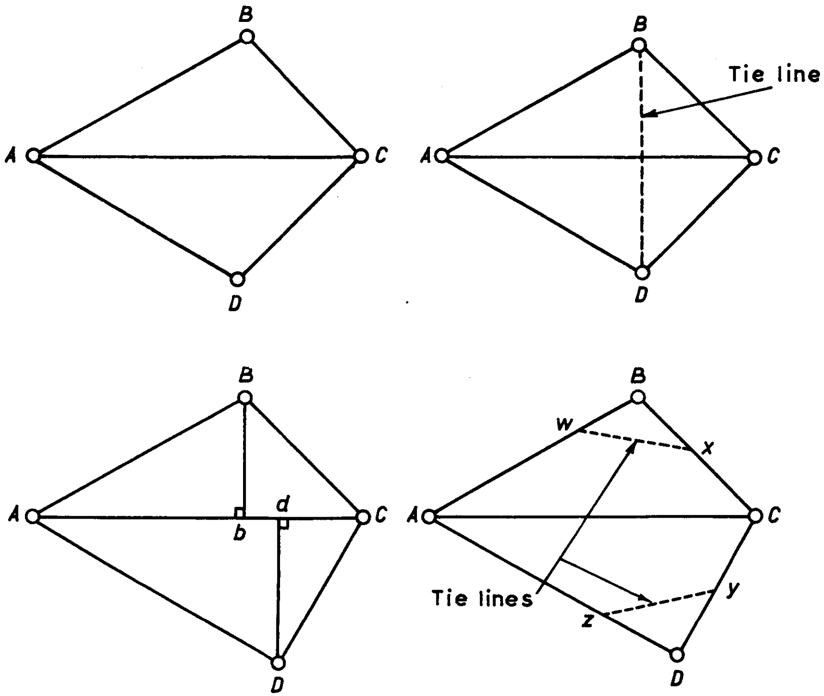


Fig. 1.2 Basic figures in chain surveying

1.41 Corrections to the ground measurements

Standardisation

Where the length of the chain or tape does not agree with its nom-

* See conversion factors, pp. v – vii.

inal value, a correction must be made to the recorded value of a measured quantity.

The following rules apply:

(1) If the tape is too long, the measurement will be too short – the correction will be positive.

(2) If the tape is too short, the measurement will be too long – the correction will be negative.

If the length of tape of nominal length l is $l \pm \delta l$,

$$\text{the error per unit length} = \pm \frac{\delta l}{l}$$

If the measured length is d_m and the true length is d_t, then

$$d_t = d_m \pm d_m \frac{\delta l}{l}$$

$$= \underline{d_m\left(1 \pm \frac{\delta l}{l}\right)} \tag{1.5}$$

Alternatively,

$$\frac{d_t}{d_m} = \frac{l \pm \delta l}{l} = \frac{\text{actual length of tape}}{\text{nominal length of tape}} \tag{1.6}$$

$$\underline{d_t = d_m\left(1 \pm \frac{\delta l}{l}\right)} \tag{1.5}$$

Example 1.5 A chain of nominal length 100 links, when compared with a standard, measures 101 links. If this chain is used to measure a line AB and the recorded measurement is 653 links, what is the true length AB?

$$\text{Error per link} = \frac{1}{100} = 0{\cdot}01$$

$$\therefore \quad \text{true length} = 653(1 + 0{\cdot}01)$$

$$= 653 + 6{\cdot}53 = \underline{659{\cdot}53 \text{ links}}.$$

Alternatively,

$$\text{true length} = 653 \times \frac{101}{100} = \underline{659{\cdot}53 \text{ links}}.$$

Effect of standardisation on areas

Based on the principle of similar figures,

$$\text{true area } (A_T) = \text{apparent area } (A_M) \times \left(\frac{\text{true length of tape}}{\text{apparent length of tape}}\right)^2$$

i.e.

$$A_T = A_M\left(\frac{l \pm \delta l}{l}\right)^2 \tag{1.7}$$

or $$A_T = A_M\left(1 \pm \frac{\delta l}{l}\right)^2 \tag{1.8}$$

Effect of standardisation on volumes

Based on the principle of similar volumes,

$$\text{true volume } V_T = \text{apparent volume} \times \left(\frac{\text{true length of tape}}{\text{apparent length of tape}}\right)^3$$

i.e. $$V_T = V_M\left(\frac{l \pm \delta l}{l}\right)^3 \tag{1.9}$$

or $$V_T = V_M\left(1 \pm \frac{\delta l}{l}\right)^3 \tag{1.10}$$

N.B. Where the error in standardisation is small compared to the size of the area, the % error in area is approximately $2 \times$ % error in length.

Example 1.6 A chain is found to be 0·8 link too long and on using it an area of 100 acres is computed.

$$\text{The true area} = 100\left(\frac{100{\cdot}8}{100}\right)^2$$

$$= 100 \times 1{\cdot}008^2 = \underline{101{\cdot}61 \text{ acres}}$$

alternatively,

$$\text{linear error} = 0{\cdot}8\%$$

$$\therefore \quad \text{area error} = 2 \times 0{\cdot}8 = 1{\cdot}6\%$$

$$\therefore \quad \text{acreage} = 100 + 1{\cdot}6 \text{ acres} = \underline{101{\cdot}6 \text{ acres}}$$

This is derived from the binomial expansion of $(1 + x)^2$

$$= 1 + 2x + x^2$$

i.e. if x is small x^2 may be neglected

$$\therefore (1 + x)^2 \simeq 1 + 2x$$

Correction for slope (Fig. 1.3)

This may be based on (1) the angle of inclination, (2) the difference in level between the ends of the line.

Fig. 1.3 (page 9)

Length AC measured (l)

Horizontal length AB required (h)

Difference in level between A and C (d)

Angle of inclination (α)

Correction to measured length (c)

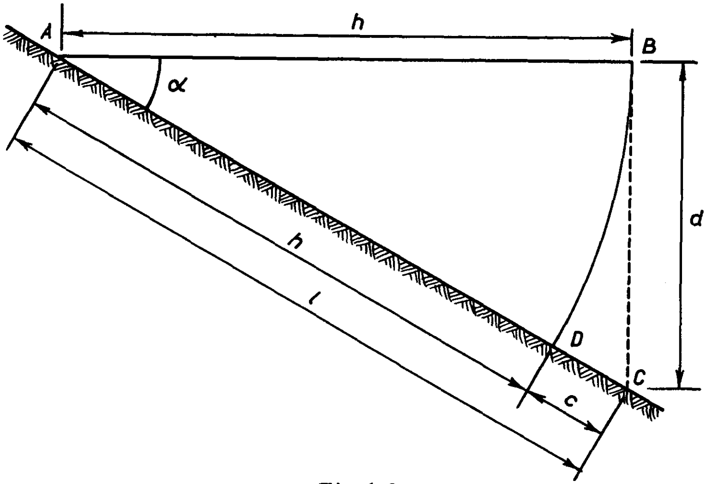


Fig. 1.3

(1) *Given the angle of inclination* α

$$AB = AC \cos\alpha$$

i.e. $$h = l\cos\alpha \tag{1.11}$$

$$c = l - h$$
$$= l - l\cos\alpha$$
$$= l(1-\cos\alpha) = l \text{ versine } \alpha \tag{1.12}$$

N.B. The latter equation is a better computation process.

Example 1.7 If $AC = 126{\cdot}3$ m, $\alpha = 2°34'$,

by Eq. (1.11) $AB = 126{\cdot}3 \cos 2°34'$

$= 126{\cdot}3 \times 0{\cdot}9990 = \underline{126{\cdot}174 \text{ m}}$

or by Eq. (1.12) $c = 126{\cdot}3\,(1 - 0{\cdot}999)$

$= 126{\cdot}3 \times 0{\cdot}001 = \underline{0{\cdot}126 \text{ m}}$

$\therefore$ $AB = 126{\cdot}3 - 0{\cdot}126 = \underline{126{\cdot}174 \text{ m}}$

Example 1.8 In chaining, account should be taken of any significant effect of the slope of the ground on the accuracy of the horizontal length. Calculate the minimum angle of inclination that gives rise to relative accuracies of 1/1000 and 1/3000.

From Eq. (1.12),

$$c = l - h = l(1-\cos\alpha)$$

$\therefore$ If $$\frac{c}{l} = \frac{1}{1000} = 1 - \cos\alpha$$

$$\cos\alpha = 1 - 0{\cdot}001 = 0{\cdot}999$$

$$\underline{\alpha = 2^\circ 34'} \quad \text{(i.e. 1 in 22)}$$

Also, if

$$\frac{c}{l} = \frac{1}{3000} = 1 - \cos\alpha$$

$$\cos\alpha = 1 - 0{\cdot}000\,3\dot{3}$$

$$= 0{\cdot}999\,67$$

$$\underline{\alpha = 1^\circ 29'} \quad \text{(i.e. 1 in 39)}$$

If the difference in level, d, is known

$$h = (l^2 - d^2)^{\frac{1}{2}} = \{(l-d)\times(l+d)\}^{\frac{1}{2}} \tag{1.13}$$

or

$$l^2 = h^2 + d^2$$

$$= (l-c)^2 + d^2$$

$$= l^2 - 2lc + c^2 + d^2$$

$$\therefore \quad c^2 - 2lc = -d^2$$

$$c(c-2l) = -d^2$$

$$c = \frac{-d^2}{c-2l}$$

$$c \simeq \frac{d^2}{2l} \qquad \text{as } c \text{ is small compared with } 2l \tag{1.14}$$

Rigorously, using the binomial expansion,

$$c = l - (l^2 - d^2)^{\frac{1}{2}}$$

$$= l - l\left(1 - \frac{d^2}{l^2}\right)^{\frac{1}{2}}$$

$$= l\left\{1 - \left(1 - \frac{d^2}{2l^2} + \frac{d^4}{8l^4} \cdots\right)\right\}$$

$$= \frac{d^2}{2l} - \frac{d^4}{8l^3} + \cdots \tag{1.15}$$

The use of the first term only gives the following relative accuracies (the units may be ft or metres).

Gradient	Error per 100 ft or m	Relative accuracy
1 in 4	0·051 ft (or m)	1/2 000
1 in 8	0·003 1 ft (or m)	1/30 000
1 in 10	0·001 3 ft (or m)	1/80 000
1 in 20	0·000 1 ft (or m)	1/1 000 000

Thus the approximation is acceptable for:

Chain surveying under all general conditions.

Traversing, gradients up to 1 in 10.

Precise measurement (e.g. base lines), gradients up to 1 in 20.

For setting out purposes

Here the horizontal length (h) is given and the slope length (l) is required.

$$l = h \sec \alpha$$

$$c = h \sec \alpha - h$$

$$= \underline{h(\sec \alpha - 1)} \qquad (1.16)$$

Writing $\sec \alpha$ as a series $1 + \frac{\alpha^2}{2} + \frac{5\alpha^4}{24} + \ldots$, where α is in radians, see p. 72.

$$c = h\left\{1 + \frac{\alpha^2}{2} + \frac{5\alpha^4}{24} + \ldots - 1\right\}$$

$$\simeq \underline{\frac{h\alpha^2}{2}} \qquad (\alpha \text{ in radians}) \qquad (1.17)$$

$$\simeq \frac{h}{2}(0{\cdot}017\,45\alpha)^2$$

$$\simeq \underline{1{\cdot}53\,h \times 10^{-4} \times \alpha^2} \qquad (\alpha \text{ in degrees}) \quad (1.18)$$

$$\simeq \underline{1{\cdot}53 \times 10^{-2} \times \alpha^2} \qquad \text{per 100 ft (or m)} \quad (1.19)$$

Example 1.9 If $h = 100$ ft (or m), $\alpha = 5°$;

by Eq. (1.16) $c = 100(1{\cdot}003\,820 - 1)$

$= \underline{0{\cdot}382\,0 \text{ ft (or m) per 100 ft}}$ (or m)

or by Eq. (1.18) $c = 1{\cdot}53 \times 100 \times 10^{-4} \times 5^2$

$= 1{\cdot}53 \times 25 \times 10^{-2}$

$= \underline{0{\cdot}382\,5 \text{ ft (or m) per 100 ft}}$ (or m)

Correction per 100 ft (or m)

1°	0·015 ft (or m)	6°	0·551 ft (or m)
2°	0·061 ft (or m)	7°	0·751 ft (or m)
3°	0·137 ft (or m)	8°	0·983 ft (or m)
4°	0·244 ft (or m)	9°	1·247 ft (or m)
5°	0·382 ft (or m)	10°	1·543 ft (or m)

If the difference in level, d, is given,

$$l^2 = h^2 + d^2$$

$$(h + c)^2 = h^2 + d^2$$

$$h^2 + 2hc + c^2 = h^2 + d^2$$

$$c(2h + c) = d^2$$

$$c = \frac{d^2}{2h + c}$$

$$c \simeq \frac{d^2}{2h} \tag{1.20}$$

or rigorously

$$c = \frac{d^2}{2h} + \frac{d^4}{8h^3} + \dots \tag{1.21}$$

N.B. If the gradient of the ground is known as 1 vertical to n horizontal the angle of inclination $\alpha \simeq \dfrac{57}{n}$ (1 rad $\simeq 57{\cdot}3^\circ$)

$$\text{e.g. 1 in 10 gives } \simeq \frac{57}{10} = 5{\cdot}7^\circ$$

To find the horizontal length h given the gradient 1 in n and the measured length l

$$\frac{h}{l} = \frac{n}{\sqrt{n^2+1}} = \frac{n\sqrt{n^2+1}}{n^2+1}$$

$$h = \frac{ln\sqrt{n^2+1}}{n^2+1} \tag{1.22}$$

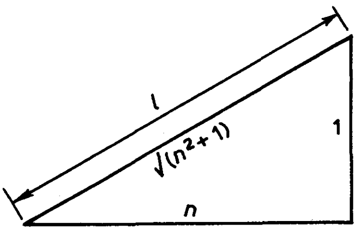


Fig.1.4

As an alternative to the above,

$$h = l - c$$

$$= l - \frac{d^2}{2l}$$

but if the gradient is given as 1 in n, then

$$d \simeq \frac{l}{n}$$

$$\therefore \quad h \simeq l - \frac{l^2}{2n^2 l}$$

$$\simeq l\left(1 - \frac{1}{2n^2}\right) \tag{1.23}$$

This is only applicable where $n > 20$.

Example 1.10 If a length of 300 ft (or m) is measured on a slope of 1 in 3, the horizontal length is given as:

by Eq. (1.22) $$h = \frac{300 \times 3\sqrt{10}}{10} = 90 \times 3{\cdot}1623$$

$$= \underline{284{\cdot}61 \text{ ft (or m)}}$$

To find the inclined length l given the horizontal length h and the gradient (1 in n)

$$\frac{l}{h} = \frac{\sqrt{n^2+1}}{n}$$

$$\therefore \quad l = \frac{h\sqrt{n^2+1}}{n} \qquad (1.24)$$

Example 1.11 If $h = 300$ ft (or m) and the gradient is 1 in 6,

by Eq. (1.24) $\quad l = \dfrac{300\sqrt{37}}{6} = 50 \times 6{\cdot}083$

$$= \underline{304{\cdot}15 \text{ ft}} \text{ (or m)}$$

1.42 The maximum length of offsets from chain lines

A point P is measured from a chain line ABC in such a way that B_1P is measured instead of BP, due to an error α in estimating the perpendicular, Fig.1.5.

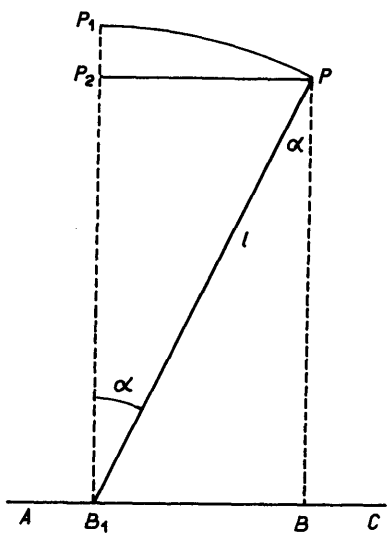


Fig.1.5

On plotting, P_1 is fixed from B_1.
Thus the displacement on the plan due to the error in direction α

$$PP_1 = B_1P\,\alpha(\text{radians})$$

$$= \frac{l\,\alpha''}{206\,265}$$

(N.B. 1 radian = 206 265 seconds of arc)

If the maximum length PP_1 represents the minimum plotable point, i.e. 0·01 in which represents $\frac{0\cdot01}{12}x$ ft, where x is the representative fraction $1/x$, then

$$0\cdot00083\dot{3}\,x = \frac{l\alpha}{206\,265}$$

$$l = \frac{171\cdot82\,x}{\alpha''}$$

Assuming the maximum error $\alpha = 4°$, i.e. 14 400″,

$$l = \frac{171\cdot82\,x}{14\,400} \simeq \underline{0\cdot012\,x} \qquad (1.25)$$

If the scale is 1/2 500, then $x = 2500$, and

$$\underline{l = 2500 \times 0\cdot012} = 30\text{ ft } (\simeq 10\text{ m})$$

If the point P lies on a fence approximately parallel to ABC, Fig. 1.6, then the plotted point will be in error by an amount P_1P_2 $= l(1 - \cos\alpha)$. (Fig. 1.5).

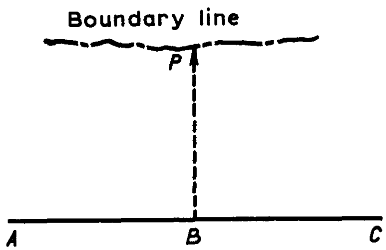


Fig. 1.6

$$\therefore \quad l = \frac{0\cdot01\,x}{12(1-\cos\alpha)} \qquad (1.26)$$

Example 1.12 If $\alpha = 4°$, by Eq. 1.26

$$l = \frac{0\cdot01\,x}{12\times(1-0\cdot9976)}$$

$$= 0\cdot35\,x \qquad (1.27)$$

Thus, if $x = 2500$,

$$l = 875\text{ ft } (267\text{ m})$$

The error due to this source is almost negligible and the offset is only limited by practical considerations, e.g. the length of the tape.

It is thus apparent that in fixing the position of a point that is critical, e.g. the corner of a building, the length of a perpendicular offset is limited to $0\cdot012\,x$ ft, and beyond this length tie lines are required,

the direction of the measurement being ignored, Fig.1.7.

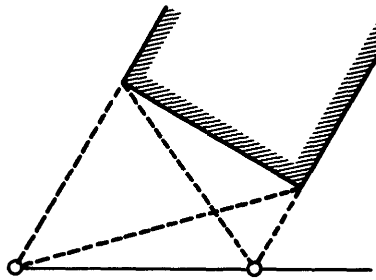


Fig.1.7

1.43 Setting out a right angle by chain

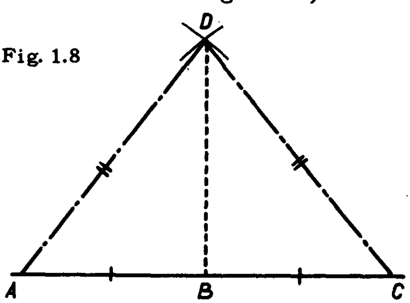
From a point on the chain line (Fig.1.8)

(a) (i) Measure off $BA = BC$

(ii) From A and C measure off $AD = CD$

(Proof: triangles ADB and DCB are congruent, thus $A\hat{B}D = D\hat{B}C = 90°$ as ABC is a straight line)

Fig. 1.8



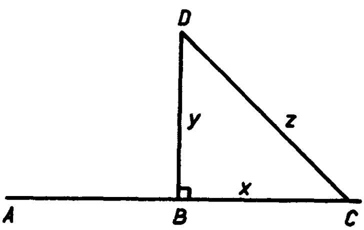


Fig. 1.9

(b) *Using the principle of Pythagoras,*

$$z^2 = x^2 + y^2 \quad \text{(Fig.1.9)}$$

By choosing suitable values the right angle may be set out. The basic relationship is

$$x : y : z :: 2n+1 : 2n(n+1) : 2n(n+1) + 1. \qquad (1.28)$$

If $n = 1$,

$$2n + 1 = 3$$
$$2n(n + 1) = 4$$
$$2n(n + 1) + 1 = 5.$$

Check:

$$\{2n(n + 1) + 1\}^2 = (2n^2 + 2n + 1)^2$$
$$(2n + 1)^2 + \{2n(n + 1)\}^2 = 4n^2 + 4n + 1 + 4n^4 + 8n^3 + 4n^2$$
$$= 4n^2 + 8n^3 + 8n^2 + 4n + 1$$

$$= (2n^2 + 2n + 1)^2.$$

Check:

$$\underline{5^2 = 3^2 + 4^2}$$

or $\underline{25 = 9 + 16.}$

Similarly, if $n = 3/4$,

$$2n + 1 = \frac{6}{4} + 1 = \frac{10}{4} = \frac{40}{16}$$

$$2n(n + 1) = \frac{6}{4}\left(\frac{3}{4} + 1\right) = \frac{6}{4} \times \frac{7}{4} = \frac{42}{16}$$

$$2n(n + 1) + 1 = \frac{42}{16} + \frac{16}{16} = \frac{58}{16}$$

Thus the ratios become 40 : 42 : 58 and this is probably the best combination for 100 unit measuring equipment; e.g. on the line ABC, Fig. 1.10, set out $BC = 40$ units. Then holding the ends of the chain at B and C the position of D is fixed by pulling taut at the 42/58 on the chain.

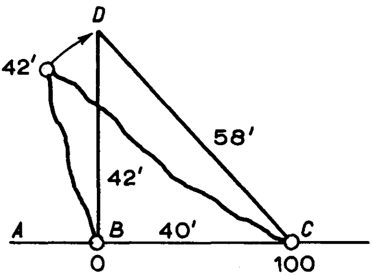


Fig. 1.10

Alternative values for n give the following:

$n = 2$	5, 12, 13	(Probably the best ratio for 30 m tapes)
$n = 3$	7, 24, 25	
$n = 4$	9, 40, 41.	

1.44 To find the point on the chain line which produces a perpendicular from a point outside the line

(1) *When the point is accessible* (Fig. 1. 11). From the point D swing the chain of length $> DB$ to cut the chain line at a and b. The required position B is then the mid-point of ab.

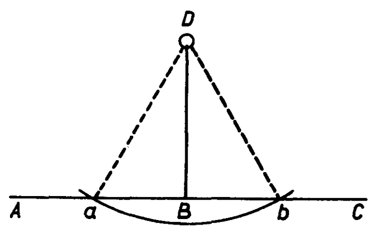


Fig. 1.11

(2) *When the point is not accessible* (Fig. 1.12). From D set out lines Da and Db and, from these lines, perpendicular ad and bc. The intersection of these lines at X gives the the line DX which when produced gives B, the required point.

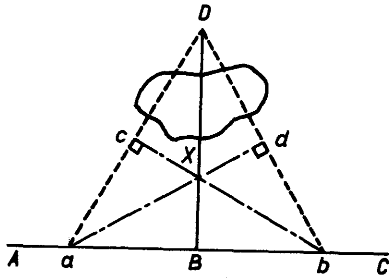


Fig. 1.12

To set out a line through a given point parallel to the given chain line (Fig. 1.13). Given the chain line AB and the given point C. From the given point C bisect the line CB at X. Measure AX and produce the line to D such that $AX = XD$. CD will then be parallel to AB.

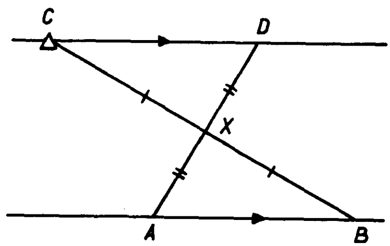


Fig. 1.13

1.45 Obstacles in chain surveying

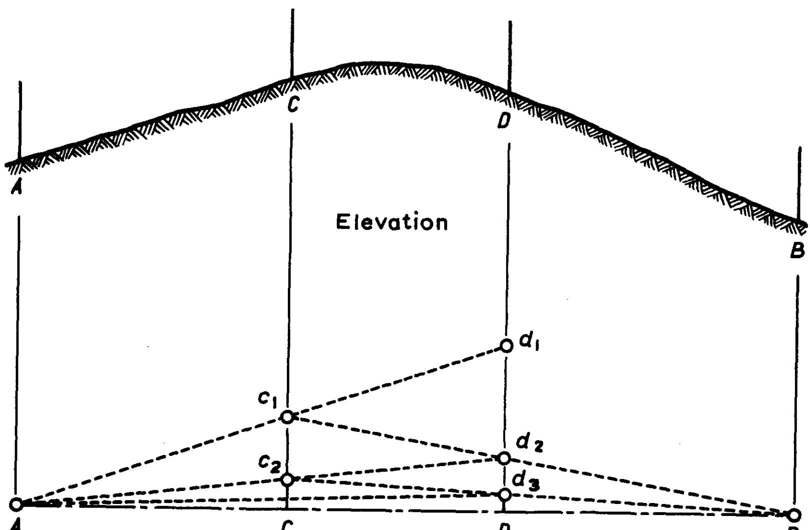


Fig. 1.14

(1) *Obstacles to ranging*

(a) *Visibility from intermediates* (Fig. 1.14). Required to line C and D on the line AB.

Place ranging pole at d_1 and line in c_1 on line Ad_1. From B observe c_1 and move d_2 on to line Bc_1. Repetition will produce c_2, c_3 and d_2, d_3 etc until C and D lie on the line AB.

(b) *Non-visibility from intermediates* (Fig. 1.15). Required to measure a long line AB in which A and B are not intervisible and intermediates on these lines are not possible.

Set out a 'random line' AC approximately on the line AB.

From B find the perpendicular BC to line AC as above. Measure AC and BC. Calculate AB.

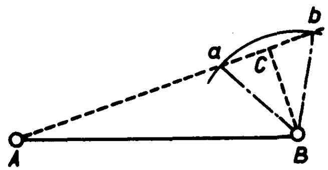


Fig. 1.15

(2) *Obstacles to chaining*

(a) *No obstacle to ranging*

(i) *Obstacle can be chained around.* There are many possible variations depending on whether a right angle is set out or not.

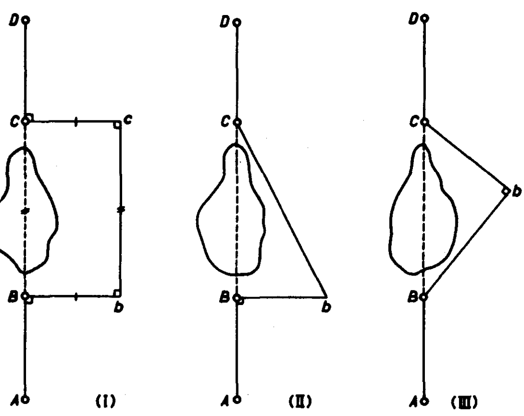


Fig. 1.16 By setting out right angles

I Set out equal perpendiculars Bb and Cc; then $bc = BC$.

II Set out Bb. Measure Bb and bC. Compute BC.

III Set out line Bb. At b set out the right angle to give C on the chain line. Measure Bb and bC. Compute BC.

IV and V Set out parallel lines bc as described above to give similar figures, triangles BCX and bcX.

Then
$$BC = \frac{bc \times BX}{bX} \tag{1.29}$$

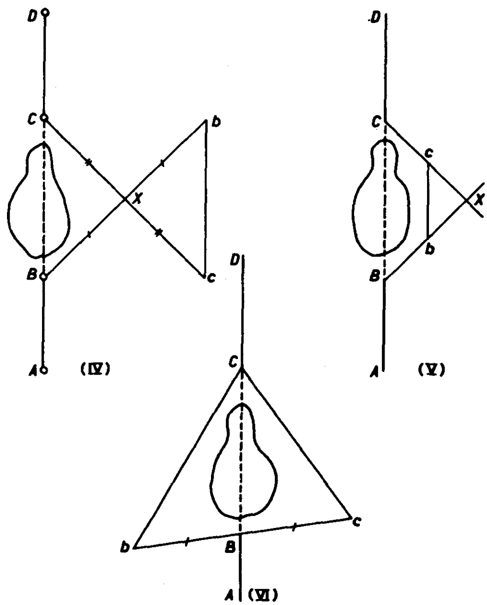


Fig. 1.17

VI Set out line bc so that $bB = Bc$. Compute BC thus,

$$BC^2 = \frac{(bC)^2\,Bc + (Cc)^2\,bB}{bc} - bB.Bc \tag{1.30}$$

but $bB = Bc$,

$$\therefore \quad BC^2 = \frac{Bb(bC^2 - Cc^2)}{2Bb} - Bb \times Bb$$

$$= \tfrac{1}{2}(bC^2 + Cc^2) - Bb^2 \tag{1.31}$$

Proof.

In Fig. 1.18 using the cosine rule (assuming $\theta > 90°$), see p. 81

$$p^2 = x^2 + d^2 + 2xd\cos\theta \quad \text{and}$$

$$q^2 = y^2 + d^2 - 2yd\cos\theta$$

$$\therefore \quad 2d\cos\theta = \frac{p^2 - x^2 - d^2}{x}$$

$$= \frac{y^2 + d^2 - q^2}{y}$$

$$\therefore \quad p^2y - x^2y - d^2y = xy^2 + d^2x - q^2x$$

$$d^2(x + y) = q^2x + p^2y - xy(x + y)$$

$$d^2 = \frac{q^2x + p^2y}{x + y} - xy$$

$$d = \sqrt{\frac{q^2x + p^2y}{x + y} - xy} \quad (1.32)$$

If $x = y$,

$$d = \sqrt{\frac{(p^2 + q^2)}{2} - x^2} \quad (1.33)$$

(ii) *Obstacle cannot be chained around.* A river or stream represents this type of obstacle. Again there are many variations depending on whether a right angle is set out or not.

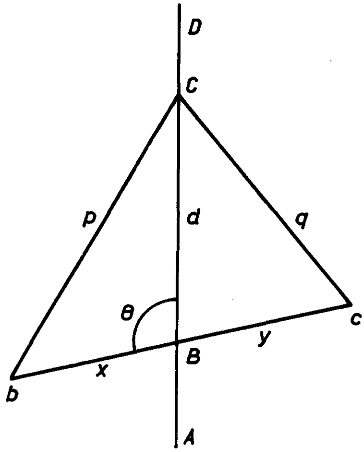


Fig. 1.18

By setting out right angles (Fig. 1.19).

A random line DA_1 is set out and from perpendiculars at C and B points C_1 and B_1 are obtained.

By similar triangles DC_1C and $C_1B_1B_2$,

$$\frac{DC}{CB} = \frac{CC_1}{BB_1 - CC_1}$$

$$\therefore \quad DC = \frac{CB \times CC_1}{BB_1 - CC_1}$$

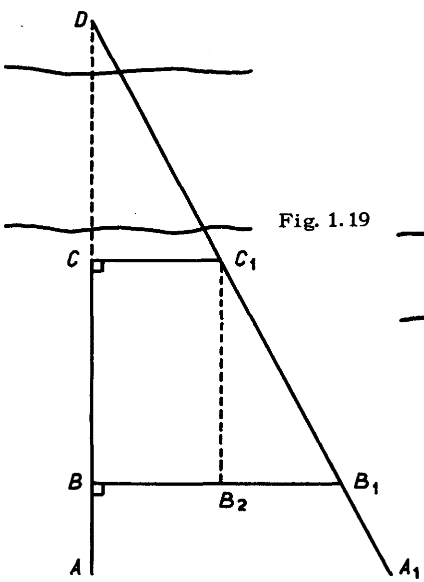


Fig. 1.19

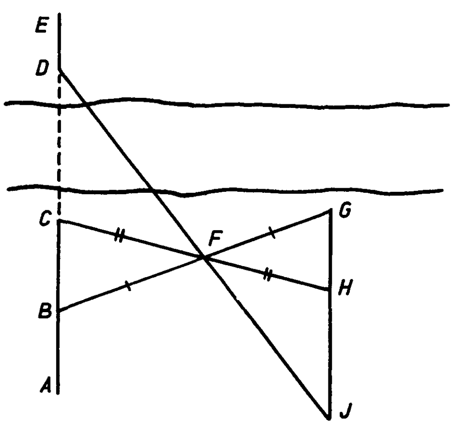


Fig. 1.20

Without setting out a right angle (Fig. 1.20).

A point F is chosen. From points B and C on line AE, BF and CF are measured and produced to G and H. $BF = FG$ and $CF = FH$. The intersection of DF and GH produce to intersect at J. Then $HJ = CD$.

(iii) *Obstacles which obstruct ranging and chaining.* The obstruction, e.g. a building, prevents the line from being ranged and thus produced beyond the obstacle.

By setting out right angles (Fig. 1.21)

On line ABC right angles are set out at B and C to produce B_1 and C_1, where $BB_1 = CC_1$.

B_1C_1 is now produced to give D_1 and E_1 where right angles are set out to give D and E, where $D_1D = E_1E = BB_1 = CC_1$. D and E are thus on the line ABC produced and $D_1C_1 = DC$.

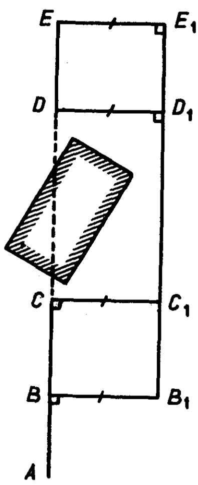


Fig. 1.21

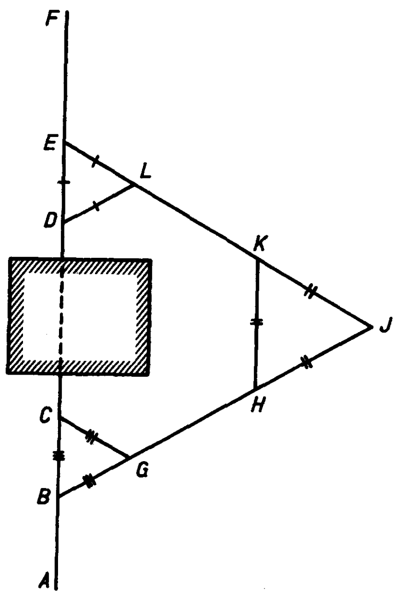


Fig. 1.22

Without setting out right angles (Fig. 1.22)

On line ABC, CB is measured and G set out to form an equilateral triangle, i.e. $CB = CG = BG$. BG is produced to J.

An equilateral triangle HKJ sets out the line JE such that $JE = BJ$.

A further equilateral triangle ELD will restore the line ABC produced.

The missing length $BE = BJ = EJ$.

Exercises 1 (a)

1. The following measurements were made on inclined ground. Reduce the slope distances to the horizontal giving the answer in feet.

(a) 200·1 yd at 1 in 2½
(b) 485·5 links at 1 in 5·75
(c) 1/24th of a mile at 1 in 10·25

(Ans. (a) 557·4 ft (b) 315·7 ft (c) 218·9 ft)

2. Calculate the acreage of an area of 4 in² on each of the plans drawn to scale, 2 chains to 1 in., 1/63 360, 1/2500 and 6 in. to 1 mile respectively.

(Ans. 1·6, 2560, 3·986, 71·1 acres)

3. A field was measured with a chain 0·3 of a link too long. The area thus found was 30 acres. What is the true area?

(I.C.E. Ans. 30·18 acres)

4. State in acres and decimals thereof the area of an enclosure measuring 4 in. square on each of three plans drawn to scale of 1/1584, 1/2500, 1/10 560 respectively.

(Ans. 6·4, 15·9, 284·4 acres)

5. A survey line was measured on sloping ground and recorded as 386·6 ft (117·84 m). The difference of elevation between the ends was 19·3 ft (5·88 m).

The tape used was later found to be 100·6 ft (30·66 m) when compared with a standard of 100 ft (30·48 m).

Calculate the corrected horizontal length of the line.

(Ans. 388·4 ft (118·38 m))

6. A plot of land in the form of a rectangle in which the length is twice the width has an area of 180 000 ft².

Calculate the length of the sides as drawn on plans of the following scales.

(a) 2 chains to 1 inch. (b) 1/25 000. (c) 6 inch to 1 mile.

(Ans. (a) 4·55 × 2·27 in. (b) 0·29 × 0·14 in. (c) 0·68 × 0·34 in.)

7. (a) Express the following gradients in degrees to the horizontal: 1 in 3, 1 in 200, 1 in 0·5, being vertical to horizontal in each case.

(b) Express the following scales as fractions: 6 in. to 1 mile, 1 in. to 1 mile, 1 in. to 1 chain, 1/8 in. to 1 ft.

(c) Express the following scales as inches to 1 mile: 1/2500, 1/500, 1/1080.

(M.Q.B./UM Ans. (a) 18°26′, 0°17′, 63°26′
(b) 1/10 560, 1/63 360, 1/792, 1/96
(c) 25·34, 126·72, 58·67)

8. Find, without using tables, the horizontal length in feet of a line recorded as 247·4 links when measured

(a) On ground sloping 1 in 4,

(b) on ground sloping at $18°26'$ ($\tan 18°26' = 0{\cdot}333$).

(Ans. (a) 158·40 (b) 154·89 ft)

9. Show that for small angles of slope the difference between the horizontal and sloping lengths is $h^2/2l$ (where h is the difference of vertical height of the two ends of a line of sloping length l).

If errors in chaining are not to exceed 1 part in 1000, what is the greatest slope that can be ignored?

(L.U./E Ans. 1 in 22·4)

1.5 Corrections to be Applied to Measured Lengths

For every linear measurement the following corrections must be considered, the need for their application depending on the accuracy required.

1. *In all cases*

   (a) Standardisation. (b) Slope.

2. *For relative accuracies of 1/5000 plus*

   (a) Temperature. (b) Tension.

   (c) Sag. (where applicable)

3. *For special cases, 1/50 000 plus*

   (a) Reduction to mean sea level. (b) Reduction to grid.

Consideration has already been given, p. 6/9, to both standardisation and reduction to the horizontal as they apply to chain surveying but more care must be exercised in precise measurement reduction.

1.51 Standardisation

The measuring band in the form of a tape or wire must be compared with a standard under specified conditions of temperature (t_s) and tension (T_s). If there is any variation from the nominal length then a standardisation correction is needed as already shown. A combination of temperature and standardisation can be seen under correction for temperature.

1.52 Correction for slope

Where the inclination of the measured length is obtained by measurement of the vertical angle the following modification should be noted.

Let the height of the instrument be h_1
the height of the target h_2

the measured vertical angle θ
the slope of the measured line α
the length of the measured line l

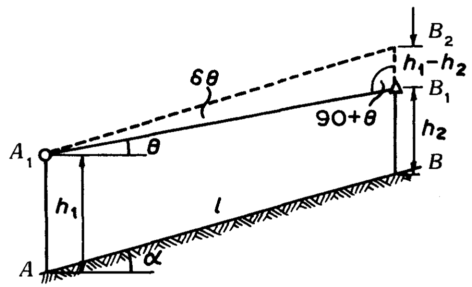


Fig. 1.23

In Fig. 1.23, $\alpha = \theta + \delta\theta$.
In triangle $A_1B_2B_1$ by the sine rule (see p. 80),

$$\sin\delta\theta = \frac{(h_1 - h_2)\sin(90 + \theta)}{l}$$

$$= \frac{(h_1 - h_2)\cos\theta}{l} \qquad (1.34)$$

$$\therefore \quad \delta\theta'' = \frac{206\,265\,(h_1 - h_2)\cos\theta}{l} \qquad (1.35)$$

N.B. The sign of the correction conforms precisely to the equation.

(1) If $h_1 = h_2$, $\delta\theta = 0 \quad \alpha = \theta$
(2) If $h_1 < h_2$ and θ is +ve, $\delta\theta$ is −ve (Fig. 1.24a)
(3) If $h_1 > h_2$ and θ is −ve, $\delta\theta$ is −ve (Fig. 1.24d)
if α is +ve, $\delta\theta$ is +ve (Fig. 1.24b)
(4) If $h_1 < h_2$ and θ is −ve, $\delta\theta$ is +ve (Fig. 1.24c)

Example 1.13

If h_1 = 4·5 ft (1·37 m), h_2 = 5·5 ft (1·68 m), $\theta = +4°30'$
l = 350 ft (106·68 m)

$$\text{then } \delta\theta = \frac{206\,265\,(4{\cdot}5 - 5{\cdot}5)\cos 4°30'}{350}$$

$$= -588''$$

$$= -0°09'48''$$

$$\therefore \quad \alpha = +4°30'00'' - 0°09'48''$$

$$= +4°20'12''$$

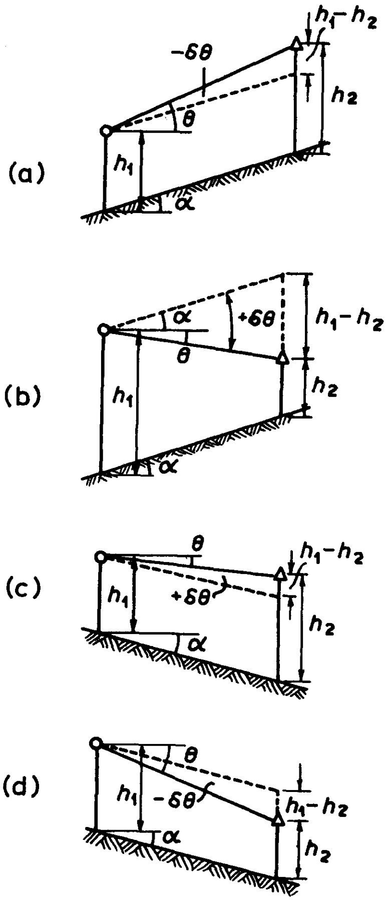


Fig. 1.24

Correction to measured length (by Eq. 1.12),

$$
\begin{aligned}
c &= -l(1 - \cos\theta^\circ) \\
&= -350\,(1 - \cos 4^\circ 20'\, 12'') \\
&= -350\,(1 - 0{\cdot}997\,14) \\
&= -350 \times 0{\cdot}002\,86 = -1{\cdot}001 \text{ ft } (0{\cdot}3051 \text{ m})
\end{aligned}
$$

$\therefore$ Horizontal length $= 348{\cdot}999$ ft ($106{\cdot}3749$ m)

If the effect was ignored;

Horizontal length $= 350 \cos 4^\circ 30'$

$= 348{\cdot}922$ ft ($106{\cdot}3514$ m)

$\therefore$ Error $= 0{\cdot}077$ ft ($0{\cdot}0235$ m)

1.53 Correction for temperature

The measuring band is standardised at a given temperature (t_s). If in the field the temperature of the band is recorded as (t_m) then the band will expand or contract and a correction to the measured length is given as

$$c = la(t_m - t_s) \tag{1.36}$$

where l = the measured length

α = the coefficient of linear expansion of the band metal.

The coefficient of linear expansion (α) of a solid is defined as 'the increase in length per unit length of the solid when its temperature changes by one degree'.

For steel the average value of α is given as

$$6{\cdot}2 \times 10^{-6} \text{ per } ^\circ\text{F}$$

Since a change of 1 °F = a change of 5/9 °C, using the value above gives

$$\alpha = 6{\cdot}2 \times 10^{-6} \text{ per } 5/9\ ^\circ\text{C}$$
$$= 11{\cdot}2 \times 10^{-6} \text{ per } ^\circ\text{C}$$

The range of linear coefficients α is thus given as:

	per 1 °F	per 1 °C
Steel	5·9 to 6·8	10·6 to 12·2 ($\times 10^{-6}$)
Invar	3 to 4	5·4 to 7·2 ($\times 10^{-7}$)

To find the new standard temperature t_s' which will produce the nominal length of the band.

Standard length at $t_s = l \pm \delta l$

To reduce the length by δl:

$$\delta l = (l \pm \delta l) \, . \, \alpha \, . \, t$$

where t = number of degrees of temperature change required

$$\therefore \quad t = \frac{\delta l}{(l \pm \delta l)\alpha}$$

$$t_s' = t_s \mp \frac{\delta l}{(l \pm \delta l)\alpha} \tag{1.37}$$

As δl is small compared with l, for practical purposes

$$t_s' = t_s \pm \frac{\delta l}{l\alpha} \tag{1.38}$$

Example 1.14 A traverse line is 500 ft (152·4 m) long. If the tape used in the field is 100 ft (30·48 m) when standardised at 63 °F (17·2 °C), what correction must be applied if the temperature at the time of measurement is 73 °F (22·8 °C)?

$$(\text{Assume } \alpha = 6{\cdot}2 \times 10^{-6} \text{ per deg F} = 11{\cdot}2 \times 10^{-6} \text{ per deg C})$$

From Eq. (1.36)

$$c_{(ft)} = 500 \times 6{\cdot}2 \times 10^{-6} \times (73 - 63) = \underline{+0{\cdot}0310 \text{ ft}}$$

or

$$c_{(m)} = 152{\cdot}4 \times 11{\cdot}2 \times 10^{-6} \times (22{\cdot}8 - 17{\cdot}2) = \underline{+0{\cdot}0096 \text{ m}}$$

Example 1.15 If a field tape when standardised at 63 °F measures 100·005 2 ft, at what temperature will it be exactly the nominal value?

$$(\text{Assume } \alpha = 6{\cdot}5 \times 10^{-6} \text{ per deg F})$$

$$\delta l = +0{\cdot}0052 \text{ ft}$$

∴ from Eq. (1.37)

$$t_s' = 63 - \frac{0{\cdot}0052}{100 \times 6{\cdot}5 \times 10^{-6}} = 63\,°\text{F} - 8\,°\text{F} = \underline{55\,°\text{F}}$$

In its metric form the above problem becomes: If a field tape when standardised at 17·2 °C measures 100·005 2 m, at what temperature will it be exactly the nominal value?

$$(\text{Assume } \alpha = 11{\cdot}2 \times 10^{-6} \text{ per deg C})$$

$$\delta l = +0{\cdot}0052 \text{ m}$$

∴ from Eq. (1.37)

$$t_s' = 17{\cdot}2 - \frac{0{\cdot}0052}{100 \times 11{\cdot}2 \times 10^{-6}} = 17{\cdot}2\,°\text{C} - 4{\cdot}6\,°\text{C} = \underline{12{\cdot}6\,°\text{C}}\ (= 54{\cdot}7\,°\text{F})$$

1.54 Correction for tension

The measuring band is standardised at a given tension (T_s). If in the field the applied tension is (T_m) then the tape will, due to its own elasticity; expand or contract in accordance with Hooke's Law.

A correction factor is thus given as

$$c = \frac{L(T_m - T_s)}{A.E} \tag{1.39}$$

where L = the measured length (the value of c is in the same unit as L),

A = cross-sectional area of the tape,

E = Young's modulus of elasticity i.e. stress/strain.

The units used for T, A and E must be compatible, e.g.

	T (lbf)	A (in^2)	E (lbf/in^2)	
or	T_1 (kgf)	A_1(cm^2)	E_1(kgf/cm^2)	(metric)
or	T_2(N)	A_2(m^2)	E_2(N/m^2)	(new S.I. units)

Conversion factors

$$1 \text{ lb} = 0{\cdot}453\,592 \text{ kg}$$

$$1 \text{ in}^2 = 6{\cdot}4516 \times 10^{-4} \text{ m}^2$$

$$\therefore \quad 1 \text{ lb/in}^2 = 703{\cdot}070 \text{ kg/m}^2$$

Based on the proposed use of the International System of Units (S.I. units) the unit of force is the Newton (N), i.e. the force required to accelerate a mass of 1 kg 1 metre per second per second.

The force 1 lbf = mass × gravitational acceleration

$= 0{\cdot}453\,592 \times 9{\cdot}806\,65$ m/s^2 (assuming standard value)

$= 4{\cdot}448\,22$N

1 kgf $= 9{\cdot}806\,65$N (1 kg = $2{\cdot}204\,62$ lb)

whilst for stress 1 lbf/in^2 = $6894{\cdot}76$ N/m^2

For steel, $E \simeq 28$ to 30×10^6 lbf/in^2 (British units)

$\simeq 20$ to 22×10^5 kgf/cm^2 (Metric units)

$\simeq 19{\cdot}3$ to $20{\cdot}7 \times 10^{10}$ N/m^2 (S.I. units)

For invar, $E \simeq 20$ to 22×10^6 lbf/in^2

$\simeq 14$ to $15{\cdot}5 \times 10^5$ kgf/cm^2

$\simeq 13{\cdot}8$ to $15{\cdot}2 \times 10^{10}$N/m^2

N.B. (1) If $T_m = T_s$ no correction is necessary.

(2) It is generally considered good practice to over tension to minimise deformation of the tape, the amount of tension being strictly recorded and the correction applied.

(3) The cross-sectional area of the tape may be physically measured using a mechanical micrometer, or it may be computed from the total weight W of the tape of length L and a value ρ for the density of the material.

$$A = \frac{W}{L\rho} \qquad (1.40)$$

Example 1.16 A tape is 100 ft at a standard tension of 25 lbf and measures in cross-section 0·125 in. × 0·05 in. If the applied tension is 20 lbf and $E = 30 \times 10^6$ lbf/in², calculate the correction to be applied.

By Eq. 1.39 $$c = \frac{100 \times (20 - 25)}{(0{\cdot}125 \times 0{\cdot}05) \times (30 \times 10^6)} = \underline{-0{\cdot}0027 \text{ ft}}$$

Converting the above units to the metric equivalents gives

$$c = \frac{30{\cdot}48\,\text{m} \times (9{\cdot}072 - 11{\cdot}340)\,\text{kgf}}{(40{\cdot}32 \times 10^{-7})\,\text{m}^2 \times (21{\cdot}09 \times 10^9)\,\text{kgf/m}^2}$$
$$= \underline{-0{\cdot}008\,13\,\text{m}} \quad (\text{i.e. } -0{\cdot}002\,7\,\text{ft})$$

Based on the International System of Units,

$$2{\cdot}268\,\text{kgf} = 2{\cdot}268 \times 9{\cdot}806\,65\,\text{N} = 22{\cdot}241\,\text{N}$$

or $$5\,\text{lbf} = 5 \times 4{\cdot}448\,22\,\text{N} = 22{\cdot}241\,\text{N}.$$

For stress,

$$(21{\cdot}09 \times 10^{10})\,\text{kgf/m}^2 = 21{\cdot}09 \times 10^9 \times 9{\cdot}806\,65 = 20{\cdot}684 \times 10^{10}\,\text{N/m}^2$$

or

$$(30 \times 10^6)\,\text{lbf/in}^2 = 30 \times 10^6 \times 6894{\cdot}76 = 20{\cdot}684 \times 10^{10}\,\text{N/m}^2$$

Thus, in S.I. units,

$$c = \frac{30{\cdot}48 \times 22{\cdot}241\,\text{N}}{(40{\cdot}32 \times 10^{-7}) \times (20{\cdot}684 \times 10^{10})\,\text{N/m}^2}$$
$$= \underline{-0{\cdot}008\,13\,\text{m}}$$

Measurement in the vertical plane

Where a metal tape is freely suspended it will elongate due to the applied tension produced by its own weight.

The tension is not uniform and the stress varies along its length.

Given an unstretched tape AB and a stretched tape AB_1, Fig. 1.25, let P and Q be two close points on the tape which become P_1Q_1 under tension.

If $AP = x$, $AP_1 = x + s$, where s is the amount of elongation of AP. Let $PQ = dx$, then $P_1Q_1 = dx + ds$ and the strain in $P_1Q_1 = \dfrac{ds}{dx}$ as ds is the increase in length and dx is the original length.

If T is the tension at P_1, in a tape of cross-section A, and E is Young's modulus, then

$$T = EA\frac{ds}{dx} \tag{1}$$

Given that the load per unit length at P_1 is w then in P_1Q_1 the load $= w.dx$ being the difference in tension between P_1 and Q_1

$\therefore$ if the tension at Q is $T + dT$

$$T - (T + dT) = w\,dx$$

$$\text{i.e.} \quad dT = -w\,dx \qquad (2)$$

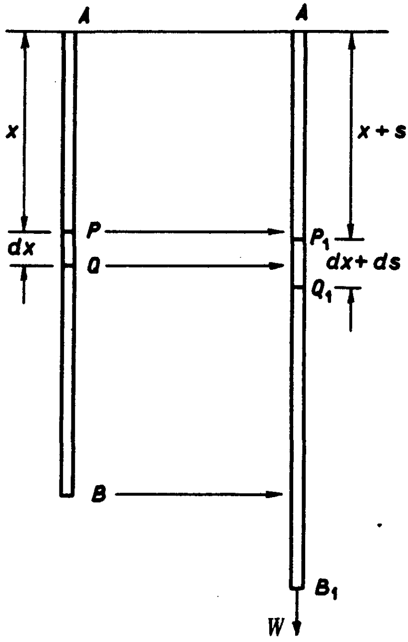


Fig.1.25 Elongation in a suspended tape

In practice the value w is a function of x and by integrating the two equations the tension and extension are derived.

Assuming the weight per unit length of the tape is w with a suspended weight W, then from (2)

$$dT = -w\,dx$$

$$T = -wx + c \qquad (3)$$

$$\text{and (1)} \quad T = EA\frac{ds}{dx}$$

$$\therefore \quad EA\frac{ds}{dx} = -wx + c$$

$$\therefore \quad EAs = -\tfrac{1}{2}wx^2 + cx + d \qquad (4)$$

When $T = W$, $x = l$ and when $x = 0$, $s = 0$

$$\therefore \quad W = -wl + c \qquad \text{i.e. } c = W + wl$$

$$\text{and} \quad d = 0.$$

Therefore putting constants into equations (3) and (4) gives

$$T = -wx + W + wl$$

$$T = W + w(l - x). \qquad (1.41)$$

and

$$EAs = -\tfrac{1}{2}wx^2 + Wx + wlx$$

$$= Wx + \tfrac{1}{2}w(2lx - x^2)$$

$$\therefore\ s = \frac{1}{EA}\left[Wx + \tfrac{1}{2}w(2lx - x^2)\right] \qquad (1.42)$$

If $x = l$, then

$$s = \frac{1}{EA}\left[Wl + \tfrac{1}{2}wl^2\right] \qquad (1.43)$$

and if $W = 0$,

$$s = \frac{wl^2}{2EA} \qquad (1.44)$$

Example 1.17 Calculate the elongation at (1) 1000 ft and (2) 3000 ft of a 3000 ft mine-shaft measuring tape hanging vertically due to its own weight.

The modulus of elasticity is 30×10^6 lbf/in²; the weight of the tape is 0·05 lbf/ft and the cross-sectional area of the tape is 0·015 in².

From Eq. (1.42)

$$s = \frac{1}{EA}\left[Wx + \tfrac{1}{2}w(2lx - x^2)\right]$$

As $W = 0$,

$$s = \frac{w}{2EA}[2lx - x^2]$$

when $x = 1000$ ft

$l = 3000$ ft

$$s = \frac{0{\cdot}05}{2 \times 30 \times 10^6 \times 0{\cdot}015}[2 \times 3000 \times 1000 - 1000^2]$$

$$= \frac{0{\cdot}05 \times 5 \times 10^6}{2 \times 30 \times 10^6 \times 0{\cdot}015} = \underline{0{\cdot}278\text{ ft}}$$

when $x = l = 3000$ ft.

From Eq. (1.44) $s = \dfrac{wl^2}{2EA}$

$$= \frac{0{\cdot}05 \times 3000^2}{2 \times 30 \times 10^6 \times 0{\cdot}015} = \underline{0{\cdot}500\text{ ft}}$$

Example 1.18 If the same tape is standardised as 3000 ft at 45 lbf tension what is the true length of the shaft recorded at 2998·632 ft?

In Eq.(1.44) $$s = \frac{wl^2}{2EA} = \frac{\frac{1}{2}WL}{EA}$$

i.e. $$T = \tfrac{1}{2}W$$

where W = total weight of tape = 3000 × 0·05 = 150 lbf

Applying the tension correction, Eq.(1.39),

$$c = \frac{L(T_m - T_s)}{EA}$$

$$= \frac{3000\,(75 - 45)}{30 \times 10^6 \times 0{\cdot}015} = \frac{30 \times 10^2 \times 30}{30 \times 10^6 \times 0{\cdot}015} = +0{\cdot}200 \text{ ft}$$

∴ true length

$$= 2998{\cdot}632 + 0{\cdot}2 = \underline{2998{\cdot}832 \text{ ft}}$$

1.55 Correction for sag

The measuring band may be standardised in two ways, (a) on the flat or (b) in catenary.

If the band is used in a manner contrary to the standard conditions some correction is necessary.

(1) *If standardised on the flat and used in catenary* the general equation for correction is applied, viz.

$$c = -\frac{w^2l^3}{24T^2} \tag{1.45}$$

or $$c = -\frac{W^2l}{24T^2} \tag{1.46}$$

where w = weight of tape or wire per unit length
$W = wl$ = total weight of tape in use,
T = applied tension.

N.B. the units w and T must be compatible (lbf, kgf or N)

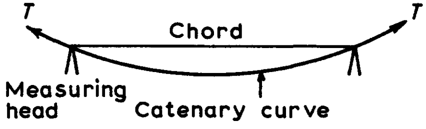


Fig.1.26 Measurement in catenary

(2) *If standardised in catenary*

(a) The length of the chord may be given relative to the length

of the tape or

(b) the length of the tape in catenary may be given.

(i) If the tape is used on the flat a positive sag correction must be applied

(ii) If the tape is used in catenary at a tension T_m which is different from the standard tension T_s, the correction will be the difference between the two relative corrections, i.e.

$$c = -\frac{W^2 l}{24}\left[\frac{1}{T_m^2} - \frac{1}{T_s^2}\right] \tag{1.47}$$

If $T_m > T_s$ the correction will be positive.

(iii) If standardised in catenary using a length l_s and then applied in the field at a different length l_m, the correction to be applied is given as

$$c = \frac{l_m}{l_s}\left(\frac{l_s^3 w^2}{24T^2} - \frac{l_m^3 w^2}{24T^2}\right)$$

$$= \frac{l_m w^2}{l_s 24T^2}\ (l_s^3 - l_m^3) \tag{1.48}$$

Alternatively, the equivalent tape length on the flat may be computed for each length and the subsequent catenary correction applied for the new supported condition, i.e. if l_s is the standard length in catenary, the equivalent length on the ground $= l_s + c_s$, where c_s = the catenary correction.

If l_m is the applied field length, then its equivalent length on the flat $= \frac{l_m}{l_s}(l_s + c_s)$

Applying the catenary correction to this length gives

$$l_m + c = \frac{l_m}{l_s}\ (l_s + c_s) - c_m$$

$$= l_m + \frac{l_m c_s}{l_s} - c_m$$

Thus the required correction

$$c = \frac{l_m c_s}{l_s} - c_m$$

$$= \frac{l_m}{l_s}\left(\frac{l_s^3 w^2}{24T^2}\right) - \frac{l_m^3 w^2}{24T^2}$$

$$= \frac{l_m w^2}{l_s 24T^2}\ (l_s^3 - l_m^3) \quad \text{as Eq. (1.48) above.}$$

The sag correction is an acceptable approximation based on the assumption that the measuring heads are at the same level. If the heads are at considerably different levels, Fig. 1.27, the correction should be

$$c = c_1 \cos^2\theta \left(1 \pm \frac{wl}{T} \sin\theta\right) \qquad (1.49)$$

the sign depending on whether the tension is applied at the upper or lower end of the tape.

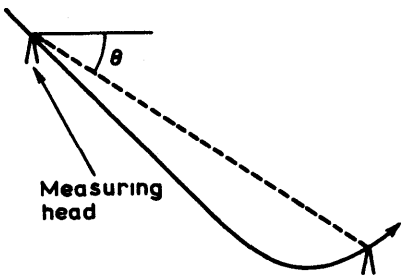


Fig. 1.27

For general purposes $c = c_1 \cos^2\theta$

$$= \frac{w^2 l^3 \cos^2\theta}{24T^2} \qquad (1.50)$$

The weight of the tape determined in the field

The catenary sag of the tape can be used to determine the weight of of the tape, Fig. 1.28

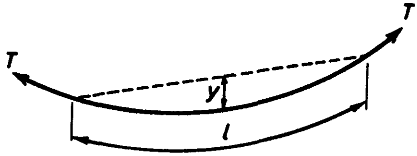


Fig. 1.28 Weight of the tape determined in the field

If y is the measured sag at the mid-point, then the weight per unit length is given as

$$w = \frac{8Ty}{l^2} \qquad (1.51)$$

or the amount of sag $$y = \frac{wl^2}{8T} \qquad (1.52)$$

where w = weight/unit length, y = vertical sag at the mid-point,
T = applied tension, l = length of tape between supports.

Example 1.19 Calculate the horizontal length between two supports, approximately level, if the recorded length is 100·237 ft, the tape weighs 15 ozf and the applied tension is 20 lbf.

From Eq. (1.46) $$c = -\frac{W^2 l}{24T^2}$$

The value of l is assumed to be 100 for ease of computation.

Then $$c = -\frac{\left(\frac{15}{16}\right)^2 \times 100}{24 \times 20^2}$$

$$= -0{\cdot}0092 \text{ ft}$$

True length $$= 100{\cdot}2370 - 0{\cdot}0092$$

$$= \underline{100{\cdot}2278 \text{ ft}}$$

Example 1.20 A 100 ft tape standardised in catenary at 25 lbf is used in the field with a tension of 20 lbf. Calculate the sag correction if $w = 0{\cdot}021$ lbf/ft.

From Eq. (1.47) $$c = -\left(\frac{l^3 w^2}{24}\,\frac{1}{T_m^2} - \frac{1}{T_s^2}\right)$$

$$= -\frac{100^3 \times 0{\cdot}021^2}{24}\left(\frac{1}{20^2} - \frac{1}{25^2}\right)$$

$$= -0{\cdot}01656 \qquad \text{i.e. } \underline{-0{\cdot}0166 \text{ ft.}}$$

Example 1.21 A tape 100 ft long is suspended in catenary with a tension of 30 lbf. At the mid-point the sag is measured as 0·55 ft. Calculate the weight per ft of the tape.

From Eq. (1.51),

$$w = \frac{8Ty}{l^2} = \frac{8 \times 30 \times 0{\cdot}55}{10\,000} = \underline{0{\cdot}0132 \text{ lbf/ft.}}$$

Based on S.I. units these problems become

1.19(a) Calculate the horizontal length between two supports approximately level if the recorded length is 30·5522 m; the tape weighs 0·425 kgf and the applied tension is 9·072 kgf.

Converting the weight and tension into units of force,

$$c = \frac{30{\cdot}5522}{24}\,\frac{(0{\cdot}425 \times 9{\cdot}80665)^2}{(9{\cdot}072 \times 9{\cdot}80665)^2}$$

Thus there is no significance in changing the weight W and tension T into units of force, though the unit of tension must be the newton.

$$c = \frac{30{\cdot}552\,2}{24}\left(\frac{0{\cdot}425}{9{\cdot}072}\right)^2$$

$$= \underline{-0{\cdot}002\,8\text{ m}} \quad (-0{\cdot}009\,2\text{ ft}).$$

1.20(a) A 30·48 m tape standardised in catenary at 111·21 N is used in the field with a tension of 88·96 N. Calculate the sag correction if $w = 0{\cdot}031\,2$ kgf/m.

Conversion of the mass/unit length w into a total force gives

$$30{\cdot}48 \times 0{\cdot}031\,2 \times 9{\cdot}806\,65 = 9{\cdot}326\text{ N}.$$

∴ Eq. (1.47) becomes

$$c = \frac{-lW^2}{24}\left(\frac{1}{T_m^2} - \frac{1}{T_s^2}\right)$$

$$= \frac{-30{\cdot}48 \times 9{\cdot}326^2}{24}\left(\frac{1}{88{\cdot}96^2} - \frac{1}{111{\cdot}21^2}\right)$$

$$= \underline{-0{\cdot}005\,04\text{ m}} \quad (-0{\cdot}016\,6\text{ ft}).$$

1.21(a) A tape 30·48 m long is suspended in catenary with a tension of 133·446 N. At the mid-point the sag is measured as 0·168 m. Calculate the weight per metre of the tape.

Eq. (1.51) becomes

$$w\text{ (kgf/m)} = \frac{8 \times T \times y}{9{\cdot}806\,65\ l^2} = \frac{0{\cdot}816\,Ty}{l^2}$$

$$= \frac{0{\cdot}816 \times 133{\cdot}446 \times 0{\cdot}168}{30{\cdot}48^2}$$

$$= \underline{0{\cdot}019\,6\text{ kgf/m}} \quad (0{\cdot}013\,2\text{ lbf/ft})$$

Example 1.22 A tape nominally 100 ft is standardised in catenary at 10 lbf and is found to be 99·933 ft. If the weight per foot is 0·01 lbf, calculate the true length of a span recorded as 49·964 ft.

Standardised length = 99·933 ft

Sag correction for 100 ft

$$c_1 = \frac{0{\cdot}01^2 \times 100^3}{24 \times 10^2} = 0{\cdot}042\text{ ft}$$

True length on the flat = 99·975 ft

True length of sub-length on flat

$$= \frac{49{\cdot}964}{100} \times 99{\cdot}9747 = 49{\cdot}952$$

Sag correction for 50 ft ($c \propto l^3$)

$$= 1/8\, c_1 = \underline{-0{\cdot}005}$$

True length between supports $= 49{\cdot}947$ ft

Alternatively, by Eq. (1.48)

$$c = \frac{50 \times 0{\cdot}01^2}{100 \times 24 \times 10^2}(50^3 - 100^3)$$

$$= -0{\cdot}018 \text{ ft}$$

$\therefore$ true length between supports $= 49{\cdot}964 - 0{\cdot}018$

$$= \underline{49{\cdot}946 \text{ ft.}}$$

Example 1.23 A copper transmission line, ½in. diameter, is stretched between two points, 1000 ft apart, at the same level, with a tension of ½ ton, when the temperature is 90 °F. It is necessary to define its limiting positions when the temperature varies. Making use of the corrections for sag, temperature and elasticity normally applied to base line measurements by tape in catenary, find the tension at a temperature of 10 °F and the sag in the two cases.

Young's modulus for copper 10×10^6 lbf/in², its density 555 lb/ft³, and its coefficient of linear expansion $9{\cdot}3 \times 10^{-6}$ per °F.

(L.U.)

The length of line needed $= 1000 + \delta l$

where $\delta l =$ added length due to sag

$$= \frac{w^2 l^3}{24T^2}$$

$$w = \pi r^2 \rho \text{ lbf/ft}$$

$$= \frac{3{\cdot}142 \times 0{\cdot}25^2 \times 555}{144} = 0{\cdot}757 \text{ lbf/ft}$$

$$\therefore \quad \delta l = \frac{0{\cdot}757^2 \times 1000^3}{24 \times 1120^2} = 19{\cdot}037 \text{ ft}$$

Total length of wire $= 1019{\cdot}037$ ft

amount of sag $\quad y = \dfrac{w l^2}{8T}$

$$= \frac{0{\cdot}759 \times 1019^2}{8 \times 1120} = \underline{87{\cdot}73 \text{ ft}}$$

when temperature falls to 10 °F,

$$\text{Contraction of wire} = L\alpha t$$
$$= 1000 \times 9{\cdot}3 \times 10^{-6} \times (90 - 10) = 0{\cdot}758\,\text{ft}$$
$$\text{new length of wire} = 1019{\cdot}037 - 0{\cdot}758 = 1018{\cdot}279\,\text{ft}$$

as $\delta l \propto \dfrac{1}{T^2}$

$$T_2^2 = T_1^2 \frac{\delta l_1}{\delta l_2}$$

$$T_2 = 1120\sqrt{\frac{19{\cdot}037}{18{\cdot}279}} = 1120 \times 1{\cdot}0205 = \underline{1142\,\text{lbf}}$$

Amount of sag at 10 °F

$$\left(y \propto \frac{1}{T}\right)$$
$$= y_1 \times \frac{T_1}{T_2}$$
$$= 87{\cdot}73 \times \frac{1120}{1142} = \underline{86{\cdot}03\,\text{ft}}$$

1.56 Reduction to mean sea level (Fig. 1.29)

If the length at mean sea level is L and h = height of line above or below mean sea level, then

$$\frac{L}{l_m} = \frac{R}{R \pm h}$$
$$\therefore \quad L = \frac{l_m R}{R \pm h}$$

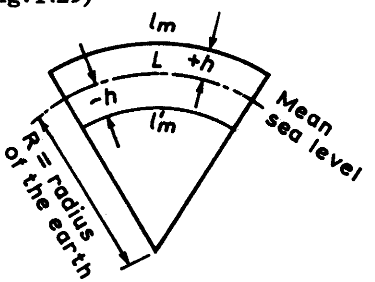


Fig. 1.29 Reduction to mean sea level

If $L = l_m \mp c$, then

$$c = l_m \mp \frac{l_m R}{R \pm h}$$
$$= l_m\left[1 \mp \frac{R}{R \pm h}\right] \tag{1.53}$$
$$= \mp \frac{l_m h}{R \pm h}$$

As h is small compared with R, $c = \pm \frac{l_m h}{R}$ (1.54)

If $R \simeq 3960$ miles,

$$c = \frac{100h}{3960 \times 5280}$$

$$\simeq 4{\cdot}8h \times 10^{-6} \text{ per 100 ft} \quad (1.55)$$

1.57 Reduction of ground length to grid length

The local scale factor depends on the properties of the projection. Here we will consider only the Modified Transverse Mercator projection as adopted by the Ordnance Survey in the British Isles.

Local scale factor (F)

$$F = F_0\left(1 + \frac{\mathrm{E}^2}{2\rho\nu}\right) \quad (1.56)$$

where F_0 = the local scale factor at the central meridian,
E = the Easting in metres from the true origin,
ρ = the radius of curvature to the meridian,
ν = the radius of curvature at right angles to the meridian.

Assuming $\rho \simeq \nu = R$, then

$$F = F_0\left(1 + \frac{\mathrm{E}^2}{2R^2}\right) \quad (1.57)$$

For practical purposes,

$$F \simeq F_0(1 + 1{\cdot}23\,\mathrm{E}^2 \times 10^{-8}) \quad (1.58)$$

$$\simeq 0{\cdot}999\,601\,3(1 + 1{\cdot}23\,\mathrm{E}^2 \times 10^{-8}) \quad (1.59)$$

N.B. E = Eastings − 400 km.

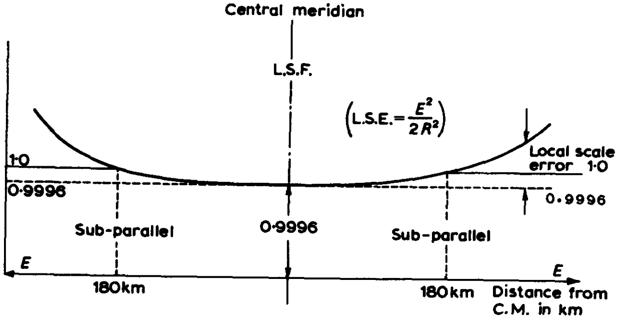


Fig.1.30

The local scale error as shown on the graph approximates to

$$\frac{E^2}{2R^2}$$

Example 1.24 Calculate (a) the local scale factors for each corner of the grid square TA (i.e. grid co-ordinates of S.W. corner 54), (b) the local scale factor at the centre of the square, (c) the percentage error in each case if the mean of the square corners is used instead.

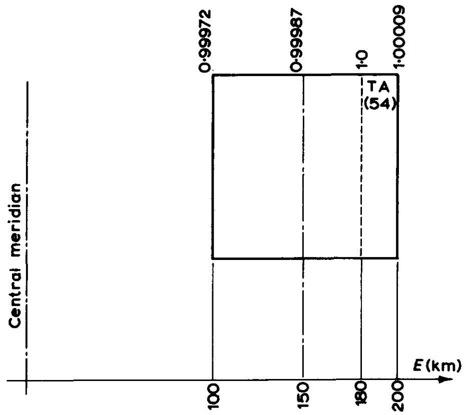


Fig. 1.31

See Chapter 3, page 160.

(a) (i) At the S.W. corner, co-ordinates are 500 km E, i.e. 100 km E of central meridian.

Therefore, from Eq.(1.58),

$$\text{L.S.F.} = 0{\cdot}999\,601\,3(1 + 100^2 \times 1{\cdot}23 \times 10^{-8})$$
$$= 0{\cdot}999\,601 + 0{\cdot}000\,123 = \underline{0{\cdot}999\,724}$$

(ii) At S.E. corner, co-ordinates are 600 km E, i.e. 200 km E of C.M.

$$\therefore \quad \text{L.S.F.} = 0{\cdot}999\,601\,3(1 + 200^2 \times 1{\cdot}23 \times 10^{-8})$$
$$= 0{\cdot}999\,601 + 2^2 \times 0{\cdot}000\,123$$
$$= 0{\cdot}999\,601 + 0{\cdot}000\,492 = 1{\cdot}000\,093$$

(b) At centre of square, 150 km from central meridian,

$$\text{L.S.F.} = 0{\cdot}999\,601 + 1{\cdot}5^2 \times 0{\cdot}000\,123 = \underline{0{\cdot}999\,878}$$

(c) (i) % error at S.W. corner

$$= \frac{0{\cdot}999\,724 - \tfrac{1}{2}({\cdot}999\,724 + 1{\cdot}000\,093)}{0{\cdot}999\,724} \times 100$$

$$= \underline{0{\cdot}018\%}$$

(ii) % error at S.E. corner

$$= \frac{1{\cdot}000\,093 - 0{\cdot}999\,908}{1{\cdot}000\,093} \times 100 = \underline{0{\cdot}019\%}$$

(iii) % error at centre

$$= \frac{0{\cdot}999\,878 - 0{\cdot}999\,908}{0{\cdot}999\,878} \times 100 = \underline{0{\cdot}003\%}$$

Example 1.25 Calculate the local scale factors applicable to a place E 415 km and to coal seams there at depths of 500 ft, 1000 ft, 1500 ft and 2000 ft respectively.

Local radius of the earth = 6362·758 km

$$\text{L.S.F.} = 0{\cdot}999\,601\,3\,(1 + E^2 \times 1{\cdot}23 \times 10^{-8}) \quad \text{(Eq. 1.59)}$$

Correction of length to mean sea level

$$L = \frac{l_m R}{R - h}$$

at 500 ft

$$L = l_m \frac{6362{\cdot}758}{6362{\cdot}758 - (500 \times 0{\cdot}3048 \times 10^{-3})}$$

$$= l_m \frac{6362{\cdot}758}{6362{\cdot}758 - 0{\cdot}152}$$

$$= l_m \frac{6362{\cdot}758}{6362{\cdot}606} = \underline{1{\cdot}000\,024\, l_m}$$

at 1000 ft

$$L = l_m \frac{6362{\cdot}758}{6362{\cdot}758 - 0{\cdot}304}$$

$$= l_m \frac{6362{\cdot}758}{6362{\cdot}454} = \underline{1{\cdot}000\,048\, l_m}$$

at 1500 ft

$$L = l_m \frac{6362{\cdot}758}{6362{\cdot}758 - 0{\cdot}457}$$

$$= l_m \frac{6362{\cdot}758}{6362{\cdot}301} = \underline{1{\cdot}000\,072\, l_m}$$

at 2000 ft

$$L = l_m \frac{6362{\cdot}758}{6362{\cdot}758 - 0{\cdot}610}$$

$$= l_m \frac{6362{\cdot}758}{6362{\cdot}148} = \underline{1{\cdot}000\,096\, l_m}$$

At mean sea level, Easting 415 km,

$$\text{L.S.F.} = 0{\cdot}999\,601\,3\,[1 + (415 - 400)^2 \times 1{\cdot}23 \times 10^{-8}]$$
$$= 0{\cdot}999\,604\,0$$

at 500 ft below,

$$\text{L.S.F.} = 0{\cdot}999\,604\,0 \times 1{\cdot}000\,024 = \underline{0{\cdot}999\,628}$$

at 1000 ft below,

$$\text{L.S.F.} = 0{\cdot}999\,604\,0 \times 1{\cdot}000\,048 = \underline{0{\cdot}999\,652}$$

at 1500 ft below,

$$\text{L.S.F.} = 0{\cdot}999\,604\,0 \times 1{\cdot}000\,072 = \underline{0{\cdot}999\,676}$$

at 2000 ft below,

$$\text{L.S.F.} = 0{\cdot}999\,604\,0 \times 1{\cdot}000\,096 = \underline{0{\cdot}999\,700.}$$

Example 1.26 An invar reference tape was compared with standard on the flat at the National Physical Laboratory at 68 °F and 20 lbf tension and found to be 100·024 0 ft in length.

The first bay of a colliery triangulation base line was measured in catenary using the reference tape and then with the invar field tape at a temperature of 60° F and with 20 lbf tension. The means of these measurements were 99·876 3 ft and 99·912 1 ft respectively.

The second bay of the base line was measured in catenary using the field tape at 56 °F and 20 lbf tension and the resulting mean measurement was 100·213 5 ft.

Given:

(a) the coefficient of expansion for invar = $3{\cdot}3 \times 10^{-7}$,
(b) the weight of the tape per foot run = 0·008 24 lbf,
(c) the inclination of the second bay = 3° 15′ 00″,
(d) the mean height of the second bay = 820 ft A.O.D.

Assuming the radius of the earth to be 20 890 000 ft, calculate the horizontal length of the second bay reduced to Ordnance Datum.

(M.Q.B./S)

To find the standardised length of the field tape.

Reference tape on the flat at 68 °F = 100·024 0 ft.

Temperature correction

$$= 100 \times 3{\cdot}3 \times 10^{-7} \times (60 - 68)$$
$$= -0{\cdot}000\,264 \quad \text{i.e.} -0{\cdot}000\,3$$

∴ Reference tape at 60 °F

$$= 100{\cdot}024\,0 - 0{\cdot}000\,3$$
$$= 100{\cdot}023\,7\text{ ft.}$$

Thus the standardisation correction is + 0·023 7 ft per 100 ft.

$$\text{Sag correction} = \frac{-w^2 l^3}{24T^2}$$

$$= \frac{-(8{\cdot}24 \times 10^{-3})^2 \times 100^3}{24 \times 20^2}$$

$$= \frac{-8{\cdot}24^2}{9600} = -0{\cdot}007\,1 \text{ ft}$$

The length of 100 ft is acceptable in all cases due to the close approximation.

The true length of the first bay thus becomes

$$99{\cdot}876\,3 - 0{\cdot}007\,1 + 0{\cdot}023\,7 = 99{\cdot}892\,9 \text{ ft.}$$

The field tape applied under the same conditions when corrected for sag gives

$$99{\cdot}912\,1 - 0{\cdot}007\,1 = 99{\cdot}905\,0 \text{ ft.}$$

The difference represents the standardisation correction

$$99{\cdot}905\,0 - 99{\cdot}892\,9 = +0{\cdot}022\,1 \text{ ft.}$$

The corrections may now be applied to the second bay.

	+	−
Standardisation		
$c = +0{\cdot}022\,1 \times \dfrac{100{\cdot}21}{99{\cdot}91}$	0·022 1	
(the proportion is not necessary because of the close proximity)		
Temperature, $L\,.\,\alpha\,.\,(t_m - t_s)$		
$c = 100 \times 3{\cdot}3 \times 10^{-7} \times (56 - 60)$		0·000 1
Sag,		
As before as length $\simeq$ 100 ft		0·007 1
Slope, $L\,(1 - \cos\theta)$		
$c = -\ 100{\cdot}21\,(1 - \cos 3^\circ\,15')$		
$= -\ 100{\cdot}21 \times 0{\cdot}001\,61$		0·161 3
Sea level, $-\ lh/R$		
$c = -\ 100 \times 820/20\,890\,000$		0·003 9
	0·022 1	0·172 4
		0·022 1
		0·150 3

Horizontal length reduced to sea level

$$= 100{\cdot}2135 - 0{\cdot}1503$$
$$= \underline{100{\cdot}0632\text{ ft.}}$$

Example 1.27 The details given below refer to the measurement of the first '100 ft' bay of a base line. Determine the correct length of the bay reduced to mean sea level.

With the tape hanging in catenary at a tension of 20 lbf and at a mean temperature of 55 °F, the recorded length was 100·0824 ft. The difference in height between the ends was 1·52 ft and the site was 1600 ft above m.s.l.

The tape had previously been standardised in catenary at a tension of 15 lbf and at a temperature of 60 °F, and the distance between zeros was 100·042 ft. $R = 20\,890\,000$ ft. Weight of tape/ft = 0·013 lbf. Sectional area of tape = 0·0056 in², $E = 30 \times 10^6$ lbf/in². Temperature coefficient of expansion of tape = 0·000 006 25 per 1 °F.

(I.C.E.)

Corrections.	Correction +	Correction −
Standardisation		
Tape is 100·042 ft at 15 lbf tension and 60 °F.		
∴ c = 0·042 ft per 100 ft	0·0420	
Temperature		
$c = L\,.\,\alpha\,.\,(t_m - t_s)$		
$= 100 \times 6{\cdot}25 \times 10^{-6} \times (55 - 60) = -0{\cdot}0031$		0·0031
Tension		
$c = \dfrac{L\,(T_m - T_s)}{A\,.\,E}$		
$= \dfrac{100 \times (20 - 15)}{0{\cdot}0056 \times 30 \times 10^6} = +0{\cdot}0030$	0·0030	
Slope		
$c = \dfrac{d^2}{2l}$		
$= \dfrac{1{\cdot}52^2}{200} = -0{\cdot}0116$		0·0116

Sag

c = difference between the corrections for field and standard tensions

$$c = -\frac{W^2 l}{24}\left[\frac{1}{T_m^2} - \frac{1}{T_s^2}\right]$$

$$W = wl = 0{\cdot}013 \times 100 = 1{\cdot}3\ \text{lbf}$$

$$\therefore \quad c = -\frac{1{\cdot}32^2 \times 100}{24}\left[\frac{15^2 - 20^2}{15^2 \times 20^2}\right]$$

$$= +0{\cdot}013\,7$$

Height

$$c = \frac{hl}{R}$$

$$= \frac{1600 \times 100}{20\,890\,000} = -0{\cdot}007\,7$$

+	−
0·013 7	
	0·007 7
+0·058 7	0·022 4
−0·022 4	
+0·036 3	

Measured length	+ 100·082 4
Total correction	+ 0·036 3
Corrected length	100·118 7 ft

1.6 The Effect of Errors in Linear Measurement

If the corrections previously discussed (pp. 23-40) are not applied correctly, then obviously errors will occur. Any errors within the formulae produce the following effects.

1.61 Standardisation

Where a tape is found to deviate from standard, the error δl can be corrected in the normal way or by altering the standard temperature as previously suggested.

1.62 Malalignment and deformation of the tape (Figs. 1.32 and 1.33)

(a) *Malalignment*. If the end of the tape is out of line by an amount d in a length l, the error will be

$$\frac{d^2}{2l} \qquad (1.60)$$

e.g., if $d = 3$ in. and $l = 100$ ft,

$$e_1 = \frac{0 \cdot 25^2}{200} = 0 \cdot 000\,3\text{ ft},$$

i.e. 1 in 330 000.

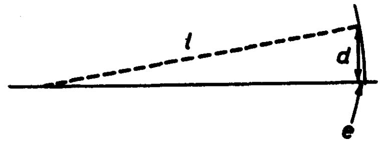


Fig. 1.32 Malalignment of the tape

(b) *Deformation in the horizontal plane.* If the tape is not pulled straight and the centre of the tape is out of line by d, then

$$e_2 = \frac{d^2}{2\left(\frac{l}{2}\right)} + \frac{d^2}{2\left(\frac{l}{2}\right)} = \frac{2d^2}{l} \quad (1.61)$$

Fig. 1.33 Deformation of the tape

e.g., if $d = 3$ in. and $l = 100$ ft,

$$e_2 = 4 \times e_1 = 0 \cdot 001\,23, \text{ i.e. 1 in 80 000.}$$

(c) *Deformation in the vertical plane.* This is the same as (b) but more difficult to detect. Any obvious change in gradient can be allowed for by grading the tape or by measuring in smaller bays between these points.

N.B. In (a) and (b) alignment by eye is acceptable for all purposes except very precise work.

1.63 Reading or marking the tape

Tapes graduated to 0·01 ft can be read by estimation to give a probable error of $\pm$ 0·001 ft.

Thus if both ends of the tape are read simultaneously the probable error in length will be $\sqrt{(2 \times 0 \cdot 001^2)}$, i.e. $0 \cdot 001 \times \sqrt{2}$, i.e. $\pm 0 \cdot 001\,4$ ft.

Professor Briggs suggests that the error in setting or marking of the end of the tape is 3 times that of estimating the reading, i.e. $\pm 0 \cdot 003$ ft per observation.

1.64 Errors due to wrongly recorded temperature

From the correction formula $c = l \,.\, \alpha \,.\, (t_m - t_s)$,

$$\delta c = l\,\alpha\,\delta t_m \quad (1.62)$$

$$\text{and} \quad \frac{\delta c}{l} = \alpha\,\delta t_m \quad (1.63)$$

It has been suggested from practical observation that errors in recording the actual temperature of the tape for ground and catenary measurement are $\pm$ 5 °F and $\pm$ 3 °F respectively.

If the error is not to exceed 1/10 000, then from Eq. 1.63

$$\frac{\delta c}{l} = \frac{1}{10\,000} = \alpha\,\delta t_m$$

i.e. $$\delta t_m = \frac{1}{10\,000\,\alpha}$$

If $\alpha = 6{\cdot}5 \times 10^{-6}$ per deg F, then

$$\delta t_m = \frac{10^6}{6{\cdot}5 \times 10^4} \simeq 15{\cdot}4^\circ$$

Thus 5° produces an error of $\simeq$ 1/30 000,
3° produces an error of $\simeq$ 1/50 000.

1.65 Errors due to variation from the recorded value of tension

These may arise from two sources:

(a) Lack of standardisation of tensioning apparatus.

(b) Variation in the applied tension during application (this is significant in ground taping).

From the correction formula (1.39) $$c = \frac{L(T_m - T_s)}{AE} = \frac{LT}{AE}$$

differentiation gives $$\delta c = \frac{L\delta T_m}{AE} \tag{1.64}$$

$$\frac{\delta c}{c} = \frac{\delta T_m}{T} \tag{1.65}$$

i.e. $$\frac{\delta c}{L} = \frac{\delta T_m}{AE} \tag{1.66}$$

If the error is not to exceed say 1 in 10 000, then

$$\frac{\delta c}{L} = \frac{\delta T_m}{AE} = \frac{1}{10\,000}$$

i.e. $$\delta T_m = \frac{AE}{10\,000}$$

If $A = 0{\cdot}003$ in², $E = 30 \times 10^6$ lbf/in², then

$$\delta T_m = \frac{0{\cdot}003 \times 30 \times 10^6}{10^4} = 9 \text{ lbf.}$$

i.e., an error of 1/10 000 is produced by a variation of 9 lbf,
an error of 1/30 000 is produced by a variation of 3 lbf.

The tape cross-section is ½in. wide to give $A = 0{\cdot}003$ in². If the width of the tape be reduced to ¼in. then, if the other dimensions

remain constant, the cross-sectional area is reduced to $\frac{1}{4}A = 0{\cdot}000\,8\ \text{in}^2$.

In this case a variation of 3 lbf will produce an error of 1/10 000 and the accuracy will be reduced as the cross-sectional area diminishes.

1.66 Errors from sag

Where the tape has been standardised on the flat and is then used in catenary with the measuring heads at different levels, the approximation formula is given as

$$c = \frac{-l^3 w^2 \cos^2\theta}{24T^2} \tag{1.50}$$

where θ is the angle of inclination of the chord between measuring heads. The value of $\cos^2\theta$ becomes negligible when θ is small.

The sources of error are derived from:

(a) an error in the weight of the tape per unit length, w,
(b) an error in the angular value, θ,
(c) an error in the tension applied, T.

By successive differentiation,

$$dc_w = \frac{-2\,l^3 w \cos^2\theta\,\delta w}{24T^2} \tag{1.67}$$

$$= \frac{2c\,\delta w}{w} \tag{1.68}$$

$$\text{i.e.}\quad \underline{\frac{\delta c_w}{c} = \frac{2\delta w}{w}} \tag{1.69}$$

This may be due to an error in the measurement of the weight of the tape or due to foreign matter on the tape, e.g. dirt.

$$\delta c_\theta = \frac{-l^3 w^2}{24T^2}\sin 2\theta\,\delta\theta \tag{1.70}$$

$$= 2c\tan\theta\,\delta\theta \tag{1.71}$$

$$\frac{\delta c_\theta}{c} = 2\tan\theta\,\delta\theta \tag{1.72}$$

$$\delta c_T = \frac{2l^3 w^2 \cos^2\theta\,\delta T}{24T^3} \tag{1.73}$$

$$= \frac{-2c\delta T}{T} \tag{1.74}$$

$$\frac{\delta c_T}{c} = \frac{-2\delta T}{T} \tag{1.75}$$

The compounded effect of a variation in tension gives

$$\frac{2l^3w^2\cos^2\theta\,\delta T}{24T^2} + \frac{l\,\delta T}{AE} \tag{1.76}$$

Example 1.28 If $l = 100$ ft, $w = 0{\cdot}01 \pm 0{\cdot}001$ lbf per ft, $\theta = 2^\circ \pm 10''$, $T = 10 \pm 1$ lbf,

$$c = \frac{100^3 \times 0{\cdot}01^2 \times \cos^2 2^\circ}{24 \times 10^2} = \frac{\cos^2 2^\circ}{24} = \underline{0{\cdot}041\,61\text{ ft.}}$$

Then $\delta c_w = \dfrac{2c\,\delta w}{w} = 2 \times 0{\cdot}041\,61 \times 0{\cdot}1 = 0{\cdot}008\,32$ ft

i.e. 10% error in weight produces an error of 1/12 000.

$$\delta c_\theta = \frac{2c\tan\theta\,\delta\theta''}{206\,265} = \frac{0{\cdot}083\,22}{206\,265} \times 0{\cdot}052\,4 \times 10$$
$$= 0{\cdot}000\,000\,21\text{ ft.}$$

This is obviously negligible.

$$\delta c_T = \frac{2c\,\delta T}{T} = 0{\cdot}083\,22 \times 0{\cdot}1 = 0{\cdot}008\,322\text{ ft}$$

i.e. 10% error in tension produces an error of 1/12 000.

Example 1.29 A base line is measured and subsequent calculations show that its total length is 4638·00 ft. It is later discovered that the tension was recorded incorrectly, the proper figure being 10 lbf less than that stated in the field book, extracts from which are given below. Assuming that the base line was measured in 46 bays of nominal length 100 ft and one bay of nominal length 38 ft, calculate the error incurred in ft.

Extract from field notes

Standardisation temperature	= 50 °F
Standardisation tension	= 20 lbf
Measured temperature	= 45 °F
Measured tension	= 40 lbf
Young's modulus of tape	= 30×10^6 lbf/in^2
Cross-sectional area of tape	= 0·125 in. × 0·05 in.
Weight of 1 in^3 of steel	= 0·28 lbf.

(N.U.)

Weight of steel tape per ft = $0{\cdot}125 \times 0{\cdot}05 \times 12 \times 0{\cdot}28 = 0{\cdot}021$ lbf.

From Eq. (1.39) $$c = \frac{L(T_m - T_s)}{AE}$$

Then the error due to wrongly applied tension $= c - c'$

$$= \frac{L(T_m - T_s)}{AE} - \frac{L(T_m' - T_s)}{AE}$$

$$= \frac{L}{AE}(T_m - T_m')$$

where T_m = true applied tension,
T_m' = assumed applied tension.

∴ Error

$$= \frac{4638(30-40)}{0{\cdot}125 \times 0{\cdot}05 \times 30 \times 10^6} = -0{\cdot}247\,36 \text{ ft.}$$

From Eq. (1.46) correction for sag $c = \dfrac{-W^2 l}{24T^2}$

∴ Error due to wrongly applied tension

$$= c - c_1$$

$$= -\frac{W^2 l}{24}\left(\frac{1}{T^2} - \frac{1}{T_1^2}\right)$$

$$W = 100 \times 0{\cdot}021 = 2{\cdot}1 \text{ lbf}$$

Error for 100 ft bay

$$= -\frac{2{\cdot}1^2 \times 100}{24}\left(\frac{1}{30^2} - \frac{1}{40^2}\right)$$

$$= -\frac{441}{24}\left(\frac{1600 - 900}{1600 \times 900}\right)$$

$$= -\frac{441}{24}\left(\frac{700}{1\,440\,000}\right)$$

$$= -0{\cdot}008\,93$$

Error for 46 bays $= \underline{-0{\cdot}410\,78 \text{ ft}}$

Error for 38 ft bay

$$W = 38 \times 0{\cdot}021 = 0{\cdot}798 \text{ lbf}$$

Error $$= -\frac{0{\cdot}798^2 \times 38}{24}\left(\frac{700}{1\,440\,000}\right)$$

$$= \underline{-0{\cdot}000\,49 \text{ ft}}$$

∴ Total error for sag $= -0{\cdot}411\,27$ ft
Total error for tension $= \underline{-0{\cdot}247\,36 \text{ ft}}$

$$\text{Total error} = -\underline{0{\cdot}658\,63\text{ ft}}$$

i.e. <u>Apparent reduced length is 0·6586 ft too large.</u>

1.67 Inaccurate reduction to the horizontal

The inclined length may be reduced by obtaining

(a) the difference in level of the measuring heads or

(b) the angle of inclination of the tape.

(a) The approximation formula is given as

$$\text{(Eq. 1.15)} \qquad c = \frac{d^2}{2l} + \left(\frac{d^4}{8l^3} + \dots\right)$$

Adopting the first term only, from the differentiation

$$\delta c = \frac{d\delta d}{l} \tag{1.77}$$

$$= \frac{2c\,\delta d}{d} \tag{1.78}$$

$$\frac{\delta c}{c} = \frac{2\delta d}{d} \tag{1.79}$$

If $\delta d/d = 1\%$, when $l = 100$ and $d = 5 \pm 0{\cdot}05$ ft,

$$\delta c = \frac{2 \times 5^2 \times 0{\cdot}01}{2 \times 100} = \underline{\pm\,0{\cdot}0025\text{ ft}} \quad \text{i.e. 1 in 40 000.}$$

As the difference in level can be obtained without difficulty to $\pm\,0{\cdot}01$ ft,

$$\underline{\delta c = \pm\,0{\cdot}0005\text{ ft},} \quad \text{i.e. } 1/200\,000.$$

(b) By trigonometrical observations (Eq. 1.12),

$$c = l(1 - \cos\theta)$$

$$\text{Then} \quad \delta c = l\sin\theta\,\delta\theta \tag{1.80}$$

$$= \frac{l\sin\theta\,\delta\theta''}{206\,265} \tag{1.81}$$

$$\text{i.e.} \quad \delta c \propto \sin\theta\,\delta\theta \tag{1.82}$$

If $l = 100$, $\theta = 30° \pm 20''$,

$$\delta c = \pm\frac{100 \times 0{\cdot}5 \times 20}{206\,265} = \pm\,0{\cdot}0048\text{ ft} \quad \text{i.e. } 1/20\,000.$$

The accuracy obviously improves as θ is reduced.

As the angle of inclination increases the accuracy in the measurement of θ must improve.

1.68 Errors in reduction from height above or below mean sea level

From the formula

$$c = \frac{lh}{R},$$

by differentiating

$$\delta c = \frac{l\delta h}{R} \tag{1.83}$$

$$= \frac{c\delta h}{h} \tag{1.84}$$

The % error in the correction is equal to the % error in the height above or below M.S.L.

1.69 Errors due to the difference between ground and grid distances

Local scale factor is given by Eq.(1.59)

$$0{\cdot}999\,601\,3\,(1 + 1{\cdot}23\,E^2 \times 10^{-8})$$

where E is the distance in km from the central meridian (i.e. the Eastings − 400 km).

As this amounts to a maximum of 0·04% it is only effective in precise surveys.

Exercises 1(b)

10. A 300 ft tape has been standardised at 80 °F and its true length at this temperature is 300·023 ft. A line is measured at 75 °F and recorded as 3486·940 ft. Find its true length assuming the coefficient of linear expansion is $6{\cdot}2 \times 10^{-6}$ per deg F.

(Ans. 3487·10 ft)

11. A base line is found to be 10 560 ft long when measured in catenary using a tape 300 ft long which is standard without tension at 60 °F. The tape in cross-section is 1/8 × 1/20 in.

If one half of the line is measured at 70 °F and the other half at 80 °F with an applied tension of 50 lbf, and the bays are approximately equal, find the total correction to be applied to the measured length.

Coefficient of linear expansion = $6{\cdot}5 \times 10^{-6}$ per deg F.
Weight of 1 in³ of steel = 0·28 lbf.
Young's modulus = 29×10^{6} lbf/in².

(Ans. − 3·042 ft)

12. A 100 ft steel tape without tension is of standard length when placed on the ground horizontally at a temperature of 60 °F. The cross-sectional area is 0·0103 in² and its weight 3·49 lbf, with a coefficient of linear expansion of $6{\cdot}5 \times 10^{-6}$ per deg F.

The tape is used in the field in catenary with a middle support such

that all the supports are at the same level.

Calculate the actual length between the measuring heads if the temperature is 75 °F and the tension is 20 lbf. (Assume Young's modulus 30×10^6 lbf/in²).

(Ans. 99·985 1 ft)

13. A nominal distance of 100 ft was set out with a 100 ft steel tape from a mark on the top of one peg to a mark on the top of another, the tape being in catenary under a pull of 20 lbf and at a mean temperature of 70 °F. The top of one peg was 0·56 ft below the top of the other. The tape had been standardised in catenary under a pull of 25 lbf at a temperature of 62 °F.

Calculate the exact horizontal distance between the marks on the two pegs and reduce it to mean sea level. The top of the higher peg was 800 ft above mean sea level.

(Radius of earth = $20{\cdot}9 \times 10^6$ ft ; density of tape 0·28 lbf/in³; section of tape = $0{\cdot}125 \times 0{\cdot}05$ in.; Young's modulus 30×10^6 lbf/in²; coefficient of expansion $6{\cdot}25 \times 10^{-6}$ per 1° F)

(I.C.E. Ans. 99·980 4 ft)

14. A steel tape is found to be 299·956 ft long at 58 °F under a tension of 12 lbf. The tape has the following specifications:

Width	0·4 in.
Thickness	0·018 in.
Young's modulus of elasticity	30×10^5 lbf/in²
Coefficient of thermal expansion	$6{\cdot}25 \times 10^{-6}$ per deg F.

Determine the tension to be applied to the tape to give a length of precisely 300 ft at a temperature of 68 °F.

(N.U. Ans. 30 lbf)

15. (a) Calculate to three decimal places the sag correction for a 300 ft tape used in catenary in three equal spans if the tape weighs 1 lb/100 ft and it is used under a tension of 20 lbf.

(b) It is desired to find the weight of a tape by measuring its sag when suspended in catenary with both ends level. If the tape is 100 ft long and the sag amounts to 9·375 in. at mid-span under a tension of 20 lbf, what is its weight in ozf per 100 ft?

(N.U. Ans. 0·031 ft, 20 ozf)

16. Describe the methods used for the measurement of the depth of vertical mine shafts and discuss the possible application of electronic distance measuring equipment.

Calculate the elongation of a shaft measuring tape due to its own weight at (1) 1000 ft and (2) 3000 ft, given that the modulus of elasticity is 30×10^6 lbf/in²; weight of the tape 0·05 lbf/ft run, and the cross-sectional area of the tape 0·015 in².

(N.U. Ans. 0·055 6 ft, 0·500 ft)

17. (a) Describe the measuring and straining tripods used in geodelic base measurement.

(b) The difference between the readings on a steel tape at the terminals of a bay between which it is freely suspended was 94·007 ft, the tension applied being 20 lbf, the temperature 39·5 °F, and the height difference between the terminals 5·87 ft. The bay was 630 ft above mean sea level.

If the tape, standardised on the flat, measured correctly at 68 °F under 10 lbf tension, and its weight was 0·0175 lbf per ft, its coefficient of expansion $0{\cdot}62 \times 10^{-5}$ per deg F and its coefficient of extension $0{\cdot}67 \times 10^{-5}$ per lb, calculate the length of the bay reduced to mean sea level. (Radius of earth = $20{\cdot}9 \times 10^{6}$ ft).

(L.U. Ans. 93·784 ft)

18. The steel band of nominal length 100 ft used in the catenary measurement of a colliery base line, has the following specification:

(i) Length 100·025 ft at 10 lbf tension and 68 °F.
(ii) Sectional area 0·004 in^2
(iii) Weight 22 ozf.
(iv) Coefficient of linear expansion $6{\cdot}25 \times 10^{-6}$ per deg F.
(v) Modulus of elasticity 30×10^{6} lbf/in^2.

The base line was measured in 10 bays and the undernoted observations were recorded in respect of the first five which were of average height 625 ft above Ordnance Datum.

Bay	Observed Length	Bay Temperature	Bay Level Difference	Tension Applied
1	100·005	52 °F	0·64	20 lbf
2	99·983	54 °F	1·23	20 lbf
3	100·067	54 °F	0·01	20 lbf
4	100·018	58 °F	0·79	20 lbf
5	99·992	60 °F	2·14	20 lbf

Correct the bays for standard, temperature, tension, sag, slope, and height above Ordnance Datum and compute the corrected length of this part of the base line. Take the mean radius of the earth to be 20 890 000 ft.

(M.Q.B./S Ans. 500·044 ft)

19. The steel band used in the catenary measurement of the base line of a colliery triangulation survey has the undernoted specification:

(i) length 50·000 3 m at a tension of 25 lbf at 60° F.
(ii) weight 2·5 lbf
(iii) coefficient of linear expansion $6{\cdot}25 \times 10^{-6}$ per deg F.

The undernoted data apply to the measurement of one bay of the base line:

(i) length 50·0027 m
(ii) mean temperature 53 °F
(iii) tension applied 25 lbf
(iv) difference in level between ends of bay 0·834 m
(v) mean height of bay above mean sea level 255·4 m.

Correct the measured length of the bay for standard, temperature, sag, slope, and height above mean sea level. Assume the mean radius of the earth is $6{\cdot}37 \times 10^6$ m.

(M.Q.B./S Ans. 49·9710 m)

20. A base line was measured with an invar tape 100 ft long which had been standardised on the flat under a tensile load of 15 lbf and at a temperature of 60 °F. Prior to the measurement of the base line the tape was tested under these conditions and found to record 0·015 ft too much on the standard length of 100 ft. The base line was then divided into bays and the results obtained from the measurement of the bays with the tape suspended are shown below:

Bay	Length (ft)	Difference in level between supports (ft)	Air temperature (°F)
1	99·768	2·15	49·6
2	99·912	1·62	49·6
3	100·018	3·90	49·8
4	100·260	4·28	50·2
5	65·715	0·90	50·3

Modulus of elasticity (E) for invar = 22×10^6 lbf/in².
Coefficient of linear expansion of invar = $5{\cdot}2 \times 10^{-7}$ per deg F.
Field pull = 25 lbf.
Cross-sectional area of tape = 0·004 in².
Weight per ft run of tape = 0·0102 lbf.
Average reduced level of base line site = 754·5 ft.
Radius of earth = $20{\cdot}8 \times 10^6$ ft.

Correct the above readings and determine to the nearest 0·001 ft the length of the base line at mean sea level.

(I.C.E. Ans. 465·397 ft)

21. The following readings were taken in measuring a base line with a steel tape suspended in catenary in five spans:

Span	Mean reading of tape (ft)	Difference in level between Index Marks (ft)	Tension (lbf)	Mean temperature (°F)
1	100·155	3·1	25	73
2	100·140	0·9	50	76
3	100·060	1·2	25	78
4	100·108	3·1	25	80
5	100·182	2·0	25	80

The tape reading was 100·005 ft when calibrated in catenary under a tension of 25 lbf at a temperature of 65 °F between two points at the same level precisely 100 ft apart.

Other tape constants are:

width of tape = 0·250 in; thickness of tape = 0·010 in;
weight of steel = 0·283 lbf/in^3; E for steel = 30×10^6 lbf/in^2;
coefficient of expansion of steel = $6{\cdot}2 \times 10^{-6}$ per deg F.

Compute the length of the base line.

(I.C.E. Ans. 500·568 ft)

22. A short base line is measured in four bays with a 100 ft invar band in catenary under a pull of 20 lbf with the following field readings:

Bay	1	2	3	4
t °F	65·2	64·0	65·5	63·8
h ft	5·08	1·31	2·31	2·13
l ft	99·6480	99·7517	99·5417	99·9377

where t is the field temperature, h is the difference in level between the ends of each bay and l is the mean reading of the invar band.

When standardised in catenary under a pull of 20 lbf at 68·5 °F the standard length of the invar band was 99·999 ft and the mean altitude of the base is 221 ft above sea level. If the coefficient of expansion of invar is 0·000 000 3 per deg F and the radius of the earth is $20{\cdot}9 \times 10^6$ ft, what is the length of the base line reduced to sea level?

(I.C.E. Ans. 398·6829 ft)

Bibliography

CLARK, D., *Plane and Geodetic Surveying*, Vols. I and II (Constable)
HOLLAND, J.L., WARDELL, K. and WEBSTER, A.G., *Surveying*, Vols. I and II. Coal Mining Series (Virtue)
SMIRNOFF, M.V., *Measurement for Engineering and Other Surveys* (Prentice–Hall)
BANNISTER, A. and RAYMOND, S., *Surveying* (Pitman)
THOMAS, W.N., *Surveying*, 5th ed. (Edward Arnold)
BRIGGS, N., *The Effects of Errors in Surveying* (Griffin)
MINISTRY OF TECHNOLOGY, *Changing to the Metric System* (H.M.S.O.)
ORDNANCE SURVEY publication, *Constants, Formulae and Methods used in Transverse Mercator* (H.M.S.O.)

2 SURVEYING TRIGONOMETRY

'Who conquers the triangle half conquers his subject'
M.H. Haddock

Of all the branches of mathematics, trigonometry is the most important to the surveyor, forming the essential basis of all calculations and computation processes. It is therefore essential that a thorough working knowledge is acquired and this chapter is an attempt to summarize the basic requirements.

2.1 Angular Measurement

There are two ways of dividing the circle:

(a) the degree system,

(b) the continental 'grade' system.

The latter divides the circle into 4 quadrants of 100 grades each and thereafter subdivides on a decimal system. It has little to commend it apart from its decimalisation which could be applied equally to the degree system. It has found little favour and will not be considered here.

2.11 The degree system

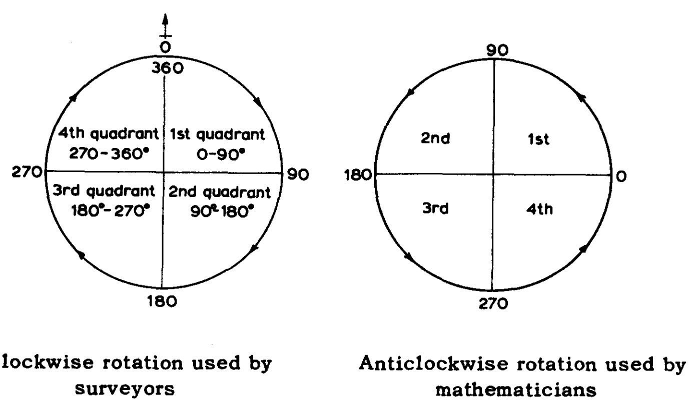


Fig. 2.1 Comparison of notations

The circle is divided into 360 equal parts or degrees, each degree into 60 minutes, and each minute into 60 seconds. The following symbols are used:—

degrees (°) minutes (′) seconds (″)

so that 47 degrees 26 minutes 6 seconds is written as

$$47°\,26'\,06''$$

N.B. The use of 06″ is preferred in surveying so as to remove any doubts in recorded or computed values.

In *mathematics* the angle is assumed to rotate *anti-clockwise* whilst in *surveying* the direction of rotation is assumed *clockwise*.

This variance in no way alters the subsequent calculations but is merely a different notation.

2.12 Trigonometrical ratios (Fig. 2.2)

Assume radius = 1

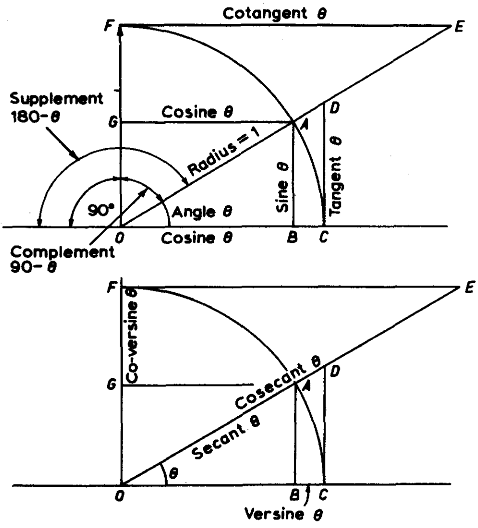


Fig. 2.2

Sine (abbreviated sin) angle $\theta = \dfrac{AB}{OA} = \dfrac{AB}{1} = AB = GO$

Cosine (abbreviated cos) angle $\theta = \dfrac{OB}{OA} = \dfrac{OB}{1} = OB = GA$

Tangent (abbreviated tan) angle $\theta = \dfrac{AB}{OB} = \dfrac{\sin\theta}{\cos\theta} = \dfrac{DC}{OC} = \dfrac{DC}{1}$

Cotangent θ (cot θ) $= \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta} = \dfrac{OB}{AB} = \dfrac{FE}{FO} = \dfrac{FE}{1}$

$$\text{Cosecant } \theta \ (\text{cosec } \theta) = \frac{1}{\sin\theta} = \frac{OA}{AB} = \frac{OE}{OF} = \frac{OE}{1}$$

$$\text{Secant } \theta \quad (\sec\theta) = \frac{1}{\cos\theta} = \frac{OA}{OB} = \frac{OD}{OC} = \frac{OD}{1}$$

$$\text{Versine } \theta \ (\text{vers } \theta) = 1 - \cos\theta = OC - OB$$

$$\text{Coversine } \theta \ (\text{covers } \theta) = 1 - \sin\theta = OF - OG.$$

N.B. In mathematical shorthand $\sin^{-1} x$ means the angle (x) whose sine is ... .

If OA = radius = 1

then, by Pythagoras, $\sin^2\theta + \cos^2\theta = 1.$ (2.1)

$$\sin^2\theta = 1 - \cos^2\theta \tag{2.2}$$

$$\cos^2\theta = 1 - \sin^2\theta \tag{2.3}$$

Dividing Eq. 2.1 by $\cos^2\theta$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

i.e.

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1. \tag{2.4}$$

Dividing Eq. 2.1 by $\sin^2\theta$,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

i.e.

$$1 + \cot^2\theta = \text{cosec}^2\theta \tag{2.5}$$

$$\sin\theta = \sqrt{(1 - \cos^2\theta)}. \tag{2.6}$$

$$\cos\theta = \sqrt{(1 - \sin^2\theta)}. \tag{2.7}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{(1 - \sin^2\theta)}} \tag{2.8}$$

$$\text{or} = \frac{\sqrt{(1 - \cos^2\theta)}}{\cos\theta} \tag{2.9}$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{(1 + \tan^2\theta)}} \tag{2.10}$$

$$\sin\theta = \frac{1}{\text{cosec}\,\theta} = \frac{1}{\sqrt{(1 + \cot^2\theta)}} \tag{2.11}$$

which shows that, by manipulating the equations, any function can be expressed in terms of any other function.

2.13 Complementary angles

The complement of an acute angle is the difference between the angle and 90°,

i.e. if angle $A = 30°$

its complement $= 90° - 30° = 60°$

The sine of an angle = cosine of its complement
The cosine of an angle = sine of its complement
The tangent of an angle = cotangent of its complement
The secant of an angle = cosecant of its complement
The cosecant of an angle = secant of its complement

2.14 Supplementary angles

The supplement of an angle is the difference between the angle and 180°,

i.e. if angle $A = 30°$

its supplement $= 180° - 30° = 150°$.

The sine of an angle = sine of its supplement
cosine of an angle = cosine of its supplement (but a negative value)
tangent of an angle = tangent of its supplement (but a negative value)

These relationships are best illustrated by graphs.

Sine Graph (Fig. 2.3)

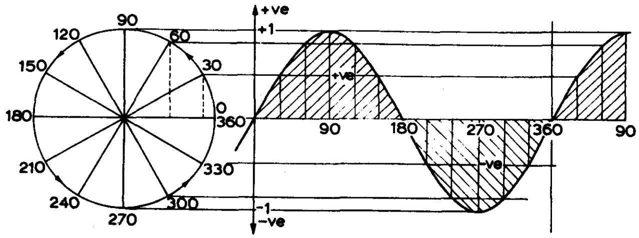


Fig. 2.3 The sine graph

Let the line OA of length 1 rotate anticlockwise. Then the height above the horizontal axis represents the sine of the angle of rotation.

At 90° it reaches a maximum = 1
At 180° it returns to the axis.
At 270° it reaches a minimum = −1

It can be seen from the graph that

$$\begin{aligned}\sin 30^\circ &= \sin(180-30), \quad \text{i.e.} \quad \sin 150^\circ\\ &= -\sin(180+30), \quad \text{i.e.} \quad -\sin 210^\circ\\ &= -\sin(360-30), \quad \text{i.e.} \quad -\sin 330^\circ\end{aligned}$$

Thus the sine of all angles $0 - 180^\circ$ are $+ve$ (positive)
the sine of all angles $180 - 360$ are $-ve$ (negative).

Cosine Graph (Fig. 2.4). This is the same as the sine graph but displaced by 90°.

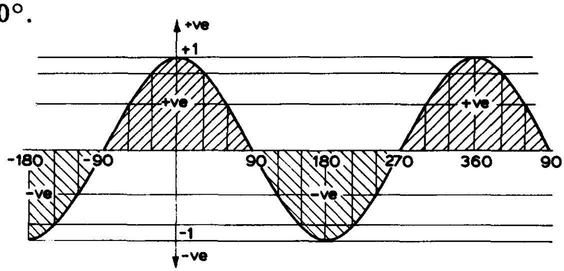


Fig. 2.4 The cosine graph

$$\begin{aligned}\cos 30^\circ &= -\cos(180-30) = -\cos 150^\circ\\ &= -\cos(180+30) = -\cos 210^\circ\\ &= +\cos(360-30) = +\cos 330^\circ\end{aligned}$$

Thus the cosine of all angles $0 - 90^\circ$ and $270^\circ - 360^\circ$ are $+ve$
$90^\circ - 270^\circ$ are $-ve$

Tangent Graph (Fig. 2.5) This is discontinuous as shown.

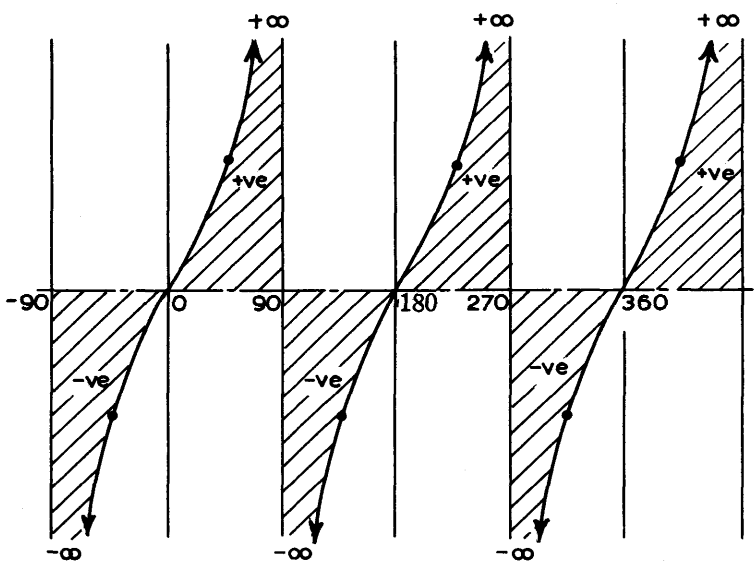


Fig. 2.5 The tangent graph

$$\begin{aligned}\tan 30^\circ &= \tan(180+30), \quad \text{i.e.} \quad \tan 210^\circ \\ &= -\tan(180-30), \quad \text{i.e.} \quad -\tan 150^\circ \\ &= -\tan(360-30), \quad \text{i.e.} \quad -\tan 330^\circ\end{aligned}$$

Thus the tangents of all angles $0 - 90^\circ$ and $180^\circ - 270^\circ$ are $+ve$
$90^\circ - 180^\circ$ and $270^\circ - 360^\circ$ are $-ve$

Comparing these values based on the clockwise notation the sign of the function can be seen from Fig. 2.6.

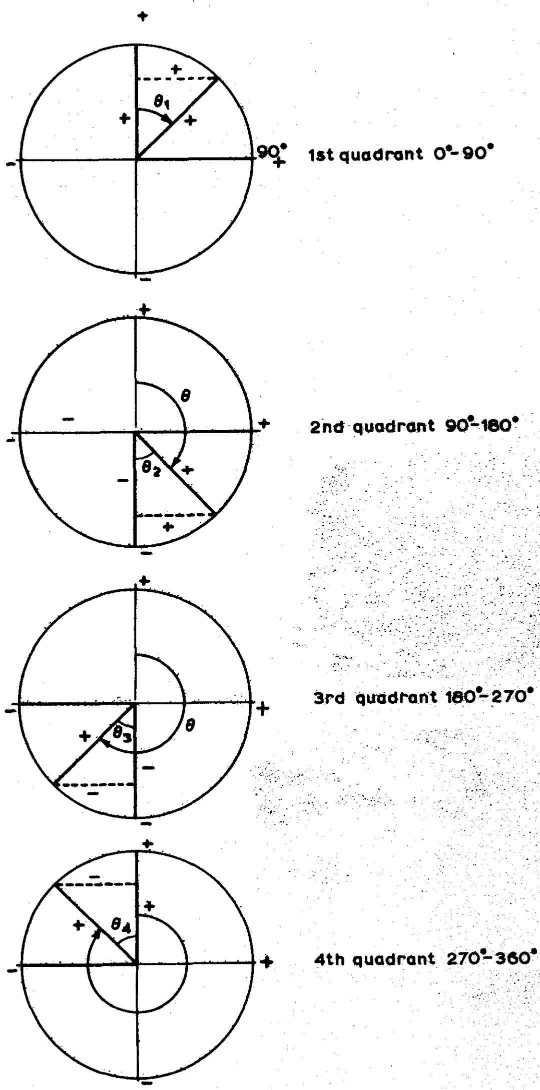


Fig. 2.6

Let the rotating arm be $+ve\ \theta^\circ$.

<u>1st quadrant</u>
$(\theta_1 = \theta)$

$$\sin\theta_1 = \frac{+}{+}, \quad \text{i.e. } +$$

$$\cos\theta_1 = \frac{+}{+}, \quad \text{i.e. } +$$

$$\tan\theta_1 = \frac{+}{+}, \quad \text{i.e. } +$$

<u>2nd quadrant</u>
$\theta_2 = (180 - \theta)$

$$\sin\theta_2 = \frac{+}{+}, \quad \text{i.e. } +$$

$$\cos\theta_2 = \frac{-}{+}, \quad \text{i.e. } -$$

$$\tan\theta_2 = \frac{-}{+}, \quad \text{i.e. } -$$

<u>3rd quadrant</u>
$\theta_3 = (\theta - 180)$

$$\sin\theta_3 = \frac{-}{+}, \quad \text{i.e. } -$$

$$\cos\theta_3 = \frac{-}{+}, \quad \text{i.e. } -$$

$$\tan\theta_3 = \frac{-}{-}, \quad \text{i.e. } +$$

<u>4th quadrant</u>
$\theta_4 = (360 - \theta)$

$$\sin\theta_4 = \frac{-}{+}, \quad \text{i.e. } -$$

$$\cos\theta_4 = \frac{+}{+}, \quad \text{i.e. } +$$

$$\tan\theta_4 = \frac{-}{+}, \quad \text{i.e. } -$$

2.15 Basis of tables of trigonometrical functions

Trigonometrical tables may be prepared, based on the following series :

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \tag{2.12}$$

where θ is expressed as radians, see p. 72.
and $3!$ is factorial 3, i.e. $3 \times 2 \times 1$
$5!$ is factorial 5, i.e. $5 \times 4 \times 3 \times 2 \times 1$.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \tag{2.13}$$

This information is readily available in many varied forms and

to the number of places of decimals required for the particular problem in hand.

The following number of places of decimals are recommended:—

for degrees only, 4 places of decimals,
for degrees and minutes, 5 places of decimals,
for degrees, minutes and seconds, 6 places of decimals.

2.16 Trigonometric ratios of common angles

The following basic angles may be calculated.

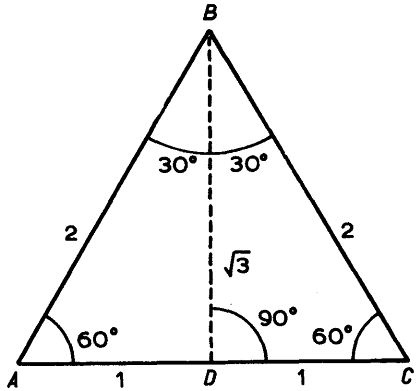


Fig. 2.7 Trigonometrical ratios of 30° and 60°

From the figure, with BD perpendicular to AC,

Let $AB = BC = AC = 2$ units,

then $AD = DC = 1$ unit,

by Pythagoras $BD = \sqrt{(2^2 - 1^2)} = \sqrt{3}$.

Thus $\sin 30^\circ = \dfrac{1}{2} = 0{\cdot}5 = \cos 60^\circ$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{1{\cdot}732}{2} = 0{\cdot}8660 = \cos 30^\circ$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1{\cdot}732}{3}$$

$$= 0{\cdot}5773 = \cot 60^\circ$$

$$\tan 60^\circ = \frac{\sqrt{3}}{1} = 1{\cdot}7320 = \cot 30^\circ$$

Similarly values for 45° may be obtained.

Using a right-angled isosceles triangle where $AC = BC = 1$,

by Pythagoras $AB = \sqrt{(1^2 + 1^2)} = \sqrt{2}$

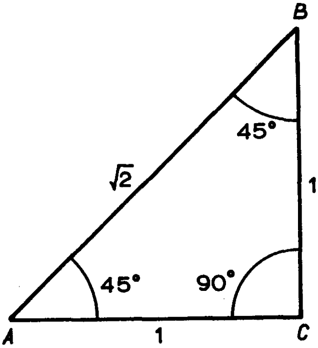


Fig. 2.8 Trigonometrical ratios of 45°

'hus $\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} = \dfrac{1{\cdot}4142}{2} = 0{\cdot}7071 = \cos 45°$

$\tan 45° = \dfrac{1}{1} = 1 = \cot 45°$

It can now be seen from the above that

$$\begin{array}{lllll}
 & \sin 120° = & \sin(180-120) = & \sin 60° = & 0{\cdot}8660 \\
\text{hereas} & \cos 120° = & -\cos(180-120) = & -\cos 60° = & -0{\cdot}5 \\
 & \tan 120° = & -\tan(180-120) = & -\tan 60° = & -1{\cdot}7320 \\
\text{lso} & \sin 210° = & -\sin(210-180) = & -\sin 30° = & -0{\cdot}5 \\
 & \cos 240° = & -\cos(240-180) = & -\cos 60° = & -0{\cdot}5 \\
 & \tan 225° = & +\tan(225-180) = & \tan 45° = & 1{\cdot}0 \\
 & \sin 330° = & -\sin(360-330) = & -\sin 30° = & -0{\cdot}5 \\
 & \cos 315° = & +\cos(360-315) = & +\cos 45° = & 0{\cdot}7071 \\
 & \tan 300° = & -\tan(360-300) = & -\tan 60° = & -1{\cdot}7320
\end{array}$$

17 Points of the compass (Fig. 2.9)

These are not used in Surveying but are replaced by Quadrant (or uadrantal) Bearings where the prefix is always N or S with the suffix or W, Fig. 2.10.

g.

NNE	=	N 22° 30′ E.
ENE	=	N 67° 30′ E.
ESE	=	S 67° 30′ E.
SSE	=	S 22° 30′ E.
SW	=	S 45° 00′ W.
NW	=	N 45° 00′ W.

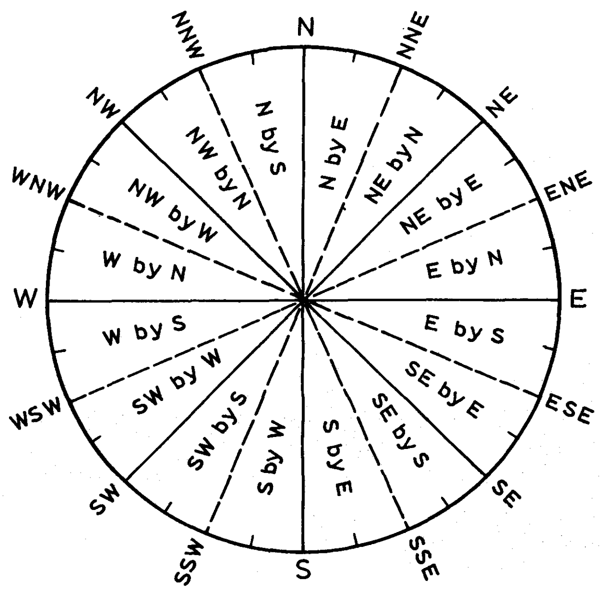


Fig. 2.9 Points of the compass

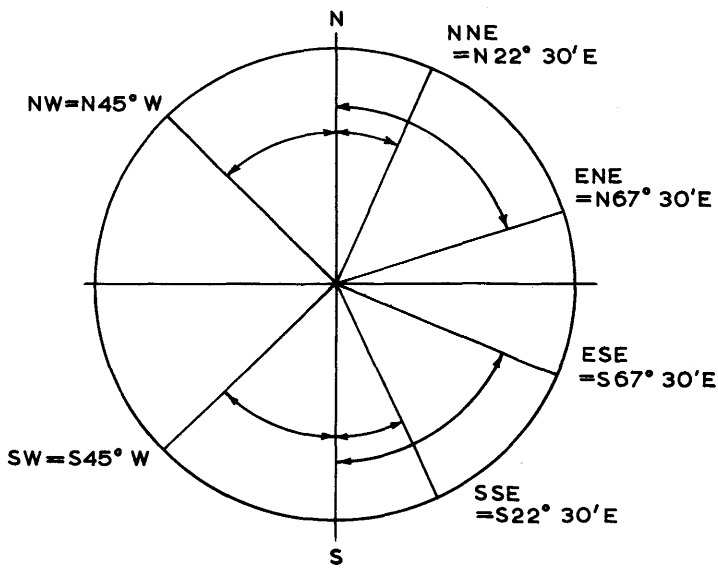


Fig. 2.10

2.18 Easy problems based on the solution of the right-angled triangle

N.B. An *angle of elevation* is an angle measured in the vertical plane where the object is above eye level, i.e. a positive vertical angle, Fig. 2.11.

An *angle of depression* is an angle measured in the vertical plane where the object is below eye level, i.e. a negative vertical angle.

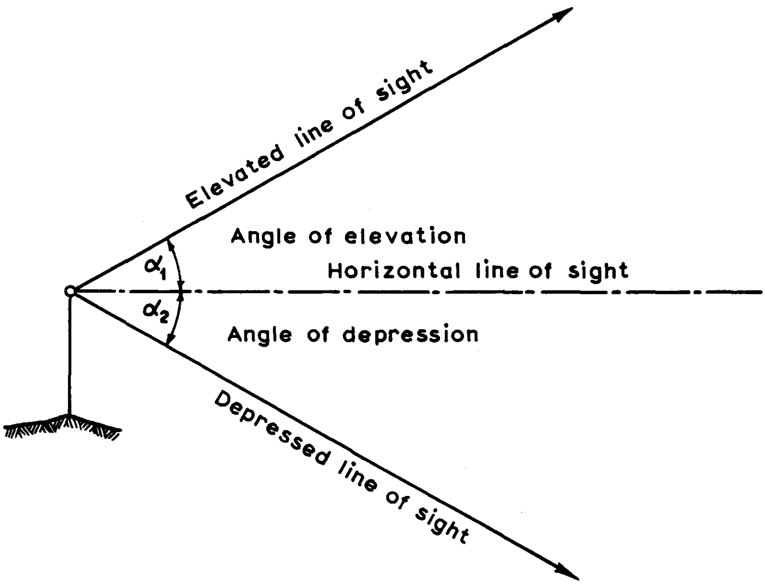


Fig. 2.11 Vertical angles

In any triangle there are six parts, 3 sides and 3 angles

The usual notation is to let the side opposite the angle A be a etc, as shown in Fig. 2.12.

The following facts are thus known about the given right-angled triangle ABC.

Angle $C = 90°$

Angle A + Angle $B = 90°$

$c^2 = a^2 + b^2$ (by Pythagoras)

$\sin A = \frac{a}{c}$

$\cos A = \frac{b}{c}$

$\tan A = \frac{a}{b}$,

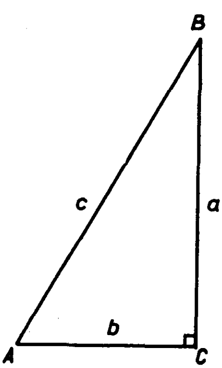


Fig. 2.12

$$\tan A = \frac{a}{c} \div \frac{b}{c} = \frac{a}{c} \times \frac{c}{b}$$

$$\text{i.e.} \quad \sin A \div \cos A.$$

To find the remaining parts of the triangle it is necessary to know *3 parts* (in the case of the right-angled triangle, one angle = 90° and therefore only 2 other facts are required).

Example 2.1. In a right-angled triangle ABC, the hypotenuse AB is 10 metres long, whilst angle A is 70°. Calculate the remaining parts of the triangle.

As the hypotenuse is AB (c) the right angle is at C (Fig. 2.13).

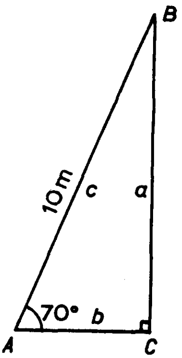


Fig. 2.13

$$\text{then} \quad \frac{a}{c} = \sin 70°$$

$$\begin{aligned} \therefore \quad a &= c \sin 70° \\ &= 10 \sin 70° \\ &= 10 \times 0{\cdot}939\,69 \\ &= 9{\cdot}397 \text{ metres} \end{aligned}$$

$$\frac{b}{c} = \cos 70°$$

$$\begin{aligned} \therefore \quad b &= c \cos 70° \\ &= 10 \times 0{\cdot}342\,02 \\ &= \underline{3{\cdot}420 \text{ metres}} \end{aligned}$$

$$\text{Angle } B = 90° - 70° = 20°$$

$$\text{Check} \quad \frac{b}{a} = \tan B = \frac{3{\cdot}420\,2}{9{\cdot}396\,9} = 0{\cdot}363\,97$$

$$\text{i.e. angle } B = 20°\,00'$$

Example 2.2 It is necessary to climb a vertical wall 45 ft (13·7 m) high with a ladder 50 ft (15·2 m) long, Fig. 2.14. Find

(a) How far from the foot of the wall the ladder must be placed,
(b) the inclination of the ladder

$$\frac{45}{50} = \sin A = 0{\cdot}9$$

$$\therefore \quad \text{angle } A = 64°\,09'\,30''$$

$$\text{thus} \quad \text{angle } B = 25°\,50'\,30''$$

$$\frac{b}{c} = \cos A$$

$$\text{thus} \quad b = 50 \cos 64°\,09'\,30'' = \underline{21{\cdot}79 \text{ ft}} \ (6{\cdot}63 \text{ m})$$

Ans. (a) 21·79 ft (b) 64° 09′ 30″ from the horizontal.

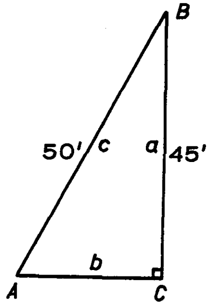


Fig. 2.14

Example 2.3. A ship sails 30 miles (48·28 km) on a bearing N 30° E. It then changes course and sails a further 50 miles (80·4 km) N 45° W.

Find (a) the bearing back to its starting point,

(b) the distance back to its starting point.

N.B. See chapter 3 on bearings

To solve this problem two triangles, *ADB* and *BCE*, are joined to form a resultant third *ACF* (Fig. 2.15).

In triangle *ADB*, *AB* is N30°E 30 miles (48·28 km). The distance travelled N = *AD*.

but $\frac{AD}{AB} = \cos 30°$

then $AD = 30 \cos 30°$

$= \underline{25{\cdot}98 \text{ miles}}$ (41·812 km)

The distance travelled E = *DB*

but $\frac{DB}{AB} = \sin 30°$

$\therefore \quad DB = 30 \sin 30°$

$= \underline{15{\cdot}00 \text{ miles}}$ (24·140 km)

Similarly in triangle *BCE* the distance travelled N = *BE*

but $BE = BC \cos 45°$

$= 50 \cos 45°$

$= 35{\cdot}35$ miles (56·890 km)

The distance travelled W (*CE*) = The distance travelled N = 35·35 miles as the bearing = 45° (sin 45° = cos 45°).

In resultant triangle *ACF*,

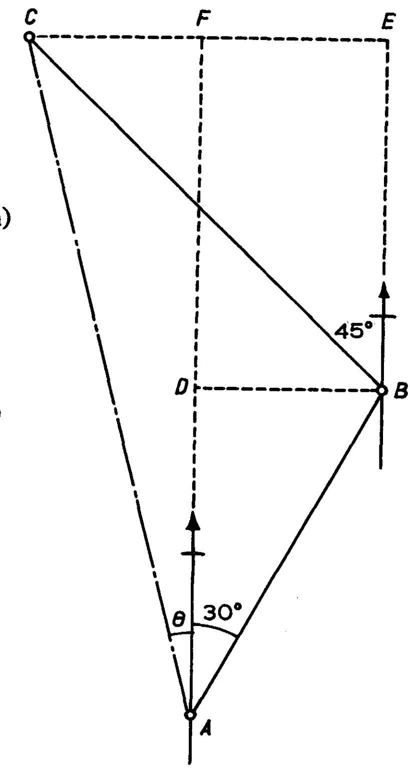


Fig. 2.15

$$CF = CE - DB = 35{\cdot}35 - 15{\cdot}00 = 20{\cdot}35 \text{ miles } (32{\cdot}750)$$
$$AF = AD + BE = 25{\cdot}98 + 35{\cdot}35 = 61{\cdot}33 \text{ miles } (98{\cdot}702)$$
$$\tan\theta = \frac{CF}{AF} = \frac{20{\cdot}35}{61{\cdot}33} = 0{\cdot}33181$$
$$\underline{\theta = 18^\circ 21' 20''} \qquad \therefore \underline{\text{bearing } AC = \text{N } 18^\circ 21' 20'' \text{ W.}}$$
$$AC = \frac{AF}{\cos\theta} = \frac{61{\cdot}33}{\cos 18^\circ 21' 20''} = \underline{64{\cdot}62 \text{ miles}} \ (104{\cdot}0 \text{ km})$$

Example 2.4. An angle of elevation of 45° was observed to the top of a tower. 42 metres nearer to the tower a further angle of elevation of 60° was observed.

Find (a) the height of the tower,
(b) the distance the observer is from the foot of the tower.

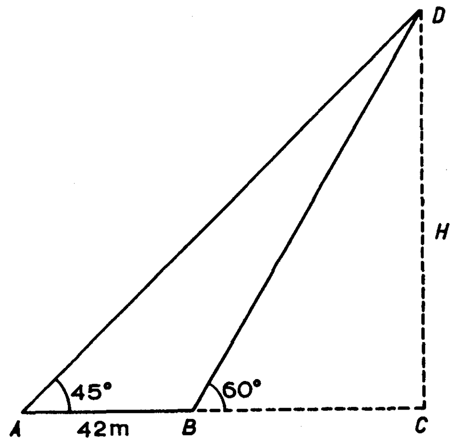


Fig. 2.16

In Fig. 2.16,
$$\frac{AC}{H} = \cot A$$
i.e. $$AC = H \cot A$$
also $$\frac{BC}{H} = \cot B$$
i.e. $$BC = H \cot B.$$
$$AC - BC = AB = H(\cot A - \cot B)$$
$$\therefore \qquad H = \frac{AB}{\cot A - \cot B}$$

$$= \frac{42}{\cot 45^\circ - \cot 60^\circ}$$

$$= \frac{42}{1 - 0{\cdot}5774}$$

$$= \frac{42}{0{\cdot}4226} = \underline{99{\cdot}38\text{ m}}$$

$$BC = H \cot B$$

$$= 99{\cdot}38 \cot 60^\circ$$

$$= \underline{99{\cdot}38 \times 0{\cdot}5774 = 57{\cdot}38\text{ m}}$$

Check $AC = DC = 99{\cdot}38$

$$BC = AC - AB$$

$$= 99{\cdot}38 - 42 = \underline{57{\cdot}38\text{ m}}$$

Exercises 2(a)

1. A flagstaff 90 ft high is held up by ropes, each being attached to the top of the flagstaff and to a peg in the ground and inclined at 30° to the vertical; find the lengths of the ropes and the distances of the pegs from the foot of the flagstaff.

(Ans. 103·92 ft, 51·96 ft)

2. From the top of a mast of a ship 75 ft high the angle of depression of an object is 20°. Find the distance of the object from the ship.

(Ans. 206·06 ft)

3. A tower has an elevation 60° from a point due north of it and 45° from a point due south. If the two points are 200 metres apart, find the height of the tower and its distance from each point of observation.

(Ans. 126·8 m, 73·2 m, 126·8 m)

4. A boat is 1500 ft from the foot of a vertical cliff. To the top of the cliff and the top of a building standing on the edge of the cliff, angles of elevation were observed as 30° and 33° respectively. Find the height of the building to the nearest foot.

(Ans. 108 ft)

5. A vertical stick 3 m long casts a shadow from the sun of 1·75 m. What is the elevation of the sun?

(Ans. 59° 45′)

6. X and Y start walking in directions N 17° W and N 73° E; find their distance apart after three hours and the direction of the line joining them. X walks at 3 km an hour and Y at 4 km an hour.

(Ans. 15 km S 70°08′ E)

7. $A, B,$ and C are three places. B is 30 km N 67½° E of A, and C

is 40 km S 22½° E of B. Find the distance and bearing of C from A.
(Ans. 50 km, S 59° 22′ E)

2.2 Circular Measure

The circumference of a circle $= 2\pi r$ where $\pi = 3{\cdot}141\,6$ approx.

2.21 The radian

The angle subtended at the centre of a circle by an arc equal in length to the radius is known as a *radian*.

Thus
$$2\pi \text{ radians} = 360°$$
$$\therefore \quad 1 \text{ radian} = \frac{360}{2\pi}$$
$$= \underline{57°\,17'\,45''} \text{ approx.}$$
$$= \underline{206\,265 \text{ seconds.}}$$

This last constant factor is of vital importance to small angle calculations and for conversion of degrees to radians.

Example 2.5. Convert 64° 11′ 33″ to radians.

$$64°\,11'\,33'' = 231\,093 \text{ seconds.}$$
$$\therefore \quad \text{no. of radians} = \frac{231\,093}{206\,265} = \underline{1{\cdot}120\,37 \text{ rad.}}$$

Tables of radian measure are available for 0°– 90° and, as the radian measure is directly proportional to the angle, any combination of values produces the same answer for any angular amount.

By tables,

64°	=	1·117 01
11′	=	0·003 20
33″	=	0·000 16
64° 11′ 33″	=	1·120 37 rad

It now follows that the length of an arc of a circle of radius r and subtending θ radians at the centre of the circle can be written as

$$\underline{\text{arc} = r.\theta \text{ rad}} \tag{2.14}$$

This is generally superior to the use of the formula

$$\underline{\text{arc} = 2\pi r \times \frac{\theta°}{360}} \tag{2.15}$$

N.B. When θ is written it implies θ radians.

To find the area of a circle.

A regular polygon $ABC \ldots A$ is drawn inside a circle, Fig. 2.17.

Draw OX perpendicular to AB

Then area of polygon =

$\frac{1}{2}OX(AB + BC + \dots) =$

$\frac{1}{2}OX$ (perimeter of polygon)

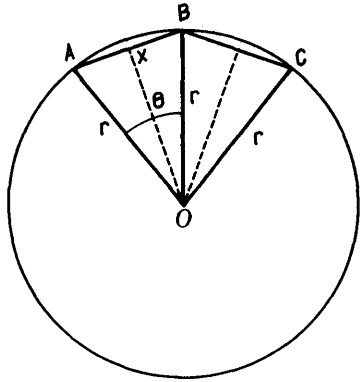


Fig. 2.17

When the number of sides of the polygon is increased to infinity (∞), OX becomes the radius, the perimeter becomes the circumference, and the polygon becomes the circle

$\therefore$ area of circle $= 1/2 \cdot r \cdot 2\pi r$

$= \pi r^2$

The area of the sector OAB.

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

$$\therefore \qquad \text{area of sector} = \frac{\pi r^2\theta}{2\pi} = \frac{1}{2}r^2\theta \qquad (2.16)$$

2.22 Small angles and approximations

For any angle $< 90°$ (i.e. $< \pi/2$ radians) $\tan\theta > \theta > \sin\theta$.

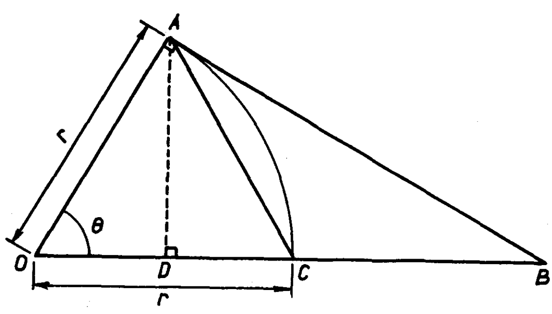


Fig. 2.18

Let angle $AOC = \theta$

$$OA = OC = r$$

and let AB be a tangent to the arc AC at A to cut OC produced at B. Draw AD perpendicular ($\perp$) to OB.

Then area of triangle $OAB = \frac{1}{2}OA\,.\,AB$

$$= \frac{1}{2}r\,.\,r\tan\theta = \underline{\frac{1}{2}r^2\tan\theta}$$

area of sector $OAC = \underline{\frac{1}{2}r^2\theta}$

area of triangle $OAC = \frac{1}{2}OC\,.\,AD$

$$= \frac{1}{2}r\,.\,r\sin\theta = \underline{\frac{1}{2}r^2\sin\theta}$$

Now triangle OAB > sector OAC > triangle OAC.

$$\therefore \qquad \frac{1}{2}r^2\tan\theta > \frac{1}{2}r^2\theta > \frac{1}{2}r^2\sin\theta$$

$$\therefore \qquad \underline{\tan\theta > \theta > \sin\theta}$$

This is obviously true for all values of $\theta < \pi/2$.

Take θ to be very small.
Divide each term by $\sin\theta$,

then $$\frac{1}{\cos\theta} > \frac{\theta}{\sin\theta} > 1$$

It is known that as $\theta \to 0$ then $\cos\theta \to 1$.

Thus $$\cos\theta \simeq 1 \quad \text{when } \theta \text{ is small}$$

$$\therefore \frac{\theta}{\sin\theta} \quad \text{must also be nearly 1.}$$

The result shows that $\sin\theta$ may be replaced by θ.
Similarly, dividing each term by $\tan\theta$,

$$1 > \frac{\theta}{\tan\theta} > \frac{\sin\theta}{\tan\theta}$$

i.e. $$1 > \frac{\theta}{\tan\theta} > \cos\theta$$

Tan θ may also be replaced by θ.

It can thus be shown that for very small angles

$$\tan\theta \simeq \theta \simeq \sin\theta.$$

The following values are taken from H.M. Nautical Almanac Office. Five-figure Tables of Natural Trigonometrical Functions. (These tables are very suitable for most machine calculations.)

Angle	Tangent	Radian	Sine
1° 00′ 00″	0·017 46	0·017 45	0·017 45
1° 30′ 00″	0·026 19	0·026 18	0·026 18
2° 00′ 00″	0·034 92	0·034 91	0·034 90
2° 30′ 00″	0·043 66	0·043 63	0·043 62
3° 00′ 00″	0·052 41	0·052 36	0·052 34
3° 30′ 00″	0·061 16	0·061 09	0·061 05
4° 00′ 00″	0·069 93	0·069 81	0·069 76
4° 30′ 00″	0·078 70	0·078 54	0·078 46
5° 00′ 00″	0·087 49	0·087 27	0·087 16

From these it can be seen that θ may be substituted for $\sin\theta$ or $\tan\theta$ to 5 figures up to 2°, whilst θ may be substituted for $\sin\theta$ up to 5° and for $\tan\theta$ up to 4° to 4 figures, thus allowing approximations to be made when angles are less than 4°

Example 2.6. If the distance from the earth to the moon be 250 000 miles (402 000 km) and the angle subtended 0° 30′, find the diameter of the moon.

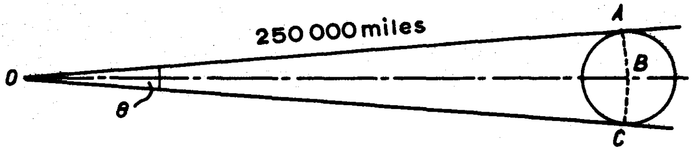


Fig. 2.19

$$\begin{aligned} \text{The diameter} &\simeq \text{arc } ABC \\ &\simeq 250\,000 \times 30' \text{ rad} \\ &\simeq 250\,000 \times \frac{30 \times 60}{206\,265} \simeq \underline{2180 \text{ miles}} \text{ (3510 km)} \end{aligned}$$

Example 2.7. Find as exactly as possible from Chambers Mathematical Tables the logarithmic sines of the following angles:

$$A = 00^\circ 02' 42'' \quad \text{and} \quad C = 00^\circ 11' 30''$$

Use these values to find the lengths of the sides AB and AC in a triangle ABC when $BC = 12{\cdot}736$ ft. Thereafter check your answer by another method, avoiding as far as possible using the tables at the same places as in the first method.

The lengths are to be stated to three places of decimals.

(M.Q.B/S)

As the sines and tangents of small angles change so rapidly, special methods are necessary.

Method 1

Chambers Mathematical Tables give the following method of finding the logarithmic sine of a small arc:

To the logarithm of the arc reduced to seconds, add 4·685 574 9 and from the sum subtract 1/3 of its logarithmic secant, the index of the latter logarithm being previously diminished by 10.

$00°\,02'\,42'' = 162''$ log 162   2·209 515 0
constant   4·685 574 9
6·895 089 9

$-\frac{1}{3}(\log \sec 02'\,42'' - 10) = \frac{1}{3} \times 0{\cdot}000\,000\,2 = 0{\cdot}000\,000\,1$
6·895 089 8

$00°\,11'\,30'' = 690''$ log 690   2·838 849 1
constant   4·685 574 9
7·524 424 0

$-\frac{1}{3}(\log \sec 11'\,30'' - 10) = \frac{1}{3} \times 0{\cdot}000\,002\,4 = 0{\cdot}000\,000\,8$
7·524 423 2

Method 2

As the sines and tangents of small angles approximate to the value,

$$\text{radian value of } A\ 00°\,02'\,42'' = \frac{162}{206\,265} = 0{\cdot}000\,785\,397\,4$$

$$\log 0{\cdot}000\,785\,397\,4 = \underline{6{\cdot}895\,089\,5}$$

$$\text{radian value of } B\ 00°\,11'\,30'' = \frac{690}{206\,265} = 0{\cdot}003\,345\,211\,2$$

$$\log 0{\cdot}003\,345\,211\,2 = \underline{7{\cdot}524\,423\,5}$$

Summary	(1)	(2)	Vegas tables (to 1″)
00° 02′ 42″	6·895 089 8	6·895 089 5	6·895 089 8
00° 11′ 30″	7·524 423 2	7·524 423 5	7·524 423 1

To find the length of sides AB and AC when $BC = 12{\cdot}736$:

$$AB = \frac{BC \sin C}{\sin A} = \frac{12{\cdot}736 \sin (02'\,42'' + 11'\,30'')}{\sin 02'\,42''} \text{ (see 2.51}$$

		Logs
12·736	=	1·105 033 1
$S/14'\,12''$	=	2·929 279 3
		4·685 574 9
		8·719 887 3
$-\frac{1}{3}(\sec - 10)$		0·000 003 5
		8·719 883 8
$S/02'\,42''$		6·895 089 8
AB		1·824 794 0

$\therefore\ AB = 66{\cdot}803$ ft

$$AC = \frac{BC \sin B}{\sin A} = \frac{12{\cdot}736 \sin 11'\,30''}{\sin 02'\,42''}$$

12·736	1·105 033 1
$S/11'\,30''$	7·524 423 2
	8·629 456 3
$S/02'\,42''$	6·895 089 8
AC	1·734 366 5

$AC = 54{\cdot}246$ ft

2.3 Trigonometrical Ratios of the Sums and Differences of two angles (Fig. 2.20)

To prove:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{2.17}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \tag{2.18}$$

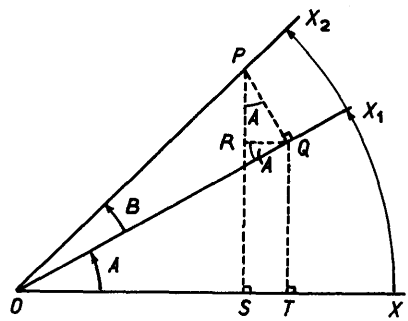


Fig. 2.20

Let the line OX trace out the angle A and then the angle B. Take a point P on the final line OX_2. Draw PS and PQ perpendicular to OX and OX_1 respectively. Through Q draw QR parallel to OX to meet PS at R. Draw QT perpendicular to OX.

$$R\hat{P}Q = R\hat{Q}O = A$$

$$\sin(A + B) = \frac{PS}{OP} = \frac{RS + PR}{OP} = \frac{RS}{OP} + \frac{PR}{OP}$$

$$= \frac{RS}{OQ} \cdot \frac{OQ}{OP} + \frac{PR}{PQ} \cdot \frac{PQ}{OP}$$

$$= \sin A \cos B + \cos A \sin B$$

$$\cos(A+B) = \frac{OS}{OP} = \frac{OT - ST}{OP} = \frac{OT}{OP} - \frac{ST}{OP}$$

$$= \frac{OT}{OQ} \cdot \frac{OQ}{OP} - \frac{ST}{PQ} \cdot \frac{PQ}{OP}$$

$$= \underline{\cos A \cos B - \sin A \sin B}$$

If angle B is now considered $-ve$,

$$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$= \underline{\sin A \cos B - \cos A \sin B} \qquad (2.19)$$

Similarly,

$$\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$$

$$= \underline{\cos A \cos B + \sin A \sin B} \qquad (2.20)$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

$$(\div \text{ by } \cos A \cos B) = \underline{\frac{\tan A + \tan B}{1 - \tan A \tan B}} \qquad (2.21)$$

Similarly, letting B be $-ve$,

$$\tan(A-B) = \underline{\frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}}$$

$$= \underline{\frac{\tan A - \tan B}{1 + \tan A \tan B}} \qquad (2.22)$$

If $\quad \sin(A+B) = \sin A \cos B + \cos A \sin B$

and $\quad \sin(A-B) = \sin A \cos B - \cos A \sin B$

then $\quad \underline{\sin(A+B) + \sin(A-B) = 2 \sin A \cos B} \qquad (2.23)$

and $\quad \underline{\sin(A+B) - \sin(A-B) = 2 \cos A \sin B} \qquad (2.24)$

Similarly,

as $\quad \cos(A+B) = \cos A \cos B - \sin A \sin B$

and $\quad \cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\underline{\cos(A+B) + \cos(A-B) = 2 \cos A \cos B} \qquad (2.25)$$

$$\underline{\cos(A+B) - \cos(A-B) = -2 \sin A \sin B} \qquad (2.26)$$

If $A = B$, then

$$\underline{\sin(A+A) = \sin 2A = 2 \sin A \cos A} \qquad (2.27)$$

$$\underline{\cos(A+A) = \cos 2A = \cos^2 A - \sin^2 A}$$

$$= \underline{1 - 2\sin^2 A} \qquad (2.28)$$

or $\quad = \underline{2\cos^2 A - 1} \qquad (2.29)$

$$\tan(A + A) = \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad (2.30)$$

2.4 Transformation of Products and Sums

From
$$\left.\begin{aligned} \sin(A + B) + \sin(A - B) &= 2 \sin A \cos B \\ \sin(A + B) - \sin(A - B) &= 2 \sin B \cos A \end{aligned}\right\}$$

if
$$A + B = C$$
and
$$A - B = D$$
then
$$A = \frac{C + D}{2} \quad \text{and} \quad B = \frac{C - D}{2}$$

$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2} \qquad (2.31)$$

Similarly,
$$\sin C - \sin D = 2 \sin \frac{C - D}{2} \cos \frac{C + D}{2} \qquad (2.32)$$

From
$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$
$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$$

$$\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2} \qquad (2.33)$$

and
$$\cos C - \cos D = -2 \sin \frac{C + D}{2} \sin \frac{C - D}{2} \qquad (2.34)$$

These relationships may thus be tabulated:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$$
$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B$$
$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B$$
$$\cos(A + B) - \cos(A - B) = -2 \sin A \sin B$$

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin A + \sin B = 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

2.5 The Solution of Triangles

The following important formulae are now proved:

Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{2.35}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{2.36}$$

$$\sin C = \frac{2}{ab}\sqrt{\{s(s-a)(s-b)(s-c)\}} \tag{2.37}$$

Area of triangle $= \frac{1}{2}ab\sin C$ (2.38)

$$= \sqrt{s(s-a)(s-b)(s-c)} \tag{2.39}$$

Half-angle formulae

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \tag{2.40}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \tag{2.41}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \tag{2.42}$$

Napier's tangent rule

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\tan\frac{B+C}{2} \tag{2.43}$$

2.51 Sine rule (Figs. 2.21 and 2.22)

Let triangle ABC be drawn with circumscribing circle.
Let AB_1 be a diameter through A (angle ABC = angle AB_1C).

In Fig. 2.21, $\dfrac{AC}{AB_1} = \sin B$

In Fig. 2.22, $\dfrac{AC}{AB_1} = \sin(180 - B)$

$= \sin B$

$\therefore \quad \dfrac{b}{2R} = \sin B$

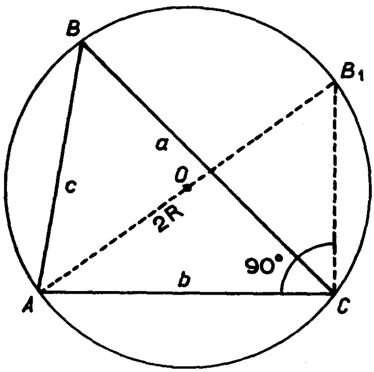


Fig. 2.21

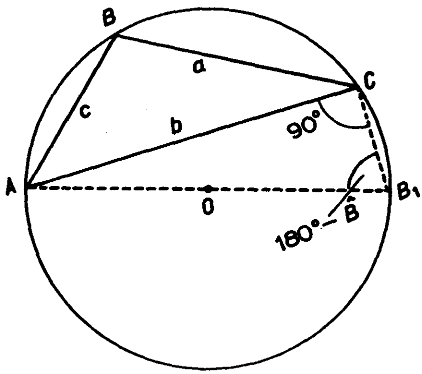


Fig. 2.22

$$\therefore \qquad \frac{b}{\sin B} = 2R$$

Similarly
$$\underline{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{b}{\sin C} = 2R} \qquad (2.35)$$

2.52 Cosine rule (Fig. 2.23)

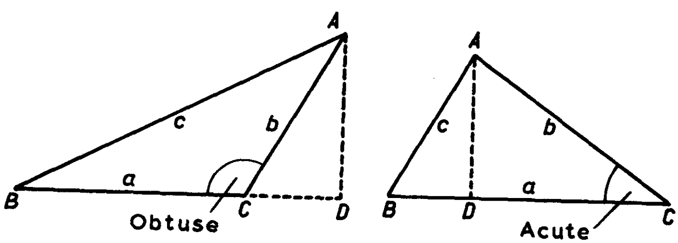


Fig. 2.23 The cosine rule

$$
\begin{aligned}
AB^2 &= AD^2 + BD^2 \quad \text{(Pythagoras)} \\
&\left.\begin{aligned} &= AD^2 + (BC - CD)^2 \\ &= b^2 \sin^2 C + (BC - b\cos C)^2 \end{aligned}\right\} \text{ with } C \text{ acute} \\
&\left.\begin{aligned} &= AD^2 + (BC + CD)^2 \\ &= b^2 \sin^2 (180 - C) + \{BC + b\cos(180 - C)\}^2 \\ &= b^2 \sin^2 C + (BC - b\cos C)^2 \end{aligned}\right\} \text{ with } C \text{ obtuse}
\end{aligned}
$$

$$\therefore \quad AB^2 = b^2 \sin^2 C + (BC - b\cos C)^2 \quad \text{in either case}$$

$$
\begin{aligned}
\therefore \quad c^2 &= b^2 \sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C. \\
&= a^2 + b^2(\sin^2 C + \cos^2 C) - 2ab\cos C \\
&= \underline{a^2 + b^2 - 2ab\cos C} \qquad (2.36)
\end{aligned}
$$

2.53 Area of a triangle

From
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\therefore \sin^2 C = 1 - \cos^2 C = 1 - \left(\frac{a^2+b^2-c^2}{2ab}\right)^2$$
$$= \left(1 + \frac{a^2+b^2-c^2}{2ab}\right)\left(1 - \frac{a^2+b^2-c^2}{2ab}\right)$$
$$= \frac{(a+b)^2 - c^2}{2ab} \times \frac{c^2-(a-b)^2}{2ab}$$
$$= \frac{(a+b+c)(-c+a+b)(c-a+b)(c+a-b)}{(2ab)^2}$$
$$= \frac{4s(s-a)(s-b)(s-c)}{a^2b^2}$$

where $2s = a+b+c$.
$$\underline{\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}} \qquad (2.37)$$

In Fig. 2.23,
$$\text{Area of triangle} = \tfrac{1}{2}AD\cdot BC$$
$$= \tfrac{1}{2}ab\sin C \qquad (2.38)$$
$$= \tfrac{1}{2}ab\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{s(s-a)(s-b)(s-c)} \qquad (2.39)$$

2.54 Half-angle formulae

From Eq.(2.28),
$$\sin^2\tfrac{1}{2}A = \tfrac{1}{2}(1-\cos A) = \tfrac{1}{2}\left(1 - \frac{b^2+c^2-a^2}{2bc}\right)$$
$$= \frac{a^2-(b-c)^2}{4bc}$$
$$= \frac{(a-b+c)(a+b-c)}{4bc}$$
$$= \frac{(s-b)(s-c)}{bc}$$
$$\therefore \qquad \underline{\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}} \qquad (2.40)$$

Similarly,

$$\cos^2 \frac{1}{2}A = \frac{1}{2}(1 + \cos A) = \frac{1}{2}\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{(b+c)^2 - a^2}{4bc}$$

$$= \frac{(b+c+a)(b+c-a)}{4bc}$$

$$= \frac{s(s-a)}{bc}$$

$$\underline{\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}} \qquad (2.41)$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \qquad (2.42)$$

The last formula is preferred as $(s - b) + (s - c) + (s - a) = s$, which provides an arithmetical check.

2.55 Napier's tangent rule

From the sine rule,

$$\frac{b}{c} = \frac{\sin B}{\sin C}$$

then

$$\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}} \quad \text{(By Eqs. 2.31/2.32)}$$

$$= \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}$$

$$\therefore \underline{\tan\frac{B-C}{2} = \frac{b-c}{b+c}\tan\frac{B+C}{2}} \qquad (2.43)$$

2.56 Problems involving the solution of triangles

All problems come within the following four cases:

(1) *Given two sides and one angle (not included)* to find the other angles.
Solution: Sine rule solution ambiguous as illustrated in Fig. 2.24.

Given AB and AC with angle B, AC may cut line BC at C_1 or C_2

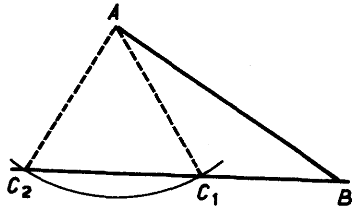


Fig. 2.24 The ambiguous case of the sine rule

(2) *Given all the angles and one side* to find all the other sides.
Solution: Sine rule

(3) *Given two sides and the included angle*
Solution: Either cos rule to find remaining side
or Napier's tangent rule (this is generally preferred using logs)

(4) *Given the three sides*
Solution: either cos rule
or half-angle formula (this is generally preferred using logs.)

Example 2.8 (Problem 1)

Let
$$c = 466{\cdot}0 \text{ m}$$
$$a = 190{\cdot}5 \text{ m}$$
$$\hat{A} = 22^\circ 15'$$

Using sine rule,

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$
$$= \frac{466{\cdot}0 \sin 22^\circ 15'}{190{\cdot}5}$$
$$= \frac{466{\cdot}0 \times 0{\cdot}378\,65}{190{\cdot}5}$$
$$= 0{\cdot}926\,25$$

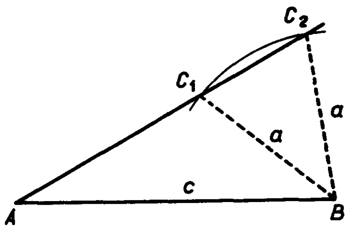


Fig. 2.25

$$\text{Angle } C_2 = 67°\,51'\,30'' \quad \text{or} \quad 180 - C$$
$$C_1 = 112\,08'\,30''$$

To find side b (this is now Problem 2),

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\therefore \quad b = \frac{a \sin B}{\sin A}$$

i.e. $$b_1 = \frac{190{\cdot}5 \sin[180 - (67°\,51'\,30'' + 22°\,15'\,00'')]}{\sin 22°\,15'\,00''}$$

$$= \frac{190{\cdot}5 \sin 89°\,53'\,30''}{\sin 22°\,15'\,00''}$$

$$= \frac{190{\cdot}5 \times 1{\cdot}0}{0{\cdot}378\,65} = \underline{503{\cdot}10 \text{ m}}$$

or $$b_2 = \frac{190{\cdot}5 \sin[(180 - (112°\,08'\,30'' + 22°\,15'\,00'')]}{\sin 22°\,15'\,00''}$$

$$= \frac{190{\cdot}5 \sin 45°\,36'\,30''}{\sin 22°\,15'\,00} = \underline{359{\cdot}51 \text{ m}}$$

Log calculation

$$\log \sin C = \log c + \log \sin A - \log a$$
$$C = 67°\,51'\,30''$$

466·0	(1) 2·668 39
sin 22° 15′	9·578 24
	(2) 2·246 63
190·5	2·279 90
sin C	(1) 9·966 73

N.B. The notation 9·578 24 is preferred to $\bar{1}{\cdot}578\,24$ – this is the form used in Chambers, Vegas, and Shortredes Tables.

Every characteristic is increased by 10 so that subtraction is simplified – the ringed figures are not usually entered.

Also $$\log b = \log a + \log \sin B + \log \operatorname{cosec} A$$

N.B. Addition using $\log \operatorname{cosec} A$ is preferable to subtracting $\log \sin A$.

i.e. $$\log b_1 = \log 190{\cdot}5 + \log \sin 89°\,53'\,30'' + \log \operatorname{cosec} 22°\,15'\,00''$$

190·5	2·279 90
sin 89° 53′ 30′	0·0
cosec 22° 15′ 00″	10·421 76
b_1	2·701 66

$\underline{b_1 = 503{\cdot}10 \text{ m}}$

$$\log b_2 = \log 190{\cdot}5 + \log \sin 45°\,36'\,30'' + \log \operatorname{cosec} 22°\,15'\,00''$$

	190·5	2·279 90
	sin 45° 36′ 30″	9·854 05
	cosec 22° 15′ 00″	10·421 76
$b_2 = \underline{359{\cdot}51\text{ m}}$	b_2	2·555 71

N.B. A gap is left between the third and fourth figures of the logarithms to help in the addition process, or it is still better to use squared paper.

Example 2.9 (Problem 3)

Let

$$a = 636\text{ m}$$
$$c = 818\text{ m}$$
$$B = 97°\,30'$$

To find b, A and C.

By cosine rule

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ &= 636^2 + 818^2 - 2 \times 636 \times 818 \times \cos 97°\,30' \\ &= 404\,496 + 669\,124 + 135\,815{\cdot}94 \\ &= 1\,209\,435{\cdot}94 \\ b &= \underline{1099{\cdot}74\text{ m}} \end{aligned}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\sin A = \frac{a \sin B}{b} = \frac{636\ \sin 97°\,30'}{1\,099{\cdot}74}$$
$$= 0{\cdot}573\,37$$
$$A = 34°\,59'\,10''$$
$$\therefore \quad B = 47°\,30'\,50''$$

The first part of the calculation is essentially simple but as the figures get large it becomes more difficult to apply and logs are not suitable. The following approach is therefore recommended.

As $c > a$, $C > A$. Then, by Eq. (2.43),

$$\begin{aligned} \tan\frac{C-A}{2} &= \frac{c-a}{c+a}\tan\frac{C+A}{2} \\ &= \frac{818-636}{818+636}\tan\frac{(180-97°\,30')}{2} \\ &= \frac{182\tan 41°\,15'}{1\,454} \\ &= 0{\cdot}109\,77 \end{aligned}$$

$$\tfrac{1}{2}(C-A) = 6^\circ 15' 50''$$

$$\tfrac{1}{2}(C+A) = 41^\circ 15' 00''$$

By adding $\quad C = 47^\circ 30' 50''$

By subtracting $\quad A = 34^\circ 59' 10''$

Now, by sine rule,

$$\begin{aligned} b &= a \sin B \operatorname{cosec} A \\ &= 636 \sin 97^\circ 30' \operatorname{cosec} 34^\circ 59' 10'' \\ &= 1099{\cdot}74 \text{ m} \end{aligned}$$

N.B. This solution is fully logarithmic and thus generally preferred. Also it does not require the extraction of a square root and is therefore superior for machine calculation.

Example 2.10 (Problem 4)

Let $\quad a = 381 \quad b = 719 \quad c = 932$

To find the angles.

From $\quad a^2 = b^2 + c^2 - 2bc \cos A$

then

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{719^2 + 932^2 - 381^2}{2 \times 719 \times 932} \\ &= \frac{516\,961 + 868\,624 - 145\,161}{1\,340\,216} \\ &= 0{\cdot}925\,54 \\ A &= 22^\circ 15' 00'' \end{aligned}$$

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{145\,161 + 868\,624 - 516\,961}{2 \times 381 \times 932} \\ &= \frac{496\,824}{710\,184} = 0{\cdot}699\,57 \\ B &= 45^\circ 36' 30'' \end{aligned}$$

$$\begin{aligned} \cos C &= \frac{a^2 + b^2 - c^2}{2ab} \\ &= \frac{145\,161 + 516\,961 - 868\,624}{2 \times 381 \times 719} \\ &= \frac{-206\,502}{547\,878} = -0{\cdot}376\,91 \end{aligned}$$

$$C = 180 - 67°51'30''$$
$$= \underline{112°08'30''}$$

Check $22°15'00'' + 45°36'30'' + 112°08'30'' = \underline{180°00'00''}$

Alternative

By half-angle formula, $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

a	$= 381$	$s-a$	$= 635$
b	$= 719$	$s-b$	$= 297$
c	$= \underline{932}$	$s-c$	$= 84$
$2s$	$= \underline{2032}$		
s	$= 1016$	s	$= 1016$

then $$\tan\frac{A}{2} = \sqrt{\frac{297 \times 84}{1016 \times 635}}$$

This is best solved by logs

$$\log\tan\frac{A}{2} = \tfrac{1}{2}[(\log 297 + \log 84) - (\log 1016 + \log 635)]$$

$$\frac{A}{2} = 11°7'30''$$

$$\underline{A = 22°15'00''}$$

297	2·472 76
84	1·924 28
	4·397 04
1016	3·006 89
635	2·802 77
	5·809 66
	2) 18·587 38
$\tan A/2$	9·293 69

$$\tan\frac{B}{2} = \sqrt{\frac{635 \times 84}{1016 \times 297}}$$

$$\frac{B}{2} = 22°48'15''$$

$$\underline{B = 45°36'30''}$$

635	2·802 77
84	1·924 28
	4·727 05
1016	3·006 89
297	2·472 76
	5·479 65
	2) 19·247 40
$\tan B/2$	9·623 70

Example 2.11 The sides of a triangle ABC measure as follows: $AB = 36 \text{ ft } 0\frac{7}{16}\text{ in.}$, $AC = 30 \text{ ft } 1\frac{3}{4}\text{ in.}$ and $BC = 6 \text{ ft } 0\frac{1}{4}\text{ in.}$

(a) Calculate to the nearest 20 seconds, the angle BAC.

(b) Assuming that the probable error in measuring any of the sides is $\pm 1/32$ in. give an estimate of the probable error in the angle A.

(M.Q.B/S)

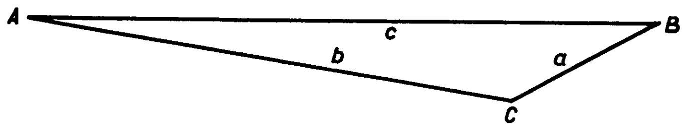


Fig. 2.26

$$AB = c = 36 \text{ ft } 0\tfrac{7}{16}\text{ in.} = 36{\cdot}036 \text{ ft} \qquad s - c = 0{\cdot}0655$$

$$AC = b = 30 \text{ ft } 1\tfrac{3}{4}\text{ in.} = 30{\cdot}146 \text{ ft} \qquad s - c = 5{\cdot}9555$$

$$BC = a = 6 \text{ ft } 0\tfrac{1}{4}\text{ in.} = \underline{6{\cdot}021 \text{ ft}} \qquad s - a = \underline{30{\cdot}0805}$$

$$2\,|\,\underline{72{\cdot}203} \qquad \underline{s = 36{\cdot}1015}$$

$$\underline{s = 36{\cdot}1015}$$

Using Eq. (2.42)

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= \sqrt{\frac{5{\cdot}955\,5 \times 0{\cdot}065\,5}{36{\cdot}101\,5 \times 30{\cdot}080\,5}}$$

$$\frac{A}{2} = 1°05'\,10''$$

$$\underline{A = 2°\,10'\,20''}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= \sqrt{\frac{30{\cdot}080\,5 \times 0{\cdot}065\,5}{36{\cdot}101\,5 \times 5{\cdot}955\,5}}$$

$$\frac{B}{2} = 5°\,28'\,05''$$

$$\underline{B = 10°\,56'\,10''}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \sqrt{\frac{30{\cdot}080\,5 \times 5{\cdot}955\,5}{36{\cdot}101\,5 \times 0{\cdot}065\,5}}$$

$$\frac{C}{2} = 83°\,26'\,45''$$

$$C = 166°\,53'\,30''$$

Check $A + B + C = 180°$

(b) The probable error of $\pm 1/32$ in. $= \pm 0{\cdot}003$ ft.

The effect on the angle A of varying the three sides is best calculated by varying each of the sides in turn whilst the remaining two sides are held constant. To carry out this process, the equation must be successively differentiated and a better equation for this purpose is the cosine rule.

Thus $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Differentiating with respect to a,

$$-\sin A\,\delta A_a = -\frac{2a\,\delta a}{2bc}$$

$$\therefore \qquad \underline{\delta A_a = \frac{a\,\delta a}{bc\,\sin A}}$$

Differentiating with respect to b,

$$-\sin A\,\delta A_b = \frac{(2bc \times 2b) - (b^2 + c^2 - a^2)(2c)}{4b^2c^2}$$

$$= \frac{1}{c} - \frac{b^2 + c^2 - a^2}{2b^2c}$$

$$= \frac{a^2 + b^2 - c^2}{2b^2c}$$

but $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \qquad \underline{\delta A_b = -\frac{a\cos C}{bc\sin A}\,\delta b} = \underline{-\delta A_a \cos C} \quad (\text{as } \delta a = \delta b)$$

Similarly, from the symmetry of the function:

$$\underline{\delta A_c = -\frac{a\cos B}{bc\,\sin A}\,\delta c} = \underline{-\delta A_a \cos B} \qquad (\text{as } \delta a = \delta c)$$

Substituting values into the equations gives:

$$\delta A_a = \frac{6{\cdot}021 \times \pm 0{\cdot}003 \times 206\,265}{30{\cdot}146 \times 36{\cdot}036 \times \sin 2°\,10'\,20''} = \pm 90{\cdot}49 \text{ sec}$$

$$\delta A_b = \delta A_a \cos 166°\,53'\,30'' = \pm 88{\cdot}13 \text{ sec}$$

$$\delta A_c = \delta A_a \cos 10°\,56'\,10'' = \pm 88{\cdot}85 \text{ sec}$$

$\therefore$ Total probable error $= \sqrt{\delta A_a^2 + \delta A_b^2 + \delta A_c^2}$

$= \sqrt{90{\cdot}49^2 + 88{\cdot}13^2 + 88{\cdot}85^2}$

$= \underline{\pm 154 \text{ seconds.}}$

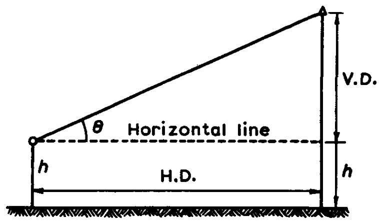
2.6 Heights and Distances

2.61 To find the height of an object having a vertical face

The ground may be (a) level or (b) sloping up or down from the observer.

(a) *Level ground* (Fig. 2.27)

Fig. 2.27



The observer of height h is a horizontal distance (H.D.) away from the object. The vertical angle (V.A.) $= \theta$ is measured. The vertical difference

$$\text{V.D.} = \text{H.D.} \tan\theta \tag{2.44}$$

Height of the object above the ground = V.D. + h

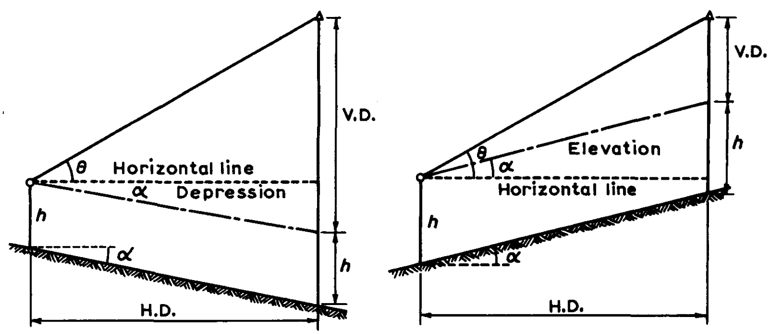


Fig. 2.28

(b) *Sloping ground* (Fig. 2.28)

The ground slope is measured as α

$$\text{V.D.} = \text{H.D.} (\tan\theta \pm \tan\alpha) \tag{2.45}$$

Height of object above the ground = V.D. + h

N.B. This assumes that the horizontal distance can be measured.

2.62 To find the height of an object when its base is inaccessible

A base line must be measured and angles are measured from its extremities.

(a) *Base line AB level and in line with object.* (Fig. 2.29)
Vertical angles α and β are measured.

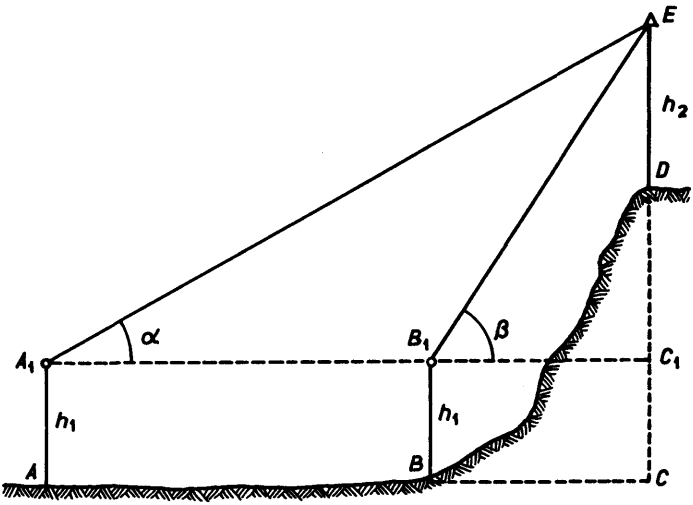


Fig. 2.29

$$A_1C_1 = EC_1 \cot\alpha$$

$$B_1C_1 = EC_1 \cot\beta$$

$$\therefore \quad A_1B_1 = A_1C_1 - B_1C_1 = EC_1(\cot\alpha - \cot\beta)$$

Thus
$$EC_1 = \frac{AB}{\cot\alpha - \cot\beta} \qquad (2.46)$$

Height of object above ground at A

$$= \underline{EC_1 + h_1}$$

Height of ground at D above ground at A

$$= \underline{EC_1 + h_1 - h_2}$$

(b) *Base line AB level but not in line with object* (Fig. 2.30)

Angles measured at A horizontal angle θ
vertical angle α
at B horizontal angle ϕ
vertical angle β

In triangle ABC;

$$AC = AB \sin\phi \operatorname{cosec}(\theta + \phi) \qquad \text{(sine rule)}$$

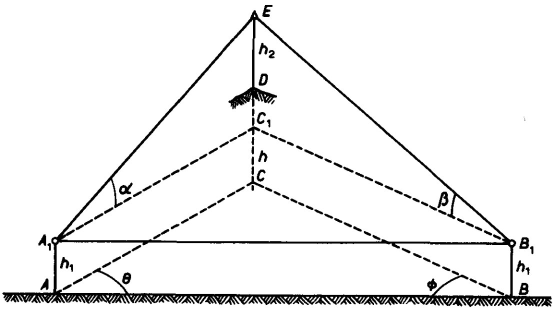


Fig. 2.30

and $\qquad BC = AB \sin\theta \operatorname{cosec}(\theta + \phi)$

Then
$$C_1E = AC \tan\alpha$$
$$= AB \sin\phi \operatorname{cosec}(\theta + \phi) \tan\alpha \qquad (2.47)$$

Also
$$C_1E = BC \tan\beta$$
$$= AB \sin\theta \operatorname{cosec}(\theta + \phi) \tan\beta \qquad (2.48)$$

Height of object (E) above ground at A
$$= C_1E + h_1$$

Height of ground (D) above ground at A
$$= C_1E + h_1 - h_2$$

(c) *Base line AB on sloping ground and in line with object* (Fig. 2.31)

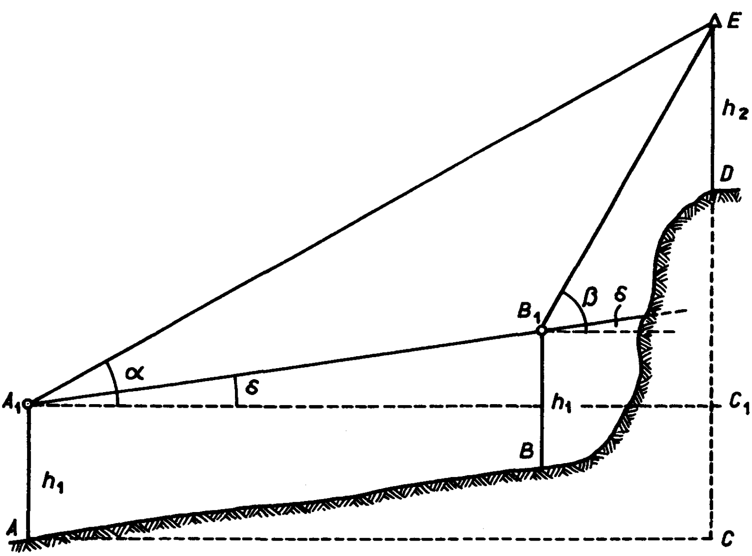


Fig. 2.31

Angles measured at A: vertical angle α (to object)
vertical angle δ (slope of ground)
Angle measured at B: vertical angle β (to object).
In triangle A_1EB_1,

$$\hat{A}_1 = \alpha - \delta$$
$$\hat{B}_1 = 180 - (\beta - \delta)$$
$$\hat{E} = \beta - \alpha$$

Then
$$A_1E = A_1B_1 \sin\{180-(\beta-\delta)\} \operatorname{cosec}(\beta - \alpha)$$
$$= \underline{AB \sin(\beta - \delta) \operatorname{cosec}(\beta - \alpha)} \qquad (2.49)$$

Height of object (E) above ground at A

$$EC = EC_1 + h_1$$
$$= A_1E \sin\alpha + h_1$$
$$= \underline{AB \sin(\beta - \delta) \operatorname{cosec}(\beta - \alpha) \sin\alpha + h_1} \qquad (2.50)$$

Height of ground (D) above ground at A

$$= EC_1 + h_1 - h_2.$$

(d) *Base line AB on sloping ground and not in line with object* (Fig.2.32)

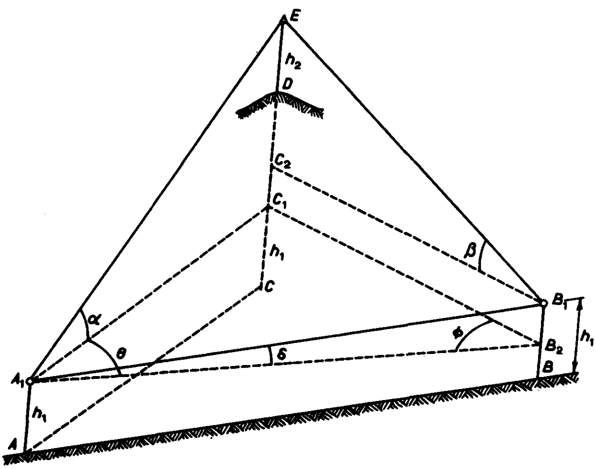


Fig. 2.32

Angles measured at A: horizontal angle θ
vertical angles α (to object)
δ (slope of ground)
at B: horizontal angle ϕ
vertical angle β

$$A_1B_2 = AB \cos\delta$$

Then
$$A_1C_1 = A_1B_2 \sin\phi \operatorname{cosec}(\theta + \phi)$$
$$EC_1 = A_1C_1 \tan\alpha$$
$$= \underline{AB \cos\delta \sin\phi \operatorname{cosec}(\theta + \phi) \tan\alpha} \qquad (2.51)$$

Height of object (E) above ground at A
$$\underline{EC = EC_1 + h_1}$$
$$B_1C_2 = B_2C_1 = A_1B_2 \sin\theta \operatorname{cosec}(\theta + \phi)$$
$$EC_2 = B_1C_2 \tan\beta$$

Similarly, height of object (E) above ground at A
$$EC = EC_2 + B_1B_2 + h_1$$
$$= \underline{AB \cos\delta \sin\theta \operatorname{cosec}(\theta + \phi) \tan\beta + AB \sin\delta + h_1} \qquad (2.52)$$

Height of ground (D) above ground at A
$$= EC_1 + h_1 - h_2$$
or
$$= EC_2 + h_1 - h_2 + AB \sin\delta$$

2.63 To find the height of an object above the ground when its base and top are visible but not accessible

(a) *Base line AB_1 horizontal and in line with object* (Fig. 2.23)
Vertical angles measured at A: α_1 and α_2
at B: β_1 and β_2

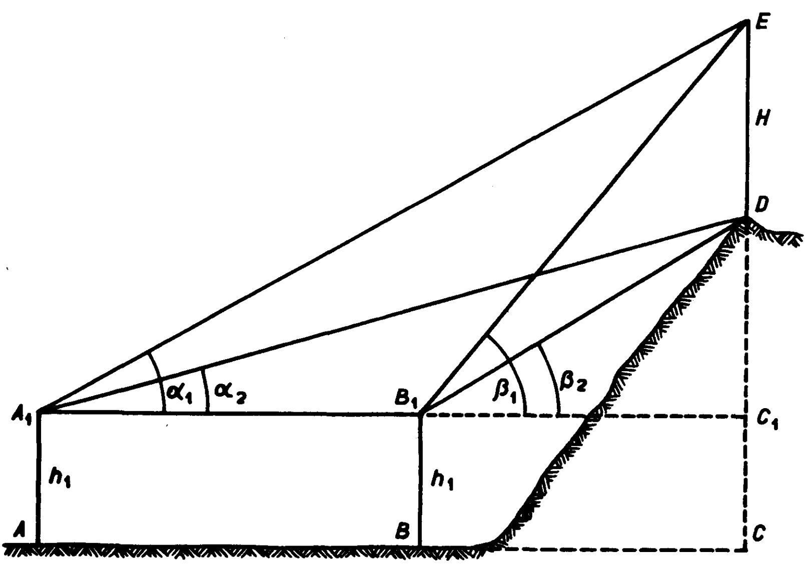


Fig. 2.33

From Eq. (2.46)

$$C_1E = \frac{AB}{\cot\alpha_1 - \cot\beta_1}$$

Also
$$C_1D = \frac{AB}{\cot\alpha_2 - \cot\beta_2}$$

Then $\quad ED = C_1E - C_1D = H$

$$\therefore \quad H = AB\left[\frac{1}{\cot\alpha_1 - \cot\beta_1} - \frac{1}{\cot\alpha_2 - \cot\beta_2}\right] \qquad (2.53)$$

(b) *Base line AB horizontal but not in line with object* (Fig. 2.34)

Angles measured at A: horizontal angle θ
vertical angles α_1 and α_2
at B: horizontal angle ϕ
vertical angles β_1 and β_2

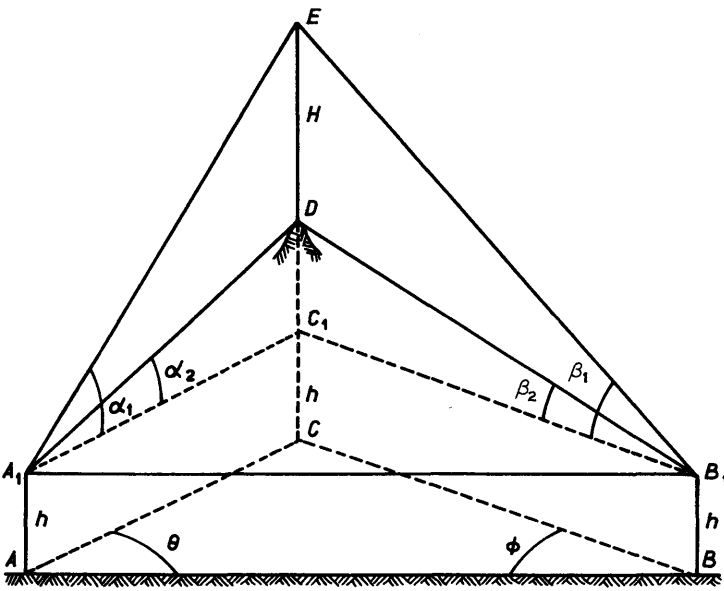


Fig. 2.34

$$AC = A_1C_1 = AB \sin\phi \operatorname{cosec}(\theta + \phi)$$
$$ED = H = AC(\tan\alpha_1 - \tan\alpha_2)$$
$$\therefore \quad H = AB \sin\phi \operatorname{cosec}(\theta + \phi)(\tan\alpha_1 - \tan\alpha_2) \qquad (2.54)$$

Similarly,
$$H = AB \sin\theta \operatorname{cosec}(\theta + \phi)(\tan\beta_1 - \tan\beta_2) \qquad (2.55)$$

(c) *Base line AB on sloping ground and in line with object* (Fig. 2.35)
Vertical angles measured at A: α_1, α_2 and δ
at B: β_1 and β_2

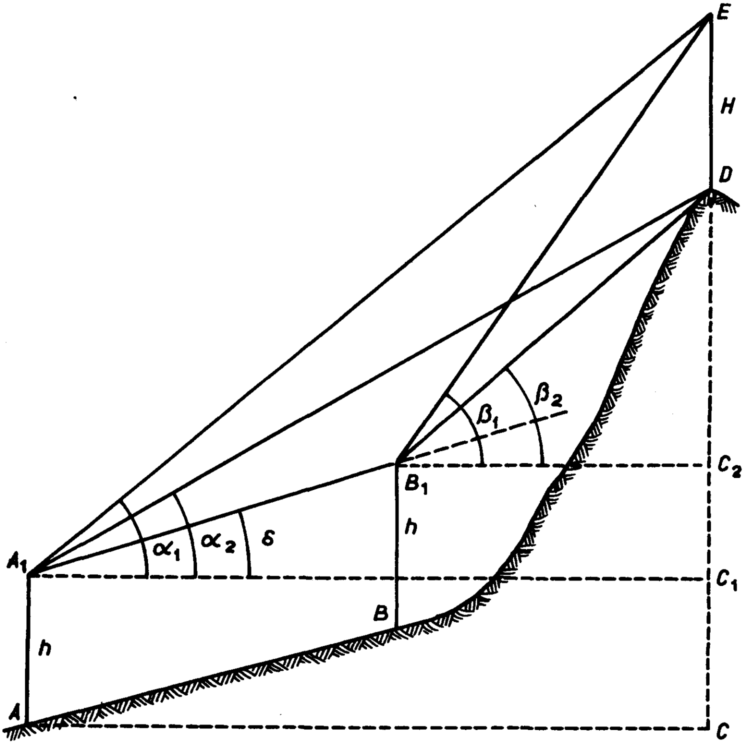


Fig. 2.35

From Eq. (2.50)

$$EC_1 = AB \sin(\beta_1 - \delta) \operatorname{cosec}(\beta_1 - \alpha_1) \sin \alpha_1$$

and $$DC_1 = AB \sin(\beta_2 - \delta) \operatorname{cosec}(\beta_2 - \alpha_2) \sin \alpha_2$$

Then $$ED = EC_1 - DC_1$$

$\therefore$ $$H = AB[\sin(\beta_1 - \delta) \operatorname{cosec}(\beta_1 - \alpha_1) \sin \alpha_1 - \sin(\beta_2 - \delta) \operatorname{cosec}(\beta_2 - \alpha_2) \sin \alpha_2] \quad (2.56)$$

(d) *Base line AB on sloping ground and not in line with object* (Fig. 2.36)

Angles measured at A: horizontal angle θ
vertical angles α_1 and α_2
δ (slope of ground)
at B: horizontal angle ϕ
vertical angles β_1 and β_2

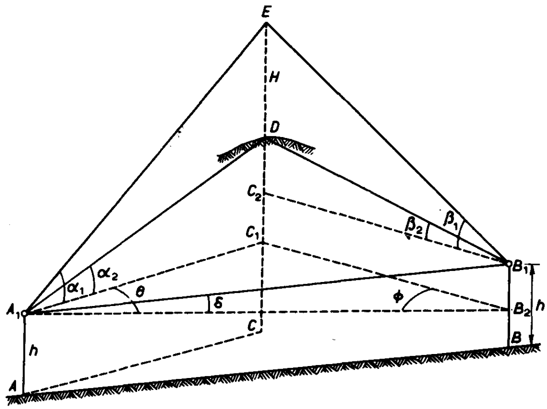


Fig. 2.36

From Eq. (2.51),

$$EC_1 = AB \cos\delta \sin\phi \operatorname{cosec}(\theta + \phi) \tan\alpha_1$$

$$DC_1 = AB \cos\delta \sin\phi \operatorname{cosec}(\theta + \phi) \tan\alpha_2$$

Then

$$ED = EC_1 - DC_1$$

$$= AB \cos\delta \sin\phi \operatorname{cosec}(\theta + \phi)(\tan\alpha_1 - \tan\alpha_2) \quad (2.57)$$

Also

$$ED = AB \cos\delta \sin\theta \operatorname{cosec}(\theta + \phi)(\tan\beta_1 - \tan\beta_2) \quad (2.58)$$

2.64 To find the length of an inclined object (e.g. an inclined flagstaff) on the top of a building (Fig. 2.37)

Base line AB is measured and if on sloping ground reduced to horizontal.

Angles measured at A: horizontal angles θ_1 and θ_2 to top and bottom of pole

vertical angles α_1 and α_2 to top and bottom of pole

at B: horizontal angles ϕ_1 and ϕ_2 to top and bottom of pole

vertical angles β_1 and β_2 to top and bottom of pole.

In plan the length ED is projected as E_1D_1 ($= E_2D$).

In elevation the length ED is projected as EE_2, i.e. the difference in height

Then

$$AE_1 = AB \sin\phi_1 \operatorname{cosec}(\theta_1 + \phi_1)$$

Also

$$AD_1 = AB \sin\phi_2 \operatorname{cosec}(\theta_2 + \phi_2)$$

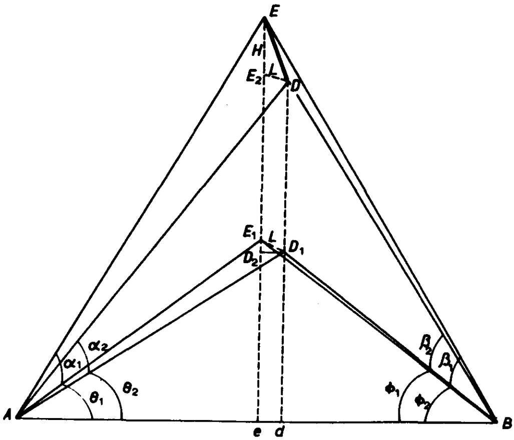


Fig. 2.37

The length $E_1D = L$ is now best calculated using co-ordinates (see Chapter 3)

Assuming bearing $AB = 180°00'$

$$Ae = AE_1 \sin(90 - \theta_1) = AE_1 \cos\theta_1$$

$$Ad = AD_1 \sin(90 - \theta_2) = AD_1 \cos\theta_2$$

Then

$$ed = D_2D_1 = Ad - Ae$$

$$= AD_1 \cos\theta_2 - AE_1 \cos\theta_1$$

and similarly, $E_1D_2 = E_1e - D_2e = AE_1 \sin\theta_1 - AD_1 \sin\theta_2$

In the triangle $E_1D_1D_2$.

The bearing of the direction of inclination (relative to AB) $= \tan^{-1} \dfrac{D_2D_1}{E_1D_2}$

Length $E_1D_1 = E_1D_2$ (sec bearing)

To find difference in height EE_2

Height of top above $A = AE_1 \tan\alpha_1$

Height of base above $A = AD_1 \tan\alpha_2$

Length $EE_2 = AE_1 \tan\alpha_1 - AD_1 \tan\alpha_2$

To find length of pole:

In triangle EDE_2,

$$ED^2 = EE_2^2 + E_2D^2$$

i.e.

$$ED = \sqrt{EE_2^2 + E_2D^2}$$

2.65 To find the height of an object from three angles of elevation only (Fig. 2.38)

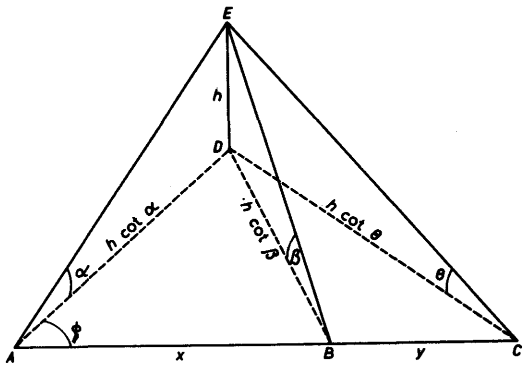


Fig. 2.38

Solving triangles *ADB* and *ADC* by the cosine rule,

$$\cos\phi = \frac{h^2\cot^2\alpha + x^2 - h^2\cot^2\beta}{2hx\cot\alpha}$$

$$= \frac{h^2\cot^2\alpha + (x+y)^2 - h^2\cot^2\theta}{2h(x+y)\cot\alpha} \qquad (2.59)$$

$$\therefore\quad (x+y)[h^2(\cot^2\alpha - \cot^2\beta) + x^2] = x[h^2(\cot^2\alpha - \cot^2\theta) + (x+y)^2]$$

i.e. $$h^2[(x+y)(\cot^2\alpha - \cot^2\beta) - x(\cot^2\alpha - \cot^2\theta)] = x(x+y)^2 - x^2(x+y)$$

i.e. $$h^2 = \frac{(x+y)[x(x+y) - x^2]}{(x+y)(\cot^2\alpha - \cot^2\beta) - x(\cot^2\alpha - \cot^2\theta)}$$

$$= \frac{(x+y)(xy)}{(x+y)(\cot^2\alpha - \cot^2\beta) - x(\cot^2\alpha - \cot^2\theta)}$$

$$h = \left[\frac{xy(x+y)}{x(\cot^2\theta - \cot^2\beta) + y(\cot^2\alpha - \cot^2\beta)}\right]^{\frac{1}{2}} \qquad (2.60)$$

If $x = y$,

$$h = \frac{\sqrt{2}x}{[\cot^2\theta - 2\cot^2\beta + \cot^2\alpha]^{\frac{1}{2}}} \qquad (2.61)$$

Example 2.12 A, B and C are stations on a straight level line of bearing 126° 03′ 34″. The distance AB is 523·54 ft and BC is 420·97 ft. With an instrument of constant height 4′ – 3″ vertical angles were successively measured to an inaccessible up-station D as follows:

At A	7° 14′ 00″
B	10° 15′ 20″
C	13° 12′ 30″

Calculate (a) the height of station D above the line ABC

(b) the bearing of the line AD

(c) the horizontal length AD.

(R.I.C.S.)

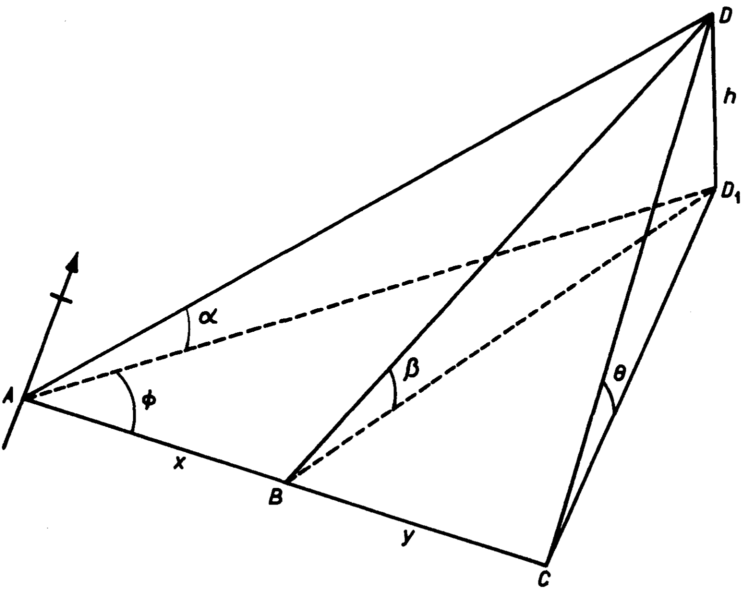


Fig. 2.39

(a) In Fig. 2.39,

$$AD = h \cot \alpha$$
$$BD = h \cot \beta$$
$$CD = h \cot \theta$$

Solving triangles AD_1B and AD_1C, using Eq. (2.60),

$$h = \left[\frac{xy(x+y)}{x(\cot^2\theta - \cot^2\beta) + y(\cot^2\alpha - \cot^2\beta)}\right]^{\frac{1}{2}}$$

$$= \left[\frac{(523{\cdot}54 \times 420{\cdot}97)(523{\cdot}54 + 420{\cdot}97)}{523{\cdot}54(\cot^2 13°\,12'\,30 - \cot^2 10°\,15'\,20'') + 420{\cdot}97(\cot^2 7°\,14'\,00'' - \cot^2 10°\,15'\,20''}\right]^{\frac{1}{2}}$$

$= 175{\cdot}16$ ft

$\therefore$ Difference in height of D above ground at A

$= 175{\cdot}16 + 4{\cdot}25 = \underline{179{\cdot}41\text{ ft}}$

Using Eq. (2.59),

$$\cos\phi = \frac{h^2(\cot^2\alpha - \cot^2\beta) + x^2}{2hx\cot\alpha}$$

$$= \frac{175{\cdot}16^2(\cot^2 7^\circ\,14'\,00 - \cot^2 10^\circ\,15'\,20'') + 523{\cdot}54^2}{2\times 175{\cdot}16\times 523{\cdot}54\times\cot 7^\circ\,14'\,00''}$$

$$= 0{\cdot}85909$$

$$\phi = 30^\circ\,47'\,10''$$

(b) Thus bearing of AD $= 126^\circ\,03'\,34' - 30^\circ\,47'\,10''$

$= \underline{095^\circ\,16'\,24''}$

(c) Length of line AD_1 $= h\cot\alpha$

$= 175{\cdot}16\cot 7^\circ\,14'$

$= \underline{1380{\cdot}07\text{ ft}}$

2.66 The broken base line problem

Where a base line AD cannot be measured due to some obstacle the following system may be adopted, Fig. 2.40.

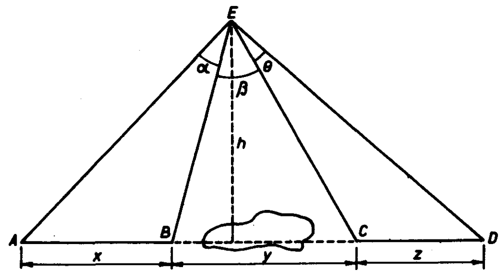


Fig. 2.40 Broken base line

Lengths x and z are measured.
Angles α, β and θ are measured at station E.
To calculate $BC = y$:

Method 1

In triangle AEB $\quad EB = \dfrac{x\sin EAB}{\sin\alpha}$

In triangle AEC $\quad EC = \dfrac{(x+y)\sin EAC}{\sin(\alpha+\beta)}$

Then $$\frac{EB}{EC} = \frac{x\sin(\alpha+\beta)}{(x+y)\sin\alpha} \tag{2.62}$$

Also in triangle EDB $$EB = \frac{(y+z)\sin EDB}{\sin(\theta+\beta)}$$

in triangle EDC $$EC = \frac{z\sin EDC}{\sin\theta}$$

Then $$\frac{EB}{EC} = \frac{(y+z)\sin\theta}{z\sin(\theta+\beta)} \tag{2.63}$$

Equating Eqs (2.62) and (2.63)

$$\frac{(y+z)\sin\theta}{z\sin(\theta+\beta)} = \frac{x\sin(\alpha+\beta)}{(x+y)\sin\alpha}$$

i.e. $$(x+y)(y+z) = \frac{xz\sin(\alpha+\beta)\sin(\theta+\beta)}{\sin\alpha\sin\theta}$$

Then $$y^2 + y(x+z) + xz\left[1 - \frac{\sin(\alpha+\beta)\sin(\theta+\beta)}{\sin\alpha\sin\theta}\right] = 0 \tag{2.64}$$

This is a quadratic equation in y. Thus

$$y = -\frac{x+z}{2} + \sqrt{\left(\frac{x+z}{2}\right)^2 - xz\left[1 - \frac{\sin(\alpha+\beta)\sin(\theta+\beta)}{\sin\alpha\sin\theta}\right]}$$

$$= -\frac{x+z}{2} + \sqrt{\left(\frac{x-z}{2}\right)^2 + xz\frac{\sin(\alpha+\beta)\sin(\theta+\beta)}{\sin\alpha\sin\theta}} \tag{2.65}$$

Method 2

Area of triangle $ABE = \frac{1}{2}xh = \frac{1}{2}AE\,.\,EB\sin\alpha$ (1)

" " " $BCE = \frac{1}{2}yh = \frac{1}{2}BE\,.\,EC\sin\beta$ (2)

" " " $CDE = \frac{1}{2}zh = \frac{1}{2}CE\,.\,ED\sin\theta$ (3)

" " " $ADE = \frac{1}{2}(x+y+z)h$

$= \frac{1}{2}AE\,.\,ED\sin(\alpha+\beta+\theta)$ (4)

Dividing (1) by (2) $$\frac{x}{y} = \frac{AE\sin\alpha}{EC\sin\beta} \tag{5}$$

Dividing (3) by (4), $$\frac{z}{x+y+z} = \frac{CE\sin\theta}{AE\sin(\alpha+\beta+\theta)} \tag{6}$$

Multiplying (5) by (6),

$$\frac{xz}{y(x+y+z)} = \frac{\sin\alpha\,\sin\theta}{\sin\beta\,\sin(\alpha+\beta+\theta)}$$

i.e. $$y^2 + y(x+z) - xz\frac{\sin\beta\,\sin(\alpha+\beta+\theta)}{\sin\alpha\,\sin\theta} = 0$$

Then $$y = -\left(\frac{x+z}{2}\right) + \sqrt{\left(\frac{x+z}{2}\right)^2 + xz\frac{\sin\beta\,\sin(\alpha+\beta+\theta)}{\sin\alpha\,\sin\theta}} \qquad (2.66)$$

Method 3 (Macaw's Method)

In order to provide a logarithmic solution an auxiliary angle is used.

From the quadratic equation previously formed,

$$y^2 + y(x+z) - xz\,\frac{\sin\beta\,\sin(\alpha+\beta+\theta)}{\sin\alpha\,\sin\theta} = 0. \qquad (2.67)$$

i.e. $$\{y + \tfrac{1}{2}(x+z)\}^2 = \tfrac{1}{4}(x+z)^2 + \frac{xz\,\sin\beta\,\sin(\alpha+\beta+\theta)}{\sin\alpha\,\sin\theta} \qquad (2.68)$$

Now let $$\tan^2 M = \frac{4xz\,\sin\beta\,\sin(\alpha+\beta+\theta)}{(x+z)^2\sin\alpha\,\sin\theta} \qquad (2.69)$$

Substituting this in Eq. (2.68), we get

$$\{y + \tfrac{1}{2}(x+z)\}^2 = \tfrac{1}{4}(x+z)^2(1 + \tan^2 M)$$

$$= \tfrac{1}{4}(x+z)^2\sec^2 M$$

$$\therefore \qquad y + \tfrac{1}{2}(x+z) = \tfrac{1}{2}(x+z)\sec M$$

$$y = \tfrac{1}{2}(x+z)(\sec M - 1)$$

$$= (x+z)\sec M\,\tfrac{1}{2}(1 - \cos M)$$

$$\underline{y = (x+z)\sec M\,\sin^2\tfrac{1}{2}M} \qquad (2.70)$$

Example 2.13 The measurement of a base line AD is interrupted by an obstacle. To overcome this difficulty two points B and C were established on the line AD and observations made to them from a station E as follows:

$$A\hat{E}B = 20°\,18'\,20''$$
$$B\hat{E}C = 45°\,19'\,40''$$
$$C\hat{E}D = 33°\,24'\,20''$$

Length $AB = 527{\cdot}43$ ft and $CD = 685{\cdot}29$ ft.
Calculate the length of the line AD.

(R.I.C.S.)

Here

$$\left.\begin{aligned}\alpha &= 20^\circ 18' 20'' \\ \beta &= 45^\circ 19' 40'' \\ \theta &= 33^\circ 24' 20''\end{aligned}\right\} \quad \begin{aligned}\alpha+\beta &= 65^\circ 38' 00'' \\ \alpha+\beta+\theta &= 99^\circ 02' 20'' \\ \beta+\theta &= 78^\circ 44' 00''\end{aligned}$$

$$\left.\begin{aligned}x &= 527{\cdot}43 \\ z &= 685{\cdot}29\end{aligned}\right\} \quad \begin{aligned}\tfrac{1}{2}(x+z) &= \tfrac{1}{2}(1212{\cdot}72) = 606{\cdot}36 \\ \tfrac{1}{2}(x \sim z) &= \tfrac{1}{2}(157{\cdot}86) = 78{\cdot}93\end{aligned}$$

By method 1

$$\begin{aligned} y &= -606{\cdot}36 + \sqrt{78{\cdot}93^2 + \frac{527{\cdot}43 \times 685{\cdot}29 \sin 65^\circ 38' \sin 78^\circ 44'}{\sin 20^\circ 18' 20'' \sin 33^\circ 24' 20''}} \\ &= -606{\cdot}36 + \sqrt{6230 + 1\,690\,057} \\ &= -606{\cdot}36 + 1302{\cdot}415 \\ &= 696{\cdot}055 \text{ ft} \end{aligned}$$

Then $AD = 1212{\cdot}72 + 696{\cdot}055 = \underline{1908{\cdot}775 \text{ ft}}$

By method 2

$$\begin{aligned} y &= -606{\cdot}36 + \sqrt{606{\cdot}36^2 + \frac{527{\cdot}43 \times 685{\cdot}29 \sin 45^\circ 19' 40'' \sin 99^\circ 02' 20''}{\sin 20^\circ 18' 20'' \sin 33^\circ 24' 20''}} \\ &= -606{\cdot}36 + \sqrt{367\,672 + 1\,328\,614} \\ &= -606{\cdot}36 + 1302{\cdot}415 \\ &= \underline{696{\cdot}055 \text{ ft}} \end{aligned}$$

By method 3

By logs	4	0·602 060 0
	x	2·722 164 8
	z	2·835 874 4
	$\sin\beta$	9·851 955 4
	$\sin(\alpha+\beta+\theta)$	9·994 573 1
	$\operatorname{cosec}\alpha$	10·459 637 2
	$\operatorname{cosec}\theta$	10·259 194 0
		6·725 458 9
	$(x+z)^2$	6·167 521 0
	$\tan^2 M$	0·557 937 9
	$\tan M$	0·278 968 9 $\rightarrow M = 62^\circ 15' 11''$
		$\tfrac{1}{2}M = 31^\circ 07' 36''$

$\sec M$	0·332 017 5
$\sin \frac{1}{2}M$	9·713 433 2
$\sin \frac{1}{2}M$	9·713 433 2
$(x + z)$	3·083 760 5
	2·842 644 4

$\therefore \qquad \underline{y = 696{\cdot}055}$

2.67 To find the relationship between angles in the horizontal and inclined planes (Fig. 2.41)

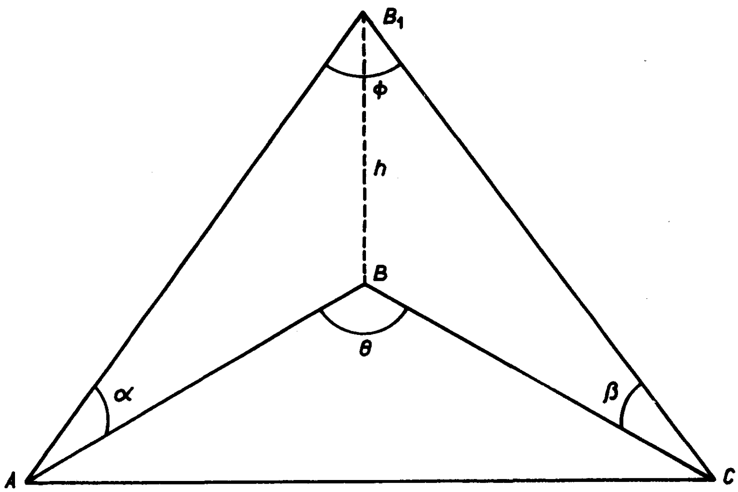


Fig. 2.41

Let (1) lines AB_1 and B_1C be inclined to the horizontal plane by α and β respectively.

(2) Horizontal angle $ABC = \theta$

(3) Angle in inclined plane $AB_1C = \phi$

(4) $B_1B = h$

Then $\qquad AB = h \cot\alpha \qquad AB_1 = h \operatorname{cosec}\alpha$

$\qquad BC = h \cot\beta \qquad B_1C = h \operatorname{cosec}\beta$

In triangle ABC,

$$AC^2 = AB^2 + BC^2 - 2AB.BC \cos\theta$$
$$= h^2 \cot^2\alpha + h^2 \cot^2\beta - 2h^2 \cot\alpha \cot\beta \cos\theta$$

Similarly in triangle AB_1C,

$$AC^2 = h^2 \operatorname{cosec}^2\alpha + h^2 \operatorname{cosec}^2\beta - 2h^2 \operatorname{cosec}\alpha \operatorname{cosec}\beta \cos\phi$$

Then

$$h^2\cot^2\alpha + h^2\cot^2\beta - 2h^2\cot\alpha\cot\beta\cos\theta = h^2\operatorname{cosec}^2\alpha + h^2\operatorname{cosec}^2\beta - 2h^2\operatorname{cosec}\alpha\operatorname{cosec}\beta\cos\phi$$

i.e.,

$$\cos\phi = \frac{(\operatorname{cosec}^2\alpha - \cot^2\alpha) + (\operatorname{cosec}^2\beta - \cot^2\beta) + 2\cot\alpha\cot\beta\cos\theta}{2\operatorname{cosec}\alpha\operatorname{cosec}\beta}$$

as $\operatorname{cosec}^2\alpha - \cot^2\alpha = \operatorname{cosec}^2\beta - \cot^2\beta = 1$

Then

$$\cos\phi = \frac{2(1 + \cot\alpha\cot\beta\cos\theta)}{2\operatorname{cosec}\alpha\operatorname{cosec}\beta}$$

$$= \sin\alpha\sin\beta + \cos\alpha\cos\beta\cos\theta \tag{2.71}$$

or

$$\cos\theta = \frac{\cos\phi - \sin\alpha\sin\beta}{\cos\alpha\cos\beta} \tag{2.72}$$

Example 2.14 From a station A observations were made to stations B and C with a sextant and an abney level.

With sextant – angle $BAC = 84°30'$

With abney level – angle of depression (AB) $8°20'$

angle of elevation (AC) $10°40'$

Calculate the horizontal angle BA_1C which would have been measured if a theodolite had been used

(R.I.C.S./M)

From equation (2.72),

$$\cos\theta = \frac{\cos\phi - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$

$$= \frac{\cos 84°30' - \sin(-8°20')\sin 10°40'}{\cos(-8°20')\cos 10°40'}$$

$$= \frac{\cos 84°30' + \sin 8°20'\sin 10°40'}{\cos 8°20'\cos 10°40'}$$

$$= \frac{0{\cdot}09585 + 0{\cdot}14493 \times 0{\cdot}18509}{0{\cdot}98944 \times 0{\cdot}98272}$$

$$= \frac{0{\cdot}12268}{0{\cdot}97234}$$

$$= 0{\cdot}12617$$

$$\underline{\theta = 82°45'10''}$$

Example 2.15 A pipe-line is to be laid along a bend in a mine roadway ABC. If AB falls at a gradient of 1 in 2 in a direction 036° 27', whilst BC rises due South at 1 in 3·5, calculate the angle of bend in the pipe.

(R.I.C.S.)

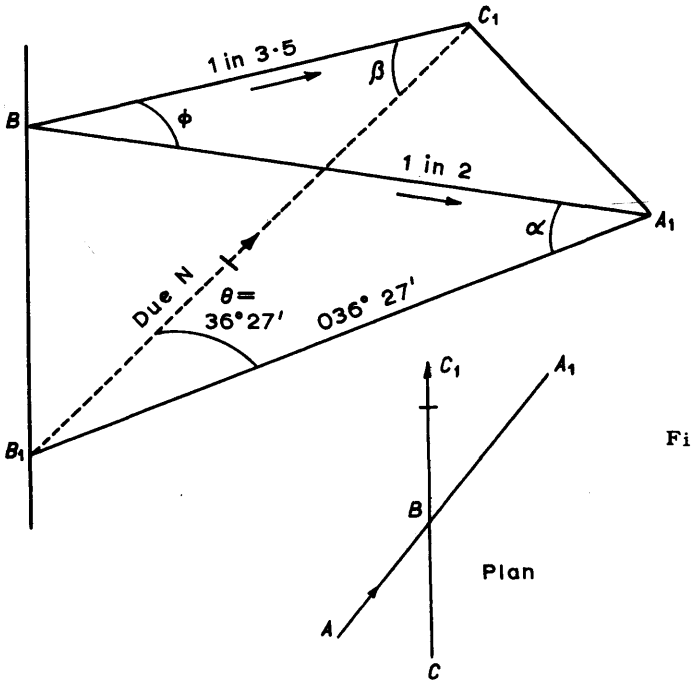


Fig. 2.42

From equation (2.71),

$$\cos\phi = \sin\alpha \sin\beta + \cos\alpha \cos\beta \cos\theta$$

where

$$\alpha = \cot^{-1}2 = 26°\,33'$$

$$\beta = \cot^{-1}3{\cdot}5 = 15°\,57'$$

$$\theta = 036°\,27' - 00° = 36°\,27'$$

$$\therefore \quad \cos\phi = \sin 26°\,33' \sin 15°\,57' + \cos 26°\,33' \cos 15°\,57' \cos 36°\,27'$$

$$\phi = 35°\,26'\,40'' \qquad \text{i.e.} \quad 35°\,27'$$

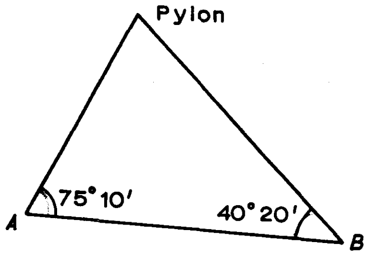
Exercises 2(b)

8. Show that for small angles of slope the difference between horizontal and sloping lengths is $h^2/2l$ (where h is the difference of vertical height of the two ends of a line of sloping length l)

If errors in chaining are not to exceed 1 part in 1000, what is the greatest slope that can be ignored?

[L.U/E Ans. 2° 34′]

9. The height of an electricity pylon relative to two stations A and B (at the same level) is to be calculated from the data given below. Find the height from the two stations if at both stations the height of the theodolite axis is $5'-0''$.



Data: $AB = 200$ ft

Horizontal angle	at A	=	75° 10′
	at B	=	40° 20′
Vertical angle	at A	=	43° 12′
	at B	=	32° 13′

(R.I.C.S. Ans. mean height 139·8 ft)

10. X, Y and Z are three points on a straight survey line such that $XY = 56$ ft and $YZ = 80$ ft.

From X, a normal offset was measured to a point A and XA was found to be 42ft. From Y and Z respectively, a pair of oblique offsets were measured to a point B, and these distances were as follows:

$$YB = 96 \text{ ft}, \qquad ZB = 88 \text{ ft}$$

Calculate the distance AB, and check your answer by plotting to some suitable scale, and state the scale used.

(E.M.E.U. Ans. 112·7 ft)

11. From the top of a tower 120 ft high, the angle of depression of a point A is 15°, and of another point B is 11°. The bearings of A and B from the tower are 205° and 137° respectively. If A and B lie in a horizontal plane through the base of the tower, calculate the distance AB.

(R.I.C.S. Ans. 612 ft)

12 A, B, C, D are four successive milestones on a straight horizontal road.

From a point O due W of A, the direction of B is 84°, and of D is 77°. The milestone C cannot be seen from O, owing to trees. If the direction in which the road runs from A to D is θ, calculate θ, and the distance of O from the road.

(R.I.C.S. Ans. $\theta = 60° 06' 50''$, $OA = 3·8738$ miles)

13. At a point A, a man observes the elevation of the top of a tower B to be 42° 15′. He walks 200 yards up a uniform slope of elevation 12° directly towards the tower, and then finds that the elevation of B has increased by 23° 09′. Calculate the height of B above the level of A.

(R.I.C.S. Ans. 823·82 ft)

14. At two points, 500 yards apart on a horizontal plane, observations of the bearing and elevation of an aeroplane are taken simultaneously. At one point the bearing is 041° and the elevation is 24°, and at another point the bearing is 032° and the elevation is 16°. Calculate the height of the aeroplane above the plane.

(R.I.C.S. Ans. 1139 ft)

15. Three survey stations X, Y and Z lie in one straight line on the same plane. A series of angles of elevation is taken to the top of a colliery chimney, which lies to one side of the line XYZ. The angles measured at X, Y and Z were:

at X, $14°02'$; at Y, $26°34'$; at Z, $18°26'$

The lengths XY and YZ are 400 ft and 240 ft respectively.

Calculate the height of the chimney above station X.

(E.M.E.U. Ans. 112·0 ft)

16. The altitude of a mountain, observed at the end A of a base line AB of 2992·5 m, was $19°42'$ and the horizontal angles at A and B were $127°54'$ and $33°09'$ respectively.

Find the height of the mountain.

(Ans. 1804 m)

17. It is required to determine the distance between two inaccessible points A and B by observations from two stations C and D, 1000 m apart. The angular measurements give $ACB = 47°$, $BCD = 58°$, $BDA = 49°$; $ADC = 59°$.

Calculate the distance AB

(Ans. 2907·4 m)

18. An aeroplane is observed simultaneously from two points A and B at the same level, A being a distance (c) due north of B. From A the aeroplane is S $\theta°$ E and from B N ϕ E.

Show that the height of the aeroplane is

$$\frac{c \tan\alpha \sin\phi}{\sin(\theta + \phi)}$$

and find its elevation from B.

(L.U. Ans. $\beta = \tan^{-1} \dfrac{\sin\phi \tan\alpha}{\sin\theta}$)

19. A straight base line $ABCD$ is sited such that a portion of BC cannot be measured directly. If AB is 575·64 ft and CD is 728·56 ft and the angles measured from station O to one side of $ABCD$ are

$$DOC = 56°40'30''$$
$$COB = 40°32'00''$$
$$BOA = 35°56'30''$$

Calculate the length BC.

(E.M.E.U. Ans. 259·32 ft)

20. It is proposed to lay a line of pipes of large diameter along a roadway of which the gradient changes from a rise of 30° to a fall of 10° coincident with a bend in the roadway from a bearing of N 22° W to N 25° E.

Calculate the angle of bend in the pipe.

(Ans. $119°39'30''$)

21. At a point A at the bottom of a hill, the elevation of the top of a tower on the hill is $51° 18'$. At a point B on the side of the hill, and in the same vertical plane as A and the tower, the elevation is $71°40'$. AB makes an angle $20°$ with the horizontal and the distance $AB = 52$ feet. Determine the height of the top of the tower above A.

(L.U. Ans. 91·5 ft)

22. Two points, A, B on a straight horizontal road are at a distance 400 feet apart. A vertical flag-pole, 100 feet high, is at equal distances from A and B. The angle subtended by AB at the foot C of the pole (which is in the same horizontal plane as the road) is $80°$.

Find (i) the distance from the road to the foot of the pole.

(ii) the angle subtended by AB at the top of the pole.

(L.U. Ans. (i) 258·5 ft, (ii) $75° 28'$)

Bibliography

USILL, G.W. and HEARN, G., *Practical Surveying* (Technical Press).
HADDOCK, M.H., *The Basis of Mine Surveying* (Chapman and Hall).
LONEY, S.C., *Plane Trigonometry* (Cambridge University Press).
BLAKEY, J., *Intermediate Pure Mathematics* (Macmillan).

3 CO-ORDINATES

A point in a plane may be defined by two systems:

(1) Polar co-ordinates.

(2) Rectangular or Cartesian co-ordinates.

3.1 Polar Co-ordinates

This system involves angular and linear values, i.e. bearing and length, the former being plotted by protractor as an angle from the meridian.

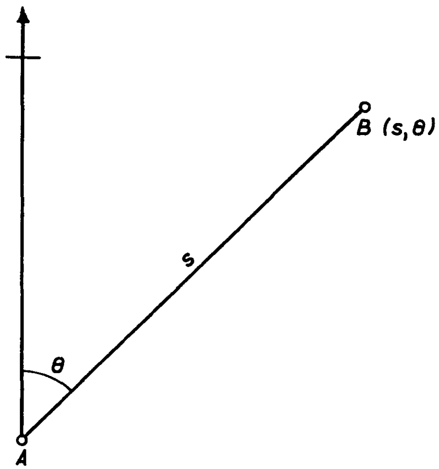


Fig. 3.1 Polar co-ordinates

A normal 6 inch protractor allows plotting to the nearest $1/4°$; a cardboard protractor with parallel rule to $1/8°$; whilst the special Bocking protractor enables $01'$ to be plotted.

The displacement of the point being plotted depends on the physical length of the line on the plan, which in turn depends on the horizontal projection of the ground length and the scale of the plotting.

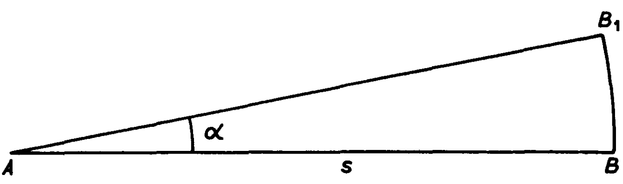


Fig. 3.2 Displacement due to angular error

If α is the angular error, then the displacement

$$BB_1 = s \tan \alpha$$

and as α is small

$$BB_1 \simeq s\alpha$$

If s = 300 ft and $\alpha = 01' \, 00''$,

$$BB_1 = \frac{300 \times 60 \times 12}{206\,265} \text{ inches}$$

$$\simeq 1 \text{ inch}$$

i.e. 1 minute of arc subtends 1 inch in 100 yards,

1 second of arc subtends 1 inch in 6000 yards, i.e. 3½ miles.

Similarly, on the metric system, if $s = 100$ metres and $\alpha = 01' \, 00''$,

then
$$BB_1 = \frac{100 \times 60}{206\,265} \text{ metres}$$

$$= 0{\cdot}029\,1 \text{ m, i.e. } 29 \text{ mm}$$

Thus, 1 minute of arc subtends approximately 30 mm in 100 m,

1 second of arc subtends approximately 1 mm in 200 m

or 1 cm in 2 km.

A point plotted on a plan may be assumed to be 0·01 in., (0·25 mm),

i.e. 0·01 in. in 1 yard (0·25 mm in 1 metre) → 1 minute of arc,

0·1 in. in 1 yard (25 mm in 1 metre) → 10 minutes of arc.

As this represents a possible plotting error on every line, it can be seen how the error may accumulate, particularly as each point is dependent on the preceding point.

3.11 Plotting to scale

The length of the plotted line is some definite fraction of the ground length, the 'scale' chosen depending on the purpose of the plan and the size of the area.

Scales may be expressed in various ways:

(1) As inches (in plan) per mile, e.g. 6 in. to 1 mile.

(2) As feet, or chains, per inch, e.g. 10 ft to 1 inch.

(3) As a representative fraction 1 in n, i.e. $1/n$, e.g. 1/2500.

3.12 Conversion of the scales

40 inches to 1 mile – 1 inch represents $\frac{1760 \times 3}{40}$ feet

$$1 \text{ in.} = 132 \text{ ft}$$
$$= 44 \text{ yd}$$
$$= 2 \text{ chn}$$

1 inch to 132 ft – 1 inch represents 132×12 inches

1 in. = 1584 in.

Thus the representative fraction is 1/1584.

3.13 Scales in common use

Ordinance Survey Maps and Plans:

Large scale: 1/500, 1/1250, 1/2500.

Medium scale: 6 in. to 1 mile (1/10 560), 2½ in. to 1 mile (1/25 000).

Small scale: 2, 1, ½, ¼ in. to 1 mile; 1/625 000, 1/1 250 000.

Engineering and Construction Surveys:

1/500, 1/2500, 10-50 ft to 1 inch, 1/4, 1/8, 1/16 in. to 1 ft.
(See Appendix, p.169)

3.14 Plotting accuracy

Considering 0·01 in. (0·25 mm) as the size of a plotted point, the following table shows the representative value at the typical scales.

O.S. Scales				suggested measurement precision limit
1/500	0·01 × 500	=	5 in.	3 in. (76 mm)
1/1250	0·01 × 1250	=	12·5 in.	1 ft (0·3 m)
1/2500	0·01 × 2500	=	25·0 in.	2 ft (0·6 m)
1/10 560	0·01 × 10 560	=	105·6 in.	5 ft (1·5 m)
1/25 000	0·01 × 25 000	=	250·0 in.	10 ft (3·0 m)
Engineering Scales				
1 in. to 10 ft	0·01 × 120	=	1·2 in.	1 in.
1 in. to 50 ft	0·01 × 600	=	6·0 in.	6 in.
1 in. to 1 chn	0·01 × 792	=	7·92 in.	6 in. or ½ link
1 in. to 2 chn	0·01 × 1584	=	15·84 in.	1 ft or 1 link.

3.15 Incorrect scale problems

If a scale of 1/2500 is used on a plan plotted to scale 1/1584 what conversion factor is required to

(a) the scaled lengths,

(b) the area computed from the scaled length?

(a) On the plan 1 in. = 1584 in. whereas the scaled value shows 1 in. = 2500 in.

All scaled values must be converted by a factor 1584/2500 = <u>0·6336.</u>

(b) All the computed areas must be multiplied by $(0{\cdot}6336)^2 =$ <u>0·4014</u>

3.2 Bearings

Four meridians may be used, Fig. 3.3:

1. True or geographical north.
2. Magnetic north.
3. Grid north.
4. Arbitrary north.

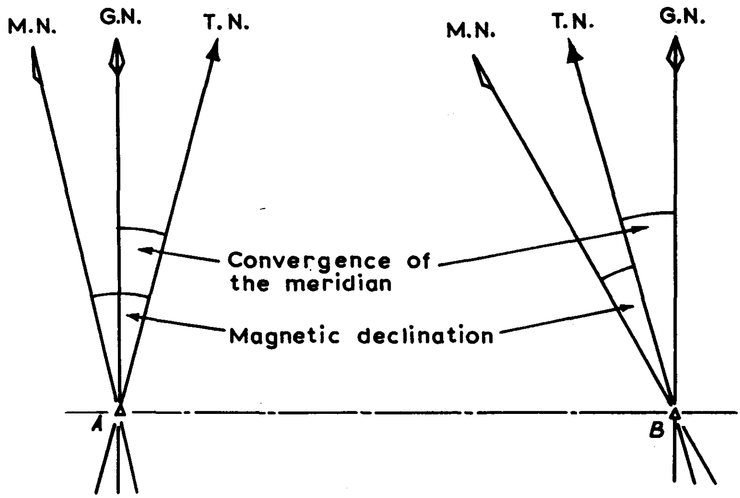


Fig. 3.3 Meridians

3.21 True north

The meridian can only be obtained precisely by astronomical observation. The difference between true bearings at A and B is the convergence of the meridians to a point, i.e. the north pole. For small surveys the discrepancy is small and can be neglected but where necessary a correction may be computed and applied.

3.22 Magnetic north

There is no fixed point and thus the meridian is unstable and subjected to a number of variations (Fig. 3.4), viz.:

(a) *Secular variation* – the annual change in the magnetic declination or angle between magnetic and true north. At present the magnetic meridian in Britain is to the west of true north but moving towards it at the approximate rate of 10 min per annum. (Values of declination and

the annual change are shown on certain O.S. sheets.)

(b) *Diurnal variation* – a daily sinusoidal oscillation effect, with the mean value at approximately 10 a.m. and 6-7 p.m., and maxima and minima at approximately 8 a.m. and 1 p.m.

(c) *Irregular variation* – periodic magnetic fluctuations thought to be related to sun spots.

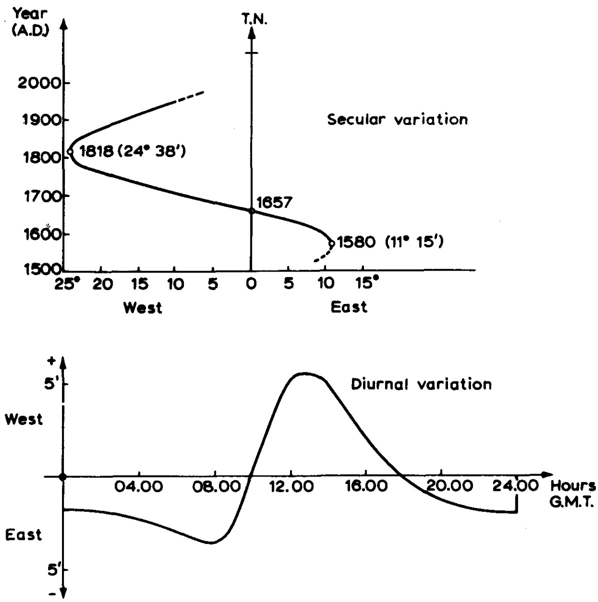


Fig. 3.4 Approximate secular and diurnal variations in magnetic declination in the London area (Abinger Observatory)

3.23 Grid north (see section 3.7).

O.S. sheets are based on a modified Transverse Mercator projection which, within narrow limits, allows:

(a) Constant bearings related to a parallel grid.

(b) A scale factor for conversion of ground distance to grid distance solely dependent on the easterly co-ordinates of the measurement site. (See page 39).

3.24 Arbitrary north

This may not be necessary for absolute reference and often the first leg of the traverse is assumed to be $0°00'$.

Example 3.1 True north is $0°37'$E of Grid North.
Magnetic declination in June 1955 was $10°27'$ W.

If the annual variation was $10'$ per annum towards North and the grid bearing of line AB $082°32'$, what will be the magnetic bearing of line AB in January 1966?

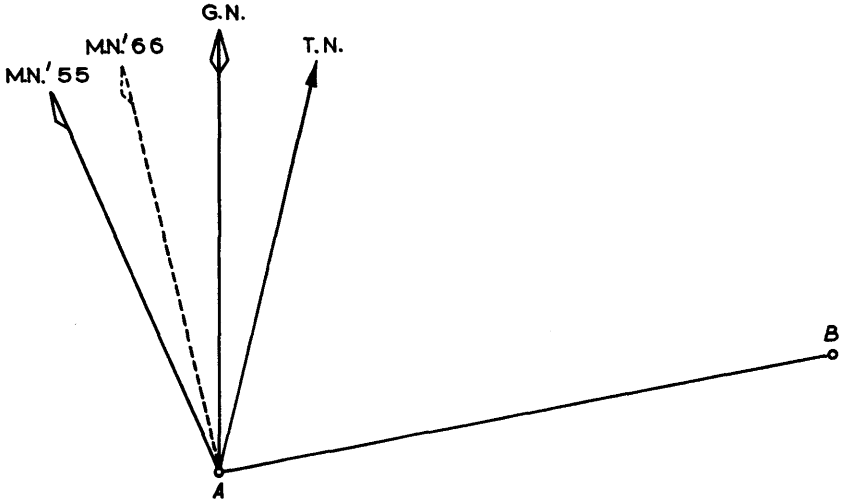


Fig. 3.5

Grid bearing	$082°32'$
Correction	$-0°37'$
True bearing	$081°55'$
Mag declination June 1955	$10°27'$
Mag. bearing June 1955	$092°22'$
Variation for January 1966 $-10\frac{1}{2} \times 10'$	$-1°45'$
Mag. bearing January 1966	$090°37'$

3.25 Types of bearing

There are two types in general use:

(a) Whole circle bearings (W.C.B.), which are measured clockwise from north or $0° - 360°$.

(b) Quadrant bearings (Q.B.), which are angles measured to the east or west of the N/S meridian.

For comparison of bearings, see Fig. 3.6.

Case (i)

Case (ii)

Case (iii)

Case (iv)

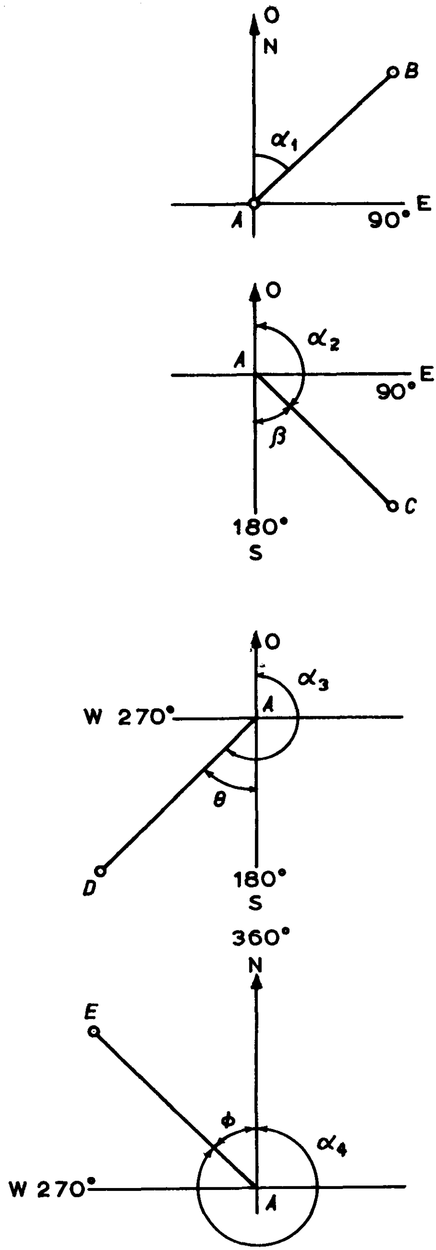


Fig. 3.6 Comparison of bearings

Case (i) Whole circle bearing in the first quadrant $0-90°$

$$\text{W.C.B. of } AB = \alpha_1^\circ$$
$$\text{Q.B. of } AB = \text{N } \alpha_1^\circ \text{ E}$$

Case (ii) $90°-180°$

$$\text{W.C.B. of } AC = \alpha_2^\circ$$
$$\text{Q.B. of } AC = \text{S } \beta^\circ \text{ E} = \text{S } (180-\alpha_2)^\circ \text{ E}$$

Case (iii) $180°-270°$

$$\text{W.C.B. of } AD = \alpha_3^\circ$$
$$\text{Q.B. of } AD = \text{S } \theta^\circ \text{ W} = \text{S } (\alpha_3-180)^\circ \text{ W}$$

Case (iv) $270°-360°$

$$\text{W.C.B. of } AE = \alpha_4^\circ$$
$$\text{Q.B. of } AE = \text{N } \phi \text{ W} = \text{N } (360-\alpha_4)^\circ \text{ W}$$

Example 3.2

072° = N 72° E		
148° = S 32° E	i.e.	180 – 148 = 32°
196° = S 16° W	i.e.	196 – 180 = 16°
330° = N 30° W	i.e.	360 – 330 = 30°

N.B. Quadrant bearings are never from the E/W line, so that the prefix is always N or S.

It is preferable to use *whole circle bearings* for most purposes, the only advantage of quadrant bearings being that they agree with the values required for trigonometrical functions $0-90°$ as given in many mathematical tables (see Chapter 2), e.g.:

(Fig. 3.7a)
$$\sin 30° = 0{\cdot}5$$
$$\cos 30° = 0{\cdot}8660$$
$$\tan 30° = 0{\cdot}5774$$

(Fig. 3.7b)
$$\sin 150° = \sin(180-150) = \underline{\sin 30°}$$
$$\cos 150° = -\cos(180-150) = \underline{-\cos 30°}$$
$$\tan 150° = \frac{\sin 150}{\cos 150} = \frac{+\sin 30}{-\cos 30} = -\tan 30°$$

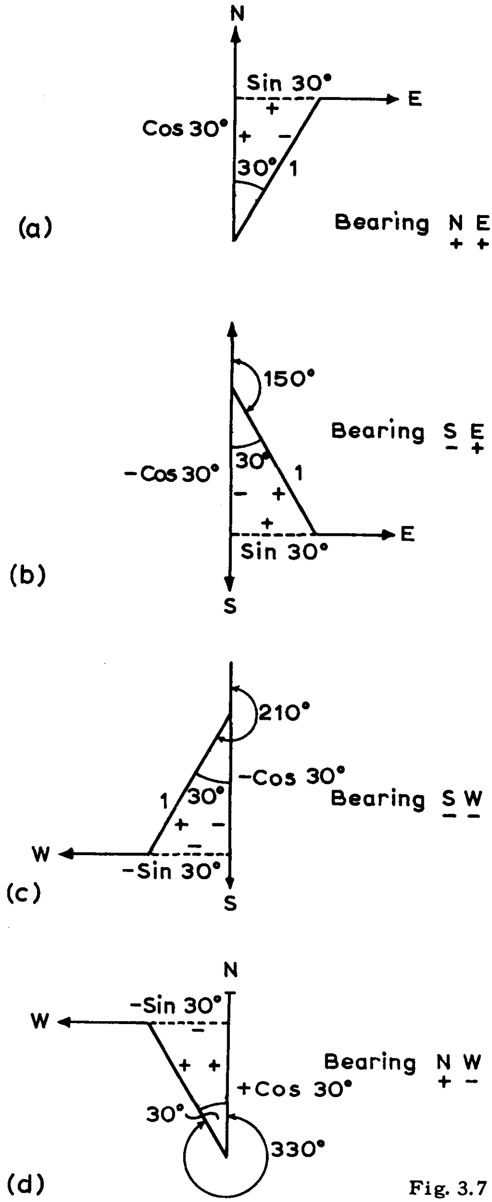


Fig. 3.7

(Fig. 3.7c) $\quad \sin 210^\circ = -\sin(210-180)$

$$= \underline{-\sin 30^\circ}$$

$$\cos 210^\circ = -\cos(210-180)$$

$$= \underline{-\cos 30^\circ}$$

$$\tan 210^\circ = \frac{\sin 210}{\cos 210} = \frac{-\sin 30}{-\cos 30}$$
$$= \underline{+\tan 30^\circ}$$

(Fig. 3.7d)
$$\sin 330^\circ = -\sin(360 - 330)$$
$$= \underline{-\sin 30^\circ}$$

$$\cos 330^\circ = \cos(360 - 330)$$
$$= \underline{+\cos 30^\circ}$$

$$\tan 330^\circ = \frac{\sin 330}{\cos 330} = \frac{-\sin 30}{+\cos 30}$$
$$= \underline{-\tan 30^\circ}$$

3.26 Conversion of horizontal angles into bearings. (Fig. 3.8)

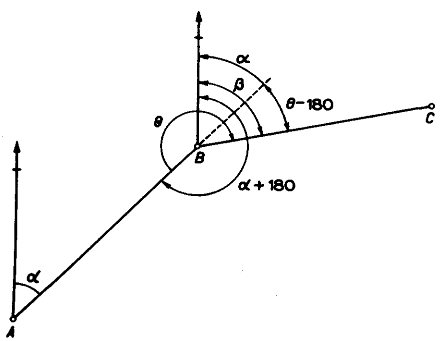


Fig. 3.8 Conversion of horizontal angles into bearings

Forward Bearing $AB = \alpha^\circ$

Back Bearing $BA = \alpha \pm 180^\circ$

Forward Bearing $BC = \alpha \pm 180 + \theta$

If the sum exceeds 360° then 360 is subtracted, i.e.

Bearing $BC(\beta) = \alpha \pm 180 + \theta - 360 = \alpha + \theta \pm 180$

This basic process may always be used but the following rules simplify the process.

(1) To the forward bearing *add* the clockwise angle.

(2) If the sum is less than 180° *add* 180°.
If the sum is more than 180° *subtract* 180°.
(In some cases the sum may be more than 540°, then subtract 540°.)

N.B. If the angles measured are anticlockwise they must be subtracted.

Example 3.3

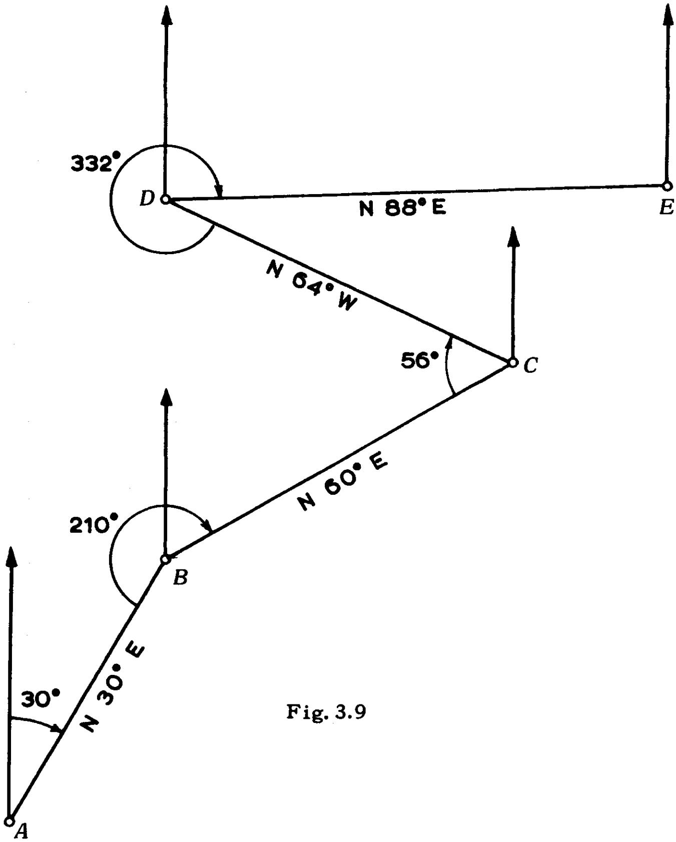


Fig. 3.9

Let	bearing AB	=	030°	N 30° E
	+ angle ABC	=	210°	
			240°	
		–	180°	
	bearing BC	=	060°	N 60° E
	+ angle BCD	=	56°	
			116°	
		+	180°	
	bearing CD	=	296°	N 64° W
	+ angle CDE	=	332°	
			628°	
		–	540°	
	bearing DE	=	088°	N 88° E

Check

$$
\begin{array}{rrl}
 & \text{bearing } AB = & 030^\circ \\
 & \text{angles} = & 210^\circ \\
 & & 56^\circ \\
 & & \underline{332^\circ} \\
 & & 628^\circ \\
-n \times 180^\circ, \text{ i.e. } -3 \times 180^\circ & - & \underline{540^\circ} \\
 & \text{bearing } DE = & 088^\circ
\end{array}
$$

The final bearing is checked by adding the bearing of the first line to the sum of the clockwise angles, and then subtracting some multiple of 180°.

Example 3.4 The clockwise angles of a closed polygon are observed to be as follows:

A	223°46′
B	241°17′
C	257°02′
D	250°21′
E	242°19′
F	225°15′

If the true bearings of *BC* and *CD* are 123°14′ and 200°16′ respectively, and the magnetic bearing of *EF* is 333°21′, calculate the magnetic declination.

(N.R.C.T.)

From the size of the angles it may be initially assumed that these are external to the polygon and should sum to $(2n+4)\,90^\circ$; i.e.

$$\{(2 \times 6) + 4\}90 = 16 \times 90 = 1440^\circ$$

$$
\begin{array}{rl}
 & 223^\circ 46' \\
 & 241^\circ 17' \\
 & 257^\circ 02' \\
 & 250^\circ 21' \\
 & 242^\circ 19' \\
 & \underline{225^\circ 15'} \\
\textit{Check} & \underline{1440^\circ 00'}
\end{array}
$$

To obtain the bearings,

$$
\begin{array}{rl}
\text{Line } BC \text{ bearing} & 123^\circ 14' \\
+\text{angle } BCD & \underline{257^\circ 02'} \\
 & 380^\circ 16' \\
- & \underline{180^\circ}
\end{array}
$$

bearing	CD	$200°16'$	(this checks with given value)
+angle	CDE	$250°21'$	
		$450°37'$	
	−	$180°$	
bearing	DE	$270°37'$	
+angle	DEF	$242°19'$	
		$512°56'$	
	−	$180°$	
bearing	EF	$332°56'$	
+angle	EFA	$225°15'$	
		$558°11'$	
	−	$540°$	
bearing	FA	$018°11'$	
+angle	FAB	$223°46'$	
		$241°57'$	
	−	$180°$	
bearing	AB	$061°57'$	
+angle	ABC	$241°17'$	
		$303°14'$	
	−	$180°$	
bearing	BC	$123°14'$	*Check*
Magnetic bearing	EF	$333°21'$	
True bearing	EF	$332°56'$	
Magnetic declination		$0°25'$ W	

3.27 Deflection angles (Fig. 3.10)

In isolated cases, deflection angles are measured and here the normal notation will be taken as:

Right angle deflection – positive.

Left angle deflection – negative.

Taking the Example 3.3,

Bearing	AB	$030°$
Deflection right		$+30°$
Deflection left		$-124°$
Deflection right		$+152°$

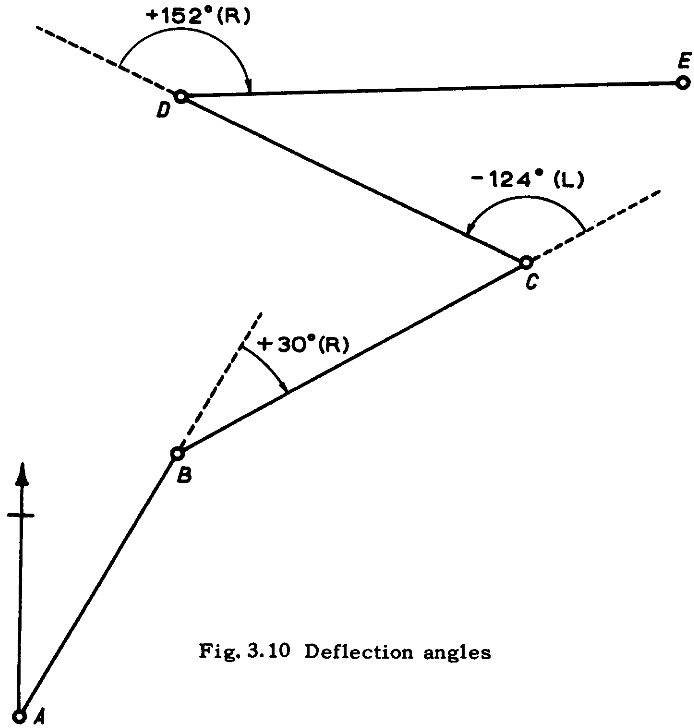


Fig. 3.10 Deflection angles

Bearing	*AB*	030°
		+ 30°
Bearing	*BC*	060°
		+ 360°
		420°
		− 124°
Bearing	*CD*	296°
		+ 152°
		448°
		− 360°
Bearing	*DE*	088°
Check	*AB*	030°
		+ 30°
		+ 152°
		+ 212°
		− 124°
	DE	088°

Exercises 3(a)

1. Convert the following whole circle bearings into quadrant bearings:

214°30′; 027°15′; 287°45′; 093°30′; 157°30′;
311°45′; 218°30′; 078°45′; 244°14′; 278°04′.

(Ans. S 34°30′ W; N 27°15′ E; N 72°15′ W; S 86°30′ E; S 22°30′ E; N 48°15′ W; S 38°30′ W; N 78°45′ E; S 64°14′ W; N 81°56′ W)

2. Convert the following quadrant bearings into whole circle bearings:

N 25°30′ E; S 34°15′ E; S 42°45′ W; N 79°30′ W;
S 18°15′ W; N 82°45′ W; S 64°14′ E; S 34°30′ W.

(Ans. 025°30′; 145°45′; 222°45′; 280°30′; 198°15′; 277°15′; 115°46′; 214°30′)

3. The following clockwise angles were measured in a closed traverse. What is the angular closing error?

163°27′36″; 324°18′22″; 62°39′27″; 330°19′18″;
181°09′15″; 305°58′16″; 188°02′03″; 292°53′02″;
131°12′50″ (Ans. 09″)

4. Measurement of the interior anticlockwise angles of a closed traverse *ABCDE* have been made with a vernier theodolite reading to 20 seconds of arc. Adjust the measurements and compute the bearings of the sides if the bearing of the line *AB* is N 43°10′20″ E.

Angle	*EAB*	135°20′40″	(R.I.C.S. Ans.	*AB*	N 43°10′20″ E
	ABC	60°21′20″		*BC*	S 17°10′52″ E
	BCD	142°36′20″		*CD*	S 20°12′56″ W
	CDE	89°51′40″		*DE*	N 69°38′36″ W
	DEA	111°50′40″		*EA*	N 01°29′08″ W)

5. From the theodolite readings given below, determine the angles of a traverse *ABCDE*. Having obtained the angles, correct them to the nearest 10 seconds of arc and then determine the bearing of *BC* if the bearing of *AB* is 45°20′40″.

Back Station	Theodolite Station	Forward Station	Readings Back Station	Readings Forward Station
E	*A*	*B*	0°00′00″	264°49′40″
A	*B*	*C*	264°49′40″	164°29′10″
B	*C*	*D*	164°29′10″	43°58′30″
C	*D*	*E*	43°58′30″	314°18′20″
D	*E*	*A*	314°18′20″	179°59′10″

(R.I.C.S. Ans. 125°00′20″)

3.3 Rectangular Co-ordinates

A point may be fixed in a plane by linear values measured parallel to the normal xy axes.

The x values are known as Departures or *Eastings* whilst the y values are known as Latitudes or *Northings*.

The following sign convention is used:

Direction

East	$+x$	$\rightarrow$	+departure	$\rightarrow$	+Easting	(+E)
West	$-x$	$\rightarrow$	−departure	$\rightarrow$	−Easting	(−E)
North	$+y$	$\rightarrow$	+latitude	$\rightarrow$	+Northing	(+N)
South	$-y$	$\rightarrow$	−latitude	$\rightarrow$	−Northing	(−N)

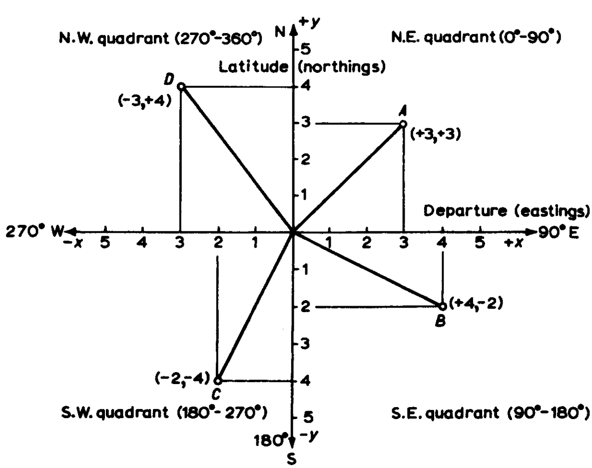


Fig. 3.11 Rectangular co-ordinates

N.B.	0°–90°	$\rightarrow$	N E	i.e.	+N +E	or	+lat	+dep.
	90°–180°	$\rightarrow$	S E	i.e.	−N +E		−lat	+dep.
	180°–270°	$\rightarrow$	S W	i.e.	−N −E		−lat	−dep.
	270°–360°	$\rightarrow$	N W	i.e.	+N −E		+lat	−dep.

This gives a mathematical basis for the determination of a point with no need for graphical representation and is more satisfactory for the following reasons:

(1) Each station can be plotted independently.

(2) In plotting, the point is not dependent on any angular measuring device.

(3) Distances and bearings between points can be computed.

Rectangular co-ordinates are sub-divided into:

(1) Partial Co-ordinates, which relate to a line.

(2) Total Co-ordinates, which relate to a point.

3.31 Partial co-ordinates, ΔE, ΔN (Fig. 3.12)

These relate one end of a line to the other end.

They represent the distance travelled East (+)/West (–) and North (+)/South (–) for a single line or join between any two points.

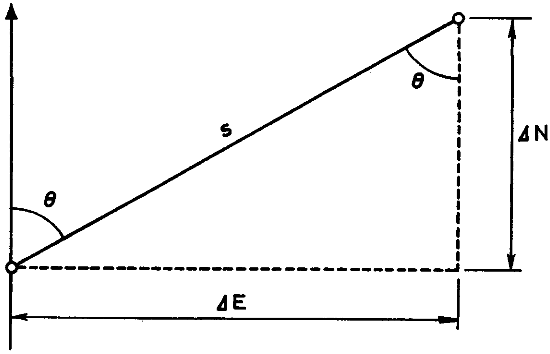


Fig. 3.12 Partial co-ordinates

Given a line of bearing θ and length s,

Partial departure = ΔE i.e. difference in Eastings

$$\Delta E_{AB} = s \sin\theta \tag{3.1}$$

Partial latitude = ΔN i.e. difference in Northings

$$\Delta N_{AB} = s \cos\theta \tag{3.2}$$

N.B. *always compute in bearings not angles* and preferably *quadrant bearings*.

3.32 Total co-ordinates (Fig. 3.13)

These relate any point to the axes of the co-ordinate system used. The following notation is used:

Total Easting of A = E_A

" Northing of A = N_A

Total Easting of B = $E_A + \Delta E_{AB}$

" Northing of B = $N_A + \Delta N_{AB}$

Total Easting of C = $E_B + \Delta E_{BC}$ = $E_A + \Delta E_{AB} + \Delta E_{BC}$

" Northing of C = $N_B + \Delta N_{BC}$ = $N_A + \Delta N_{AB} + \Delta N_{BC}$

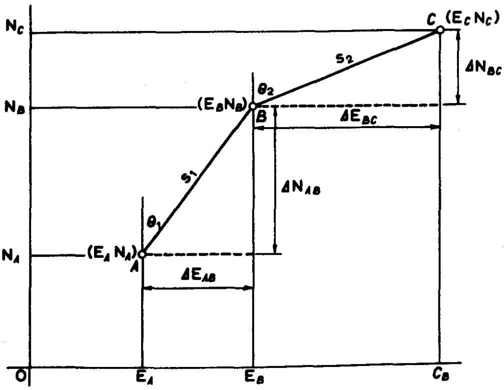


Fig. 3.13 Total co-ordinates

Thus in general terms

Total Easting of any point $= E_A + \Sigma \Delta E$ (3.3)

$=$ Total easting of the first point + the sum of the partial eastings up to that point.

Total Northing of any point $= N_A + \Sigma \Delta N$ (3.4)

$=$ Total northing of the first point + the sum of the partial northings up to that point.

N.B. If a traverse is closed polygonally then

$$\Sigma \Delta E = 0 \qquad (3.5)$$

$$\Sigma \Delta N = 0 \qquad (3.6)$$

i.e. the sum of the partial co-ordinates should equal zero.

Example 3.5

Given: (Fig. 3.14) *AB* 045° 100 m

BC 120° 150 m

CD 210° 100 m

Total co-ordinates of *A* E 50 m N 40 m

Line *AB* 045° = N 45° E 100 m

Partial departure ΔE_{AB} = 100 sin 45° = 100 × 0·707 = + 70·7 m

Total departure (E_A) *A* = + 50·0 m

Total departure (E_B) *B* = + 120·7 m

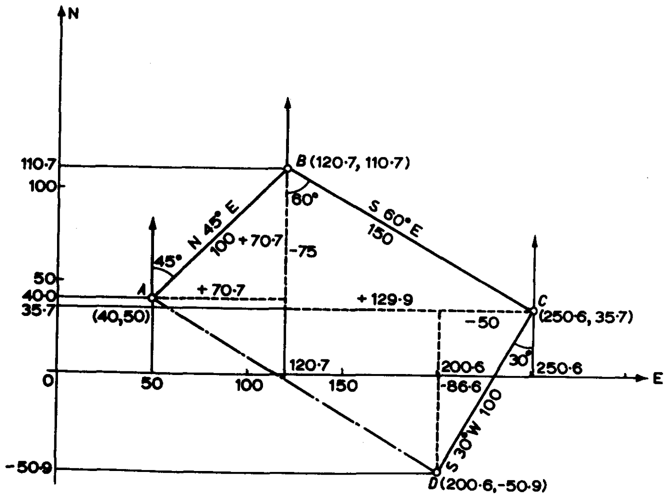


Fig. 3.14

Partial latitude ΔN_{AB} = 100 cos 45° = 100 × 0·707 = + 70·7 m

Total latitude (N_A) A = + 40·0 m

Total latitude (N_B) B = + 110·7 m

Line BC 120° = S 60° E 150 m

Partial departure ΔE_{BC} = 150 sin 60° = 150 × 0·866 = + 129·9 m

Total departure (E_B) B = + 120·7 m

Total departure (E_C) C = + 250·6 m

Partial latitude ΔN_{BC} = 150 cos 60° = 150 × 0·5 = − 75·0 m

Total latitude (N_B) B = + 110·7 m

Total latitude (N_C) C = + 35·7 m

Line CD 210° = S 30° W 100 m

Partial departure ΔE_{CD} = 100 sin 30° = 100 × 0·5 = − 50·0 m

Total departure (E_C) C = + 250·6 m

Total departure (E_D) D = + 200·6 m

Partial latitude ΔN_{CD} = 100 cos 30° = 100 × 0·866 = − 86·6 m

Total latitude (N_C) C = + 35·7 m

Total latitude (N_D) D = − 50·9 m

Check $E_D = E_A + \Delta E_{AB} + \Delta E_{BC} + \Delta E_{CD}$

$= 50·0 + 70·7 + 129·9 - 50·0 = +\underline{200·6\ \text{m}}$

$$N_D = N_A + \Delta N_{AB} + \Delta N_{BC} + \Delta N_{CD}$$
$$= 40{\cdot}0 + 70{\cdot}7 - 75{\cdot}0 - 86{\cdot}6 \quad = - \underline{50{\cdot}9\,\text{m}}$$

Exercises 3(b) (Plotting)

6. Plot the following traverse to a scale of 1 in = 100 links, and thereafter obtain the length and bearing of the line *AB* and the area in square yards of the enclosed figure.

N 21° W	120 links from *A*
N 28° E	100 links
N 60° E	117 links
N 32° E	105 links
S 15° E	200 links
S 40° W	75 links to *B*

(Ans. From scaling N 62° 45′ E 340 links; approx. area 3906 sq yd.)

7. The following table shows angles and distances measured in a theodolite traverse from a line *AB* bearing due South and of horizontal length 110 ft.

Angle	Angle value	Inclination	Inclined distance (ft)
ABC	192°00′	+15°	*BC* 150
BCD	92°15′	0°	*CD* 200
CDE	93°30′	−13°	*DE* 230
DEF	170°30′	0°	*EF* 150

Compute the whole circle bearing of each line, plot the survey to a scale of 1 in. = 100 ft and measure the horizontal length and bearing of the closing line.

(M.Q.B./M. Ans. 260 ft; 076°30′)

8. The following notes refer to an underground traverse made from the mouth, *A*, of a surface drift.

Line	Bearing	Distance (links)	
AB	038°	325	dipping at 1 in 2·4
BC	111°	208	level
CD	006°	363	level
DE	308°	234	rising at 1 in 3·2

Plot the survey to a scale of 1 chain to 1 inch.

Taking *A* as the origin, measure from your plan, the co-ordinates of *E*.

What is the difference in level between *A* and *E* to the nearest foot?

(M.Q.B./UM Ans. *E*, E 233 links N 688 links; diff. in level *AE* 78 ft)

9. Plot the following notes of an underground traverse to a scale of 1 in = 100 ft.

Line	Bearing	Distance
AB	N 28° W	354 ft dipping at 1 in 7
BC	N 83° W	133 ft level
CD	S 83° W	253 ft level
DE	N 8° E	219 ft rising at 1 in 4
EF	S 89° E	100 ft level

Points *A*,*B*,*C* and *D* are in workings of a lower seam and points *E* and *F* are in the upper seam, *DE* being a cross measure drift between the two seams.

It is proposed to drive a drift from *A* to *F*.

Find the bearing, length, and gradient of this drift.

(M.Q.B./UM Ans. N 40° W; 655 ft; +1 in 212)

10. The co-ordinates in feet, relative to a common point of origin *A*, are as follows:

	Departure	Latitude
A	0	0
B	275 E	237 N
C	552 E	230 N
D	360 E	174 S

Plot the figure *ABCD* to a scale of 1 inch to 100 ft and from the co-ordinates calculate the bearing and distance of the line *AC*.

(M.Q.B./UM Ans. N 67°24′ E; 598 ft)

11. An area in the form of a triangle *ABC* has been defined by the co-ordinates of the points *A B* and *C* in relation to the origin *O*, as follows:

A	South 2460 ft	East 3410 ft
B	North 2280 ft	East 4600 ft
C	North 1210 ft	East 1210 ft

Plot the positions of the points to a scale of 1 in. to 1000 ft, and find the area, in acres, enclosed by the lines joining *AB*, *BC* and *CA*.

(M.Q.B./M Ans. 169·826 acres)

12. There is reason to suspect a gross angular error in a five-legged closed traverse in which the recorded information was as follows:

Interior angles: *A* 110°; *B* 150°; *C* 70°; *D* 110°; *E* 110°

Sides: *AB* 180 ft; *BC* 420 ft; *CD* 350 ft; *DE* 410 ft; *EA* 245 ft

Plot the traverse to a scale of 100 ft to 1 in. and locate the gross angular error*, stating its amount.

(L.U./E)

13. A rough compass traverse of a closed figure led to the following field record:

Line	Length	Bearing
AB	422	57°
BC	405	316°
CD	348	284°
DE	489	207°
EA	514	109°

Plot the figure (scale 1 in = 50 ft) and adjust it to close using a graphical method. Letter your plan and add a north point (magnetic declination 10° W).

(L.U./E)

3.4 Computation Processes

As tables of trigonometrical functions are generally tabulated only in terms of angles 0°–90°, it is convenient to convert the *whole circle bearings* into *reduced* or *quadrant bearings*.

The signs of the partial co-ordinates are then related to the symbols of the quadrant bearings, Fig. 3.15.

$$\left.\begin{matrix}\text{E} & + \\ \text{W} & -\end{matrix}\right\}\ \text{Departures}$$

$$\left.\begin{matrix}\text{N} & + \\ \text{S} & -\end{matrix}\right\}\ \text{Latitudes}$$

Alternatively, the whole circle bearings are used and the sign of the value of the partial co-ordinate is derived from the sign of the trigonometrical function.

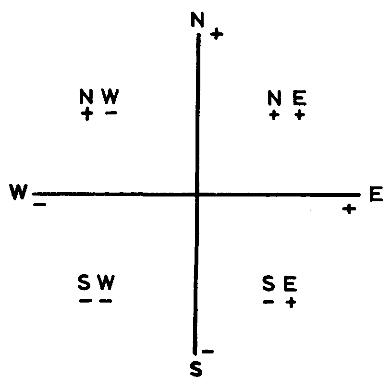


Fig. 3.15

The process may be either:

(i) by logarithms or

(ii) by machine (using natural trigonometrical functions).

*See Chapter 6 on location of errors.

3.41 Computation by logarithms

Let AB = 243° 27′ 423·62 m (A 2063·16 m E 5138·42 m N)
(243° 27′ = S 63° 27′ W)

Logs				
			E_A	2063·16 m
partial departure (ΔE)	2·578 579	⟶	ΔE_{AB}	−378·95
sin bearing AB	$\bar{1}$·951 602		E_B	1684·21 m
distance	2·626 977			
cos bearing	$\bar{1}$·650 287		N_A	5138·42
partial latitude (ΔN_{AB})	2·277 264	⟶	ΔN_{AB}	−189·35
			N_B	4949·07 m

N.B. The log distance is written down once only, being added to the log sin bearing above and the log cos bearing below, to give the partial departure and latitude respectively.

It is often considered good computing practice to separate the log figures for convenience of adding though the use of squared paper would obviate this.

3.42 Computation by machine

			E_A	2063·16
partial departure ΔE_{AB}		⟶	ΔE_{AB}	−378·95
			E_B	1684·21 m
sin bearing	0·894 545			
distance	423·62			
cos bearing	0·446 979		N_A	5138·42
partial latitude ΔN_{AB}		⟶	ΔN_{AB}	−189·35
			N_B	4949·07 m

Using a normal digital machine, the distance (being common) is set once in the machine and then separately multiplied by the appropriate trigonometrical function.

In the case of the twin-banked Brunsviga, the processes are simultaneous.

N.B. For both natural and logarithmic trigonometrical functions the following tables are recommended:

Degrees only	4 figure tables
Degrees and minutes	5 figure tables
Degrees, minutes and seconds	6 figure tables
Degrees, minutes, seconds and decimals of seconds	7 figure tables

3.43 Tabulation process (Fig. 3.16)

Nottingham Regional College of Technology

Traverse computation sheet Nº 1 Traverse A to Z

Compiled by J.H.B. (Red) Computed by F.A.S. (Green)

Date 8. 5. 1967. Checked by R.R.

ΔE = length × sin B'g ΔN = length × cos B'g

Line	Bearing	Length	sin/cos B'g	ΔE	ΔN	E	N	Point
AB	243° 27'00"	423·62				2063·16	5138·42	A
	S63°27' W		s 0·894 545 c 0·446 979	-378 95	-189·35			
						1684·21	4949·07	B
BC	042°32'00" N42°32' E	221·38	s 0·676 019 c 0·736 884	+149·66	+163·13			
						1833 87	5112·20	C
			Check Σ	-229·29	- 26·22			
				2063·16	5138·42			
				1833·87	5112·20			

Fig. 3.16 Tabulated computation

Example 3.6 Calculate the total co-ordinates, in feet, of a point B if the bearing of AB is $119° 45'$ and the distance is 850 links on a slope of $15°$ from the horizontal.

The co-ordinates of A relative to a local origin are N 5356·7 ft E 264·5 ft.

(M.Q.B./UM)

To find horizontal length (Fig. 3.17)

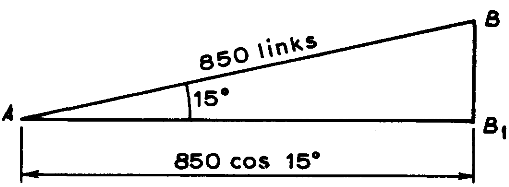


Fig. 3.17

$$AB_1 = AB \cos 15° \text{ links}$$

but 100 links = 66 ft; therefore to convert links to feet the length must be multiplied by $K = 0{\cdot}66$, i.e.

$$AB_1 = K \,.\, AB \cos 15°$$
$$= 0{\cdot}66 \times 850 \times \cos 15°$$

By logs,

0·66	$\bar{1}{\cdot}819\,54$
850	$2{\cdot}929\,42$
cos 15°	$\bar{1}{\cdot}984\,94$
AB_1	$2{\cdot}733\,90$

To find partial co-ordinates of line AB (Fig. 3.18)

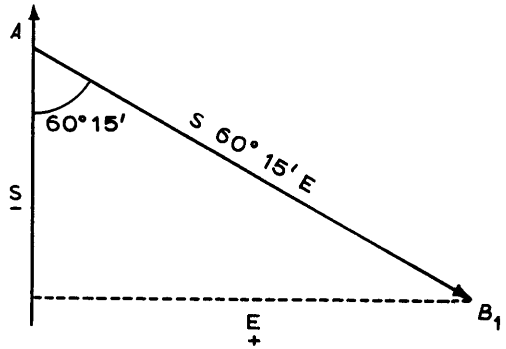


Fig. 3.18

$$119^\circ 45' = \text{S } 60^\circ 15' \text{ E}$$

By logs,

		E_A	$+264{\cdot}5$
partial departure	$2{\cdot}672\,52$ ⟶	ΔE_{AB}	$+470{\cdot}4$
		E_B	$+734{\cdot}9$
$\sin 60^\circ 15'$	$\bar{1}{\cdot}938\,62$		
AB_1	$2{\cdot}733\,90$ (see above)		
$\cos 60^\circ 15'$	$\bar{1}{\cdot}695\,67$		
		N_A	$5356{\cdot}7$
partial latitude	$2{\cdot}429\,57$ ⟶	ΔN_{AB}	$-268{\cdot}9$
		N_B	$5087{\cdot}8$

Co-ordinates of B, N 5087·8 ft E 734·9 ft

3.44 To obtain the bearing and distance between two points given their co-ordinates (Fig. 3.19)

Let the co-ordinates of A and B be $E_A N_A$ and $E_B N_B$ respectively.

$$\text{Then tan bearing } (\theta) = \frac{E_B - E_A}{N_B - N_A} \tag{3.7}$$

$$= \frac{\Delta E_{AB}}{\Delta N_{AB}} \tag{3.8}$$

N.B. For convenience this is frequently written:

$$\text{Bearing } AB = \tan^{-1}\Delta E/\Delta N \tag{3.9}$$

(the sign of the differences will indicate the quadrant bearing).

$$\text{Length } AB = \sqrt{(\Delta E_{AB}^2 + \Delta N_{AB}^2)} \tag{3.10}$$

N.B. This is not a very good solution for computation purposes and the trigonometrical solution below is preferred.

$$AB = \frac{\Delta N_{AB}}{\cos \text{bearing}(\theta)} \tag{3.11}$$

$$= \Delta N_{AB} \sec\theta \tag{3.12}$$

or

$$AB = \frac{\Delta E_{AB}}{\sin \text{bearing}\ (\theta)} \tag{3.13}$$

$$= \Delta E_{AB} \operatorname{cosec}\theta \tag{3.14}$$

If both of these determinations are used, their agreement provides a check on the determination of θ, but no check on the subtraction of the Eastings or Northings.

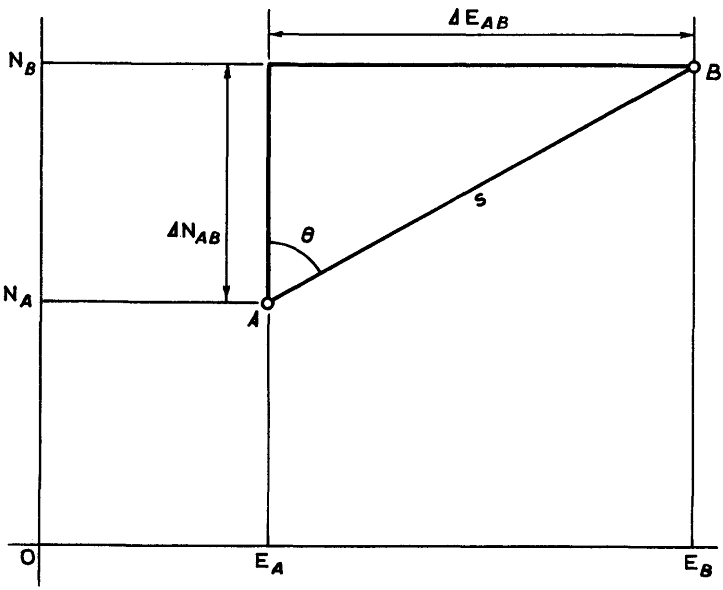


Fig. 3.19 To find the length and bearing between two points

Example 3.7

	E	N
A	632·16 m	949·88 m
B	925·48 m	421·74 m
	ΔE + 293·32 m	ΔN − 528·14 m

$$\text{Bearing } AB = \tan^{-1}\frac{+293{\cdot}32\ \text{(E)}}{-528{\cdot}14\ \text{(S)}}$$

By logs,

293·32	2·467 34	
528·14	2·722 75	
tan (θ)	$\bar{1}$·744 59	$\longrightarrow$ S 29° 03′ E
		i.e. 150° 57′

Length AB = ΔN sec θ or ΔE cosec θ

= 528·14 sec 29° 03′ 293·32 cosec 29° 03′

By logs,

	528·14	2·722 75	
	sec 29° 03′	0·058 39	
		2·781 14	$\longrightarrow$ 604·14 m
or	293·32	2·467 34	
	cosec 29° 03′	0·313 75	
		2·781 09	$\longrightarrow$ 604·07 m

The first solution is better as $\Delta N > \Delta E$, but a more compatible solution is obtained if the bearing is more accurately determined, using 7 figure logs,

	ΔE	2·467 341 7	
	ΔN	2·722 749 1	
	tan (θ)	9·744 592 6	
	Bearing (θ) =	29°02′50″	
	ΔN	2·722 749 1	
	sec θ	10·058 378 6	
		2·781 127 7	$\longrightarrow$ 604·13 m
or	ΔE	2·467 341 7	
	cosec θ	10·313 783 6	
		2·781 125 3	$\longrightarrow$ 604·12 m

Example 3.8 The following horizontal angle readings were recorded during a counter-clockwise traverse $ABCD$. If the line AD is taken as an arbitrary meridian, find the quadrantal bearings of the remaining lines.

Find also the latitudes and departures of the line CD whose length is 893·6 m.

Station at	Sight	Vernier A	Vernier B
A	D	241° 36′ 20″	061° 36′ 40″
A	B	038° 54′ 00″	218° 53′ 40″
B	A	329° 28′ 00″	149° 28′ 20″
B	C	028° 29′ 00″	208° 29′ 00″
C	B	106° 58′ 20″	286° 58′ 40″
C	D	224° 20′ 20″	044° 20′ 20″
D	C	026° 58′ 00″	206° 58′ 40″
D	A	053° 18′ 40″	233° 18′ 00″

Ans. *Mean values*

				Angle
A	D	241° 36′ 30″		
	B	038° 53′ 50″	DAB	157° 17′ 20″
B	A	329° 28′ 10″		
	C	028° 29′ 00″	ABC	59° 00′ 50″
C	B	106° 58′ 30″		
	D	224° 20′ 20″	BCD	117° 21′ 50″
D	C	026° 58′ 20″		
	A	053° 18′ 20″	CDA	26° 20′ 00″
				360° 00′ 00″

Bearing AD	=	0° 00′	
Angle DAB	=	157° 17′ 20″	
Bearing AB	=	157° 17′ 20″	(S 22° 42′ 40″ E)
Angle ABC	=	59° 00′ 50″	
		216° 18′ 10″	
		− 180°	
Bearing BC		036° 18′ 10″	(N 36° 18′ 10″ E)
Angle BCD		117° 21′ 50″	
		153° 40′ 00″	
		+ 180°	
Bearing CD		333° 40′ 00″	(N 26° 20′ 00″ W)
Angle CDA		26° 20′ 00″	
		360° 00′ 00″	

Co-ordinates CD 893·6 m (N 26° 20′ W)

$$\Delta E = 893{\cdot}6 \sin 26^\circ 20' = -396{\cdot}39 \text{ m}$$

$$\Delta N = 893{\cdot}6 \cos 26^\circ 20' = +800{\cdot}87 \text{ m}$$

Ans.

$AB = $ S 22° 42′ 40″ E

$BC = $ N 36° 18′10″ E

$CD = $ N 26° 20′00″ E

Co-ordinates of line CD $\Delta E = -396{\cdot}4$ m $\Delta N = +800{\cdot}9$ m

Example 3.9 In order to continue a base line AC to G, beyond a building which obstructed the sight, it was necessary to make a traverse round the building as follows, the angles being treated as deflection angles when traversing in the direction $ABCDEFG$.

$A\hat{C}D = 92°24'$ to the left

$C\hat{D}E = 90°21'$ to the right $CD = 56{\cdot}2$ ft

$D\hat{E}F = 89°43'$ to the right $DE = 123{\cdot}5$ ft

Calculate EF for F to be on AC produced and find $E\hat{F}G$ and CF. (L.U.)

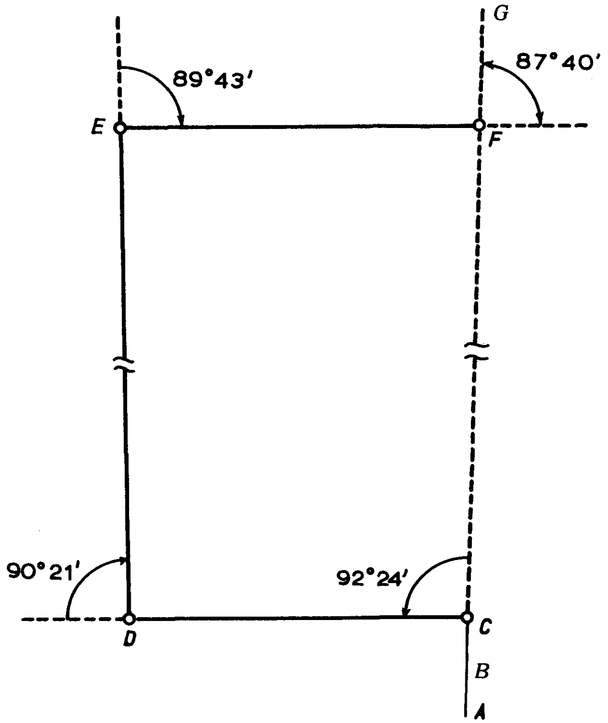


Fig. 3.20

Assuming the bearing $AC = 0°00'$

i.e.	Bearing AC	$= 360°00'$	
	$-$angle ACD (left)	$92°24'$	
	Bearing CD	$267°36'$	i.e. S 87° 36′ W
	$+$angle CDE (right)	$90°21'$	
	Bearing DE	$357°57'$	i.e. N 02° 03′ W
	$+$angle DEF (right)	$89°43'$	
		$447°40'$	
		$-360°00'$	
	Bearing EF	$087°40'$	i.e. N 87° 40′ E

Thus, to obtain the bearing of FG = bearing AC,

the deflection angle $EFG = 87°40'$ left.

Check on deflection angles

+	−
90° 21′	92° 24′
89° 43′	87° 40′
180° 04′	180° 04′

To obtain the co-ordinates of E

Line	Distance (ft)	Bearing	sin Bearing	cos Bearing	ΔE	ΔN
AC		0° 00′				
CD	56·2	S 87° 36′ W	0·999 12	0·041 88	−56·15	− 2·35
DE	123·5	N 02° 03′ W	0·035 77	0·999 36	− 4·42	+123·42
						+123·42
						− 2·35
					E_E − 60·57	N_E +121·07

Thus F must be +60·57 ft east of E. The line EF has a bearing 087° 40′

$$\text{Length } EF = \frac{\Delta E_{EF}}{\text{sin bearing}} = \frac{60{\cdot}57}{\sin 87°40'}$$

$$= 60{\cdot}62 \text{ ft}$$

To find the co-ordinates of F,

$$\Delta N_{EF} = 60{\cdot}62 \cos 87°40' = +\ 2{\cdot}47$$

$$N_E = +121{\cdot}07$$

$$N_F = +123{\cdot}54$$

$\therefore$ F is 123·54 ft above C on the bearing due N.

$\therefore$ CF = 123·54 ft.

Ans.

$$EF = 60{\cdot}6 \text{ ft}$$

$$E\hat{F}G = 87^\circ 40' \text{ deflection left}$$

$$CF = 123{\cdot}5 \text{ ft}$$

Example 3.10. The co-ordinates (metres) of the base line stations A and B are

A	26 543·36 E	35 432·31 N
B	26 895·48 E	35 983·37 N

The following clockwise angles were measured as part of a closed traverse: $ABCDEA$

ABC	183° 21′
BCD	86° 45′
CDE	329° 17′
DEA	354° 36′
EAB	306° 06′

Determine the adjusted quadrant bearings of each of the lines relative to the meridian on which the co-ordinates were based.

A	26 543·36 m		35 432·31 m
B	26 895·48 m		35 983·37 m
ΔE	352·12 m	ΔN	551·06 m

$$\tan \text{bearing } AB = \frac{352{\cdot}12}{551{\cdot}06}$$

$$\text{bearing } AB = 032^\circ 34'$$

		corr.	Corrected Angle
Σ Angles	183° 21′	−01′	183° 20′
	86° 45′	−01′	86° 44′
	329° 17′	−01′	329° 16′
	354° 36′	−01′	354° 35′
	306° 06′	−01′	306° 05′
	1260° 05′		1260° 00′

Σ Angles should equal $(2n + 4)90$

i.e. $(2 \times 5 + 4)90 = 1260^\circ$

$\therefore$ error is 05′ distributed as 01′ per angle.

Calculation of bearings

Bearing	AB	032° 34′	⟶	N 32° 34′ E
Angle	ABC	183° 20′		
		215° 54′		
		− 180°		
Bearing	BC	035° 54′	⟶	N 35° 54′ E
Angle	BCD	86° 44′		
		122° 38′		
		+ 180°		
Bearing	CD	302° 38′	⟶	N 57° 22′ W
Angle	CDE	329° 16′		
		631° 54′		
		− 540°		
Bearing	DE	091° 54′	⟶	S 88° 06′ E
Angle	DEF	354° 35′		
		446° 29′		
		− 180°		
Bearing	EA	266° 29′	⟶	S 86° 29′ W
Angle	EAB	306° 05′		
		572° 34′		
		− 540°		
Bearing	AB	032° 34′	*Check*	

Example 3.11 A disused colliery shaft C, situated in a flooded area, is surrounded by a circular wall and observations are taken from two points A and B of which the co-ordinates, in feet, relative to a local origin, are as follows:

Station	Eastings	Northings
A	3608·1	915·1
B	957·6	1808·8

C is approximately N.W. of A.

Angles measured at A to the tangential points 1 and 2 of the walls are $BAC_1 = 25° 55'$ and $BAC_2 = 26° 35'$.

Angles measured at B to the tangential points 3 and 4 of the wall are $C_3BA = 40° 29'$ and $C_4BA = 39° 31'$.

Determine the co-ordinates of the centre of the shaft in feet relative to the origin, to one place of decimals and calculate the diameter

of the circle formed by the outside of the wall.

(M.Q.B./S)

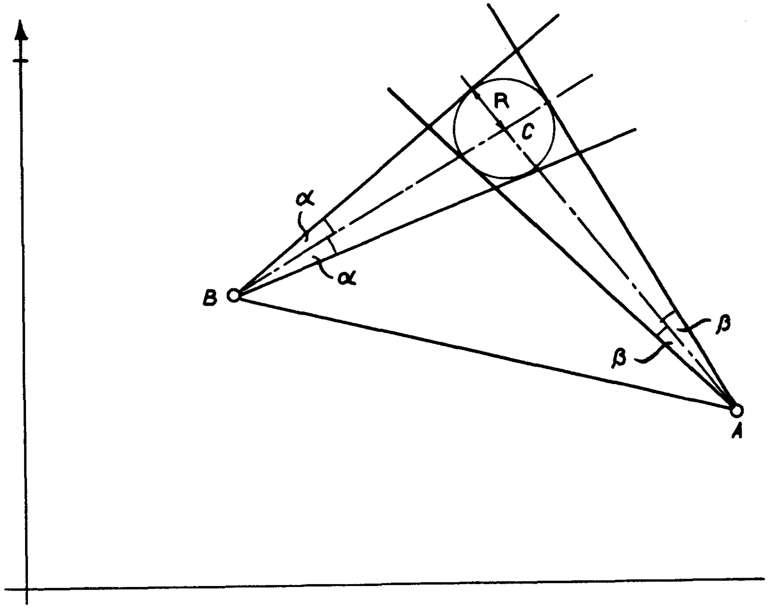


Fig. 3.21

	E	N
A	3608·1	915·1
B	957·6	1808·8
	ΔE_{AB} − 2650·5	ΔN_{AB} + 893·7

In Fig. 3.21,

$$\text{Bearing of } AB = \tan^{-1}\frac{\Delta E}{\Delta N} = \tan^{-1}\frac{-2650{\cdot}5}{+893{\cdot}7}$$

$$= \underline{\text{N } 71^\circ 22' \text{ W}} \quad \text{i.e. } 288^\circ 38'$$

$$\text{Length } AB = \Delta N \sec \text{bearing} \quad \text{or} \quad \Delta E \text{ cosec bearing}$$

$$= 893{\cdot}7 \sec 71^\circ 22' \quad \text{or} \quad 2650{\cdot}5 \text{ cosec } 71^\circ 22'$$

$$= \underline{2797{\cdot}1} \qquad \underline{2797{\cdot}1}$$

In triangle ABC

$$\text{Angle } A = \tfrac{1}{2}\{25^\circ 55' + 26^\circ 35'\} = 26^\circ 15'$$

$$\text{Angle } B = \tfrac{1}{2}\{40^\circ 29' + 39^\circ 31'\} = 40^\circ 00'$$

$$\text{Angle } C = 180^\circ - (26^\circ 15' + 40^\circ 00') = \underline{113^\circ 45'}$$

$$180^\circ 00'$$

By the sine rule,

$$BC = AB \sin A \operatorname{cosec} C$$
$$= 2797{\cdot}1 \sin 26^\circ 15' \operatorname{cosec} 113^\circ 45' = 1351{\cdot}6 \text{ ft}$$

$$AC = BC \sin B \operatorname{cosec} A$$
$$= 1351{\cdot}6 \sin 40^\circ 00' \operatorname{cosec} 26^\circ 15' = 1964{\cdot}3 \text{ ft}$$

	Bearing AB	$288^\circ 38'$
+	Angle BAC	$26^\circ 15'$
	Bearing AC	$\underline{314^\circ 53'}$
	Bearing BA	$108^\circ 38'$
−	Angle CBA	$40^\circ 00'$
	Bearing BC	$\underline{068^\circ 38'}$

To find co-ordinates of C

Line BC $068^\circ 38'$ i.e. N $68^\circ 38'$ E $1351{\cdot}6$ ft

$$\Delta E_{BC} = 1351{\cdot}6 \sin 68^\circ 38' = +1258{\cdot}7$$
$$E_C = E_B + \Delta E_{BC}$$
$$= 957{\cdot}6 + 1258{\cdot}7 = \underline{+2216{\cdot}3}$$
$$\Delta N_{BC} = 1351{\cdot}6 \cos 68^\circ 38' = +\ 492{\cdot}4$$
$$N_C = N_B + \Delta N_{BC}$$
$$= 1808{\cdot}8 + 492{\cdot}4 = \underline{+2301{\cdot}2}$$

Check

Line AC $314^\circ 53'$ i.e. N $45^\circ 07'$ W $1964{\cdot}3$ ft

$$\Delta E_{AC} = 1964{\cdot}3 \sin 45^\circ 07' = -1391{\cdot}8$$
$$E_C = E_A + \Delta E_{AC}$$
$$= 3608{\cdot}1 - 1391{\cdot}8 = \underline{2216{\cdot}3}$$
$$\Delta N_{AC} = 1964{\cdot}3 \cos 45^\circ 07' = \underline{+1386{\cdot}1}$$
$$N_C = N_A + \Delta N_{AC}$$
$$= 915{\cdot}1 + 1386{\cdot}1 = \underline{2301{\cdot}2}$$

To find the diameter of the wall.

Referring to Fig. 3.21

$$\alpha = \tfrac{1}{2}(40^\circ 29' - 39^\circ 31') = 0^\circ 29'$$
$$\therefore \quad R = BC \sin 0^\circ 29' \simeq BC \times 0^\circ 29' \text{(rad)}$$
$$= \frac{1351{\cdot}6 \times 29 \times 60}{206\,265} = \underline{11{\cdot}40 \text{ ft}}$$

Check

$$\beta = \tfrac{1}{2}(26°35' - 25°55') = 0°20'$$

$$R = AC \times 0°20'\,(\text{rad})$$

$$= \frac{1964{\cdot}3 \times 20 \times 60}{206\,265} = \underline{11{\cdot}43\text{ ft}}$$

$$\therefore \quad \text{Diameter of wall} = \underline{22{\cdot}8\text{ ft}}$$

3.5 To Find the Co-ordinates of the Intersection of Two Lines

3.51 Given their bearings from two known co-ordinate stations

As an alternative to solving the triangle and then computing the co-ordinates the following process may be applied:

Given $A\ (E_A\ N_A)$

$B\ (E_B\ N_B)$

bearings α and β

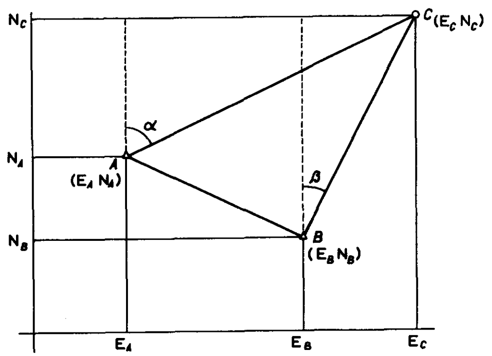


Fig. 3.22

From Fig. 3.22,

$$E_C = E_A + (N_C - N_A)\tan\alpha$$

$$= E_A + \Delta N_{AC}\tan\alpha \qquad (3.15)$$

$$= E_B + (N_C - N_B)\tan\beta$$

$$= E_B + \Delta N_{BC}\tan\beta \qquad (3.16)$$

$$\therefore \quad N_C(\tan\alpha - \tan\beta) = E_B - E_A + N_A\tan\alpha - N_B\tan\beta$$

Then the total northing of C

$$N_C = \frac{E_B - E_A + N_A\tan\alpha - N_B\tan\beta}{\tan\alpha - \tan\beta} \qquad (3.17)$$

To obtain the partial co-ordinates from equation (3.17)

Partial Northing $\Delta N_{AC} = N_C - N_A$

i.e.

$$N_C - N_A = \frac{E_B - E_A + N_A \tan\alpha - N_B \tan\beta}{\tan\alpha - \tan\beta} - N_A$$

$$= \frac{E_B - E_A + N_A \tan\alpha - N_B \tan\beta - N_A \tan\alpha + N_A \tan\beta}{\tan\alpha - \tan\beta}$$

$$= \frac{(E_B - E_A) - (N_B - N_A)\tan\beta}{\tan\alpha - \tan\beta}$$

Then $$\Delta N_{AC} = \frac{\Delta E_{AB} - \Delta N_{AB}\tan\beta}{\tan\alpha - \tan\beta} \qquad (3.18)$$

Similarly, from equation (3.17),

$$\Delta N_{BC} = N_C - N_B = \frac{E_B - E_A + N_A \tan\alpha - N_B \tan\beta}{\tan\alpha - \tan\beta} - N_B$$

Then $$\Delta N_{BC} = \frac{\Delta E_{AB} - \Delta N_{AB}\tan\alpha}{\tan\alpha - \tan\beta} \qquad (3.19)$$

The following alternative process may be used:

$$N_C = N_A + (E_C - E_A)\cot\alpha = N_A + \Delta E_{AC}\cot\alpha \qquad (3.20)$$

$$= N_B + (E_C - E_B)\cot\beta = N_B + \Delta E_{BC}\cot\beta \qquad (3.21)$$

As before, the total and partial co-ordinates are given as:

$$E_C = \frac{N_B - N_A + E_A \cot\alpha - E_B \cot\beta}{\cot\alpha - \cot\beta} \qquad (3.22)$$

and $$\Delta E_{AC} = \frac{\Delta N_{AB} - \Delta E_{AB}\cot\beta}{\cot\alpha - \cot\beta} \qquad (3.23)$$

$$\Delta E_{BC} = \frac{\Delta N_{AB} - \Delta E_{AB}\cot\alpha}{\cot\alpha - \cot\beta} \qquad (3.24)$$

N.B. Theoretically, if $\Sigma \cot > \Sigma \tan$, then it is preferable to use the cot values, though in practice only one form would be used.

Example 3.12 Let the co-ordinates be A = E4, N6 B = E13, N4

the bearings be $\alpha = 060°$ $\beta = 330°$

$\tan\alpha = 1{\cdot}7321$ $\cot\alpha = 0{\cdot}5774$

$\tan\beta = -0{\cdot}5774$ $\cot\beta = -1{\cdot}7321$

Using the tan values ;

from equation (3.17)

$$N_C = \frac{(13-4) + (6 \times 1{\cdot}7321) + (4 \times 0{\cdot}5774)}{1{\cdot}7321 + 0{\cdot}5774} = \underline{9{\cdot}397}$$

from equation (3.15)

$$E_C = 4 + (9{\cdot}397 - 6) \times 1{\cdot}7321 = \underline{9{\cdot}884}$$

or equation (3.16)

$$E_C = 13 + (9{\cdot}397 - 4) \times -0{\cdot}5774 = \underline{9{\cdot}884}$$

Using the cot values,
from equation (3.22),

$$E_C = \frac{4 - 6 + 4 \times 0{\cdot}5774 + 13 \times 1{\cdot}7321}{0{\cdot}5774 + 1{\cdot}7321} = \underline{9{\cdot}884}$$

From equation (3.20),

$$N_C = 6 + (9{\cdot}884 - 4) \times 0{\cdot}5774 = \underline{9{\cdot}397}$$

or equation (3.21)

$$N_C = 4 + (9{\cdot}884 - 13) \times -1{\cdot}7321 = \underline{9{\cdot}397}$$

If the formulae using partial values are employed the individual equation computation becomes simplified.

Using the previous values (Ex. 3.5),
from equation (3.18)

$$\Delta N_{AC} = \frac{(13 - 4) - (4 - 6) \times -0{\cdot}5774}{1{\cdot}7321 + 0{\cdot}5774} = +3{\cdot}397$$

Then
$$N_C = N_A + \Delta N_{AC}$$
$$= 6 + 3{\cdot}397 = \underline{9{\cdot}397}$$

When this value is known, equation (3.15) may be used as before.

From equation (3.23);

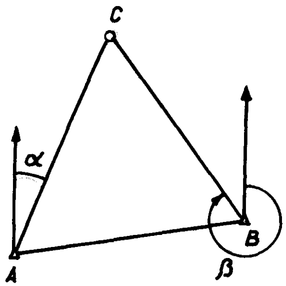
$$\Delta E_{AC} = \frac{(4 - 6) - (13 - 4) \times -1{\cdot}7321}{0{\cdot}5774 + 1{\cdot}7321} = +5{\cdot}884$$

$$E_C = E_A + \Delta E_{AC}$$
$$= 4 + 5{\cdot}884 = \underline{9{\cdot}884}$$

When this value is known, equation (3.20) may be used as before.

The above process is prefered and this can now be given in a tabulated form.

Example 3.13



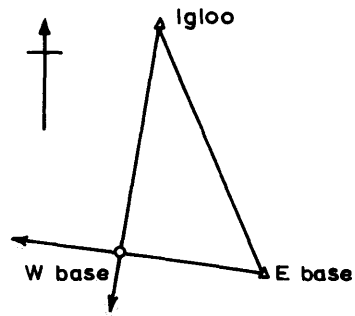
(1) $$\Delta N_{AC} = \frac{\Delta E_{AB} - \Delta N_{AB} \tan\beta}{\tan\alpha - \tan\beta}$$

(2) $$\Delta E_{AC} = \Delta N_{AC} \tan\alpha$$

(3) $$\Delta N_{BC} = \frac{\Delta E_{AB} - \Delta N_{AB} \tan\alpha}{\tan\alpha - \tan\beta}$$

(4) $$\Delta E_{BC} = \Delta N_{BC} \tan\beta$$

Oriented diagram



Stations	E	Bearings	N	
E Base (A)	+13 486·85 m	α 278° 13′ 57″	+10 327·36 m	
Igloo (B)	+12 759·21 m	β 182° 27′ 44″	+13 142·72 m	
		$\tan\alpha$ − 6·911 745 2		
ΔE_{AB}	−727·64	$\tan\beta$ + 0·043 000 4	+2815·36	ΔN_{AB}
		$\tan\alpha - \tan\beta$		
$\Delta N_{AB}\tan\beta$	+121·06			
	−848·70	÷ −6·954 7456		
ΔE_{AC}	−843·44	⟵ × $\tan\alpha$ ⟶	+122·03	ΔN_{AC}
W Base E_C	12 643·41 m		10 449·39 m	N_C
		Check		
ΔE_{AB}	−727·64			
$\Delta N_{AB}\tan\alpha$	−19 459·05			
		$\tan\alpha - \tan\beta$		
	+18 731·41	÷ −6·954 7456		
ΔE_{BC}	115·81	⟵ × $\tan\beta$ ⟶	2693·33	ΔN_{BC}
W Base E_C	+12 643·40 m		10 449·39 m	N_C

3.52 Given the length and bearing of a line *AB* and all the angles *A*, *B* and *C*, Fig 3. 23

Given: (a) Length and Bearing of AB, (b) Angles A, B and C.

$$
\begin{aligned}
E_C - E_A &= b\cos(A+\theta) \\
&= b(\cos A\cos\theta - \sin A\sin\theta) \\
&= \frac{c\sin B\cos A\cos\theta - c\sin B\sin A\sin\theta}{\sin C}
\end{aligned}
$$

but $\quad E_B - E_A = c\cos\theta = AB\sin \text{bearing}_{AB}$

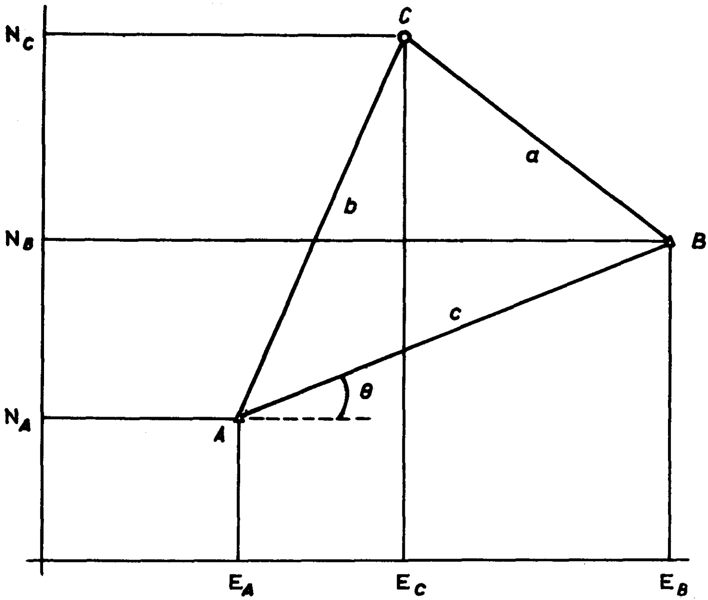


Fig. 3.23

$$N_B - N_A = c \sin\theta = AB \cos \text{bearing}_{AB}$$

and
$$C = 180 - (A + B)$$

$\therefore$
$$\sin C = \sin A \cos B + \cos A \sin B$$

Then

$$E_C - E_A = \frac{(E_B - E_A) \sin B \cos A - (N_B - N_A) \sin B \sin A}{\sin A \cos B + \cos A \sin B}$$

$$\therefore \quad E_C = \frac{E_A \sin A \cos B + E_A \cos A \sin B + E_B \sin B \cos A - E_A \sin B \cos A - N_B \sin B \sin A + N_A \sin B \sin A}{\sin A \cos B + \cos A \sin B}$$

$$= \frac{E_A \cot B + E_B \cot A + (N_A - N_B)}{\cot A + \cot B}$$

$$= \frac{E_A \cot B + E_B \cot A - \Delta N_{AB}}{\cot A + \cot B} \qquad (3.25)$$

Similarly,

$$N_C = \frac{N_A \cot B + N_B \cot A + \Delta E_{AB}}{\cot A + \cot B} \qquad (3.26)$$

Check

$$E_A(\cot B - 1) + E_B(\cot A + 1) + N_A(\cot B + 1) + N_B(\cot A - 1) - (E_C + N_C)(\cot A + \cot B) = 0 \qquad (3.27)$$

Using the values of Example 3.13,

Bearing AB = $\tan^{-1} - 727{\cdot}64/{+}2815{\cdot}36$ = N 14° 29′ 28″ W
= 345° 30′ 32″

Bearing AC = 278° 13′ 57″ $\therefore$ Angle A = 67° 16′ 35″

Bearing BC = 182° 27′ 44″

Bearing BA = 165° 30′ 32″ $\therefore$ Angle B = 16° 57′ 12″

Bearing CB = 02° 27′ 44″

Bearing CA = 098° 13′ 57″ $\therefore$ Angle C = 95° 46′ 13″

$$Check \quad \Sigma = 180°00'00''$$

$$\cot A = 0{\cdot}418\,79 \qquad \cot B = 3{\cdot}280\,40 \qquad \cot C = -0{\cdot}10\,105$$

From equation (3.25), $$E_C = \frac{E_A \cot B + E_B \cot A - \Delta N_{AB}}{\cot A + \cot B}$$

$$= \frac{(13\,486{\cdot}85 \times 3{\cdot}280\,4) + (12\,759{\cdot}21 \times 0{\cdot}418\,79) - 2815{\cdot}36}{0{\cdot}418\,79 + 3{\cdot}280\,40}$$

$$= \frac{44\,242{\cdot}26 + 5343{\cdot}43 - 2815{\cdot}36}{3{\cdot}699\,19}$$

$$\underline{E_C = 12\,643{\cdot}40}$$

From equation (3.26), $$N_C = \frac{N_A \cot B + N_B \cot A + \Delta E_{AB}}{\cot A + \cot B}$$

$$= \frac{(10\,327{\cdot}36 \times 3{\cdot}2804) + (13\,142{\cdot}72 \times 0{\cdot}41\,879) - 727{\cdot}64}{3{\cdot}699\,19}$$

$$= \frac{33\,877{\cdot}87 + 5504{\cdot}04 - 727{\cdot}64}{3{\cdot}699\,19}$$

$$\underline{N_C = 10\,449{\cdot}39}$$

Check (equation 3.27)

$$\begin{aligned}
E_A(\cot B - 1) &= 13\,486{\cdot}85 \times 2{\cdot}280\,40 = 30\,755{\cdot}41\\
E_B(\cot A + 1) &= 12\,759{\cdot}21 \times 1{\cdot}418\,79 = 18\,102{\cdot}64\\
N_A(\cot B + 1) &= 10\,327{\cdot}36 \times 4{\cdot}280\,40 = 44\,205{\cdot}23\\
N_B(\cot A - 1) &= 13\,142{\cdot}72 \times -0{\cdot}581\,21 = -7638{\cdot}68\\
&= 85\,424{\cdot}60
\end{aligned}$$

$$\begin{aligned}
(E_C + N_C)(\cot A + \cot B) &= (12643{\cdot}40 + 10449{\cdot}39)(3{\cdot}69919)\\
&= \underline{85\,424{\cdot}62}
\end{aligned}$$

Example 3.14 Given the co-ordinates of four stations,

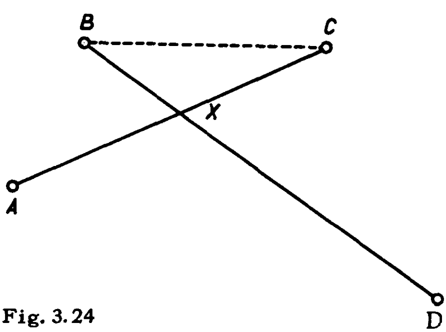


Fig. 3.24

A	E 250·00	N 100·00
B	E 320·70	N 170·70
C	E 520·70	N 170·70
D	E 652·45	S 263·12

to find the co-ordinates of the intersection of the lines *AC* and *BD*.

Method 1

$$\tan \text{bearing } BD = \frac{\Delta E}{\Delta N}$$

$$= \frac{652{\cdot}45 - 320{\cdot}70}{-263{\cdot}12 - 170{\cdot}70} = \frac{331{\cdot}75}{-433{\cdot}82}$$

$$\text{bearing } BD = \text{S } 37^\circ 24' \text{ E} = 142^\circ 36' = \text{bearing } BX$$

$$\tan \text{bearing } CA = \frac{250{\cdot}0 - 520{\cdot}7}{100{\cdot}0 - 170{\cdot}7} = \frac{-270{\cdot}7}{-70{\cdot}7}$$

$$\text{bearing } CA = \text{S } 75^\circ 22' \text{ W} = 255^\circ 22' = \text{bearing } CX$$

In triangle BCX

$BC = 520{\cdot}7 - 320{\cdot}7 = 200$ (No difference in latitude, therefore due E)

$$\text{Bearing } BC = 090^\circ$$
$$BX = 142^\circ 36'$$
$$\therefore \text{ Angle } XBC = 52^\circ 36'$$
$$\text{Bearing } CB = 270^\circ$$
$$CX = 255^\circ 22'$$
$$\text{Angle } BCX = 14^\circ 38'$$

$$\text{Length } BX = \frac{BC \sin BCX}{\sin BXC} = 200 \sin 14^\circ 38' \operatorname{cosec}(52^\circ 36' + 14^\circ 38')$$

$$\log BX = 1{\cdot}738\,75$$

To find co-ordinates of X. (Length BX known. Bearing S 37°24′E)

Logs	$1{\cdot}522\,21$	$\longrightarrow$	$+33{\cdot}28\,(\Delta E)$	E_A	$+320{\cdot}7$
sin bearing	$\bar{1}{\cdot}783\,46$			ΔE	$33{\cdot}28$
length	$1{\cdot}738\,75$			E_X	$353{\cdot}98$
cos bearing	$\bar{1}{\cdot}900\,05$				
	$1{\cdot}638\,80$	$\longrightarrow$	$-43{\cdot}53\,(\Delta N)$	N_B	$+170{\cdot}70$
				ΔN	$-43{\cdot}53$
			Ans. X = E 353·98 N 127·17	N_X	$+127{\cdot}17$

Method 2

From the previous method,

$$\tan \text{bearing } BD\ (\beta) = \frac{331{\cdot}75}{-433{\cdot}82} = -0{\cdot}764\,72$$

$$\tan \text{bearing } CA\ (\alpha) = \frac{-270{\cdot}7}{-70{\cdot}7} = 3{\cdot}828\,85$$

Using equation (3.18),

$$\Delta N_{CX} = \frac{\Delta E_{CB} - \Delta N_{CB} \tan\beta}{\tan\alpha - \tan\beta}$$

$$\Delta E_{CX} = \Delta N_{CX} \tan\alpha$$

	E	N
B	320·70	170·70
C	520·70	170·70
	ΔE_{CB} – 200·00	ΔN_{CB} 0·0

$$\tan\alpha - \tan\beta = 3{\cdot}828\,85 + 0{\cdot}764\,72 = 4{\cdot}593\,57$$

$$\Delta N_{CX} = \frac{-200{\cdot}00}{4{\cdot}593\,57} - 0 = -43{\cdot}538$$

$$N_C = 170{\cdot}70$$

$$N_X = \underline{127{\cdot}16}$$

$$\Delta E_{CX} = -43{\cdot}538 \tan\alpha$$

$$= -43{\cdot}538 \times 3{\cdot}828\,85 = -166{\cdot}70$$

$$E_C = 520{\cdot}70$$

$$E_X = \underline{354{\cdot}00}$$

Method 3

By normal co-ordinate geometry,

$$\text{the equation of line } AC = \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

i.e.
$$\frac{y - 100}{x - 250} = \frac{170{\cdot}7 - 100}{520{\cdot}7 - 250} = \frac{70{\cdot}7}{270{\cdot}7} = 0{\cdot}261\,2$$

$$\therefore \quad y - 100 = 0{\cdot}261\,2\,(x - 250) \qquad (1)$$

Similarly,

$$\text{the equation of line } BD = \frac{y - 170{\cdot}7}{x - 320{\cdot}7} = \frac{-263{\cdot}12 - 170{\cdot}7}{652{\cdot}45 - 320{\cdot}7}$$

$$= \frac{-433{\cdot}82}{331{\cdot}75} = -1{\cdot}3076$$

i.e.
$$y - 170{\cdot}7 = -1{\cdot}3076\,(x - 320{\cdot}7) \qquad (2)$$

Subtracting (1) from (2),

$$70{\cdot}7 = 1{\cdot}5688\,x - (250 \times 0{\cdot}261\,2) - (320{\cdot}7 \times 1{\cdot}307\,6)$$

$$= 1{\cdot}568\,8\,x - 65{\cdot}3 - 419{\cdot}343\,2$$

$$x = \frac{555{\cdot}343\,2}{1{\cdot}5688} = \underline{354{\cdot}00}$$

Substituting in equation (1),

$$\begin{aligned} y &= 0{\cdot}261\,2\,(x - 250) + 100 \\ &= 0{\cdot}261\,2\,(354 - 250) + 100 \\ &= (0{\cdot}261\,2 \times 104) + 100 \\ &= \underline{127{\cdot}16} \end{aligned}$$

Ans. $\underline{X = \text{E } 354{\cdot}0 \text{ N } 127{\cdot}16}$

N.B. All these methods are mathematically sound but the first has the advantages that (1) no formulae are required beyond the solution of triangles, (2) additional information is derived which might be required in setting-out processes.

Example 3.15. *Equalisation of a boundary line.* The following survey notes refer to a boundary traverse and stations A and E are situated on the boundary.

Line	Bearing	Horizontal length (ft)
AB	N 83° 14′ E	253·2
BC	S 46° 30′ E	426·4
CD	N 36° 13′ E	543·8
DE	S 23° 54′ E	1260·2

It is proposed to replace the boundary $ABCDE$ by a boundary AXE where AX is a straight line and X is situated on the line DE.

Calculate the distance EX which will give equalisation of areas on each side of the new boundary.

(M.Q.B./S)

Computation of co-ordinates with A as the origin, Fig. 3.25

Line AB N 83° 14′ E 253·2 ft

	Logs		
		E_A	0·0
ΔE	2·400 42	→	+ 251·44
$\sin\theta$	9·996 96	E_B	+ 251·44
length	2·403 46		
$\cos\theta$	9·071 24	N_A	0·0
ΔN	1·474 60	→	+ 29·83
		N_B	+ 29·83

Line BC S 46° 30′ E 426·4

		E_B	+ 251·44
ΔE	2·490 38	→	+ 309·30
$\sin\theta$	9·860 56	E_C	+ 560·74

length	2·629 82		
$\cos\theta$	9·837 81	N_B	+ 29·83
ΔN	2·467 63	→	− 293·51
		N_C	− 263·68

Line CD N 36° 13′ E 543·8

		E_C	+ 560·74
ΔE	2·506 91	→	+ 321·30
$\sin\theta$	9·771 47	E_D	+ 882·04
length	2·735 44		
$\cos\theta$	9·906 76	N_C	− 263·68
ΔN	2·642 20	→	+ 438·74
		N_D	+ 175·06

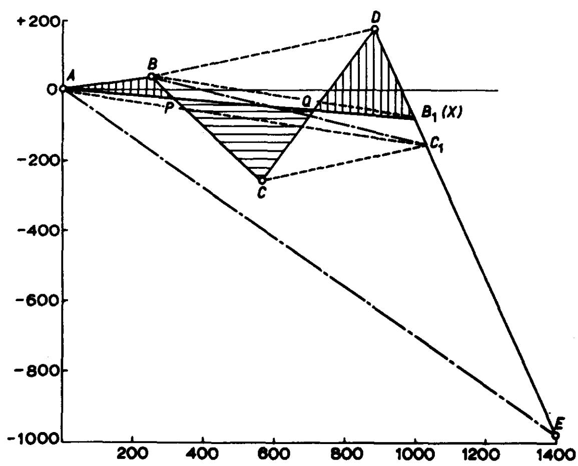


Fig. 3.25

By construction,
Join BD.
Draw line parallel to BD through C to cut ED at C_1.
Area of triangle BDC_1 = area of triangle BDC (triangles on same base and between same parallels).
Join C_1A.
Draw line parallel to C_1A through B to cut ED at B_1.
Area of triangle AB_1C = area of triangle ABC_1.
∴ Line $AB_1(X)$ equalises the irregular boundary in such a way that

triangle ABP + triangle QDB_1 = triangle PQC

Length EX = 977·84 ft (calc.) = 978 ft (scaled).

Line DE S 23° 54′ E 1260·2

			E_D	+ 882·04
ΔE	2·708 05	→		+ 510·57
sin θ	9·607 61		E_E	+ 1392·61
length	3·100 44			
cos θ	9·961 07		N_D	+ 175·06
ΔN	3·061 51	→		− 1152·15
			N_E	− 977·09

Checks

ΔE	+ 251·44	ΔN + 29·83	− 293·51
	+ 309·30	+ 438·74	− 1152·15
	+ 321·30	+ 468·57	− 1445·66
	+ 510·57		+ 468·57
ΣΔE	+ 1392·61	ΣΔN	− 977·09

Area of figure ABCDEA (see Chapter 11)

	(1) N	(2) E	(3) F.dep.	(4) B.dep.	(5) (3) − (4)
A	0·0	0·0	+ 251·44	+ 1392·61	− 1141·17
B	+ 29·83	+ 251·44	+ 560·74	0·0	+ 560·74
C	− 263·68	+ 560·74	+ 882·04	+ 251·44	+ 630·60
D	+ 175·06	+ 882·04	+ 1392·61	+ 560·74	+ 831·87
E	− 977·09	+ 1392·61	0·0	+ 882·04	− 882·04

Double areas (1) × (5)

	+	−
A	0	
B	167 26·8	
C		166 276·0
D	145 627·0	
E	861 832·0	

+1 024 185·8
− 166 276·0
2) 857 909·8
428 954·4 sq ft

From co-ordinates

$$\text{Bearing } EA = \tan^{-1}\frac{-1392{\cdot}61}{+977{\cdot}09} = \text{N } 54^\circ 56' 44'' \text{ W}$$

$$= 305^\circ 03' 16''$$

$$\begin{aligned} \text{Length } EA &= 1392{\cdot}61 \operatorname{cosec} 54^\circ 56' 44'' \\ &= 1701{\cdot}2 \text{ ft } (x) \\ \text{Bearing } ED &= \text{N } 23^\circ 54' 00'' \text{ W} = 336^\circ 06' 00'' \\ \text{Angle } AED &= 336^\circ 06' 00'' - 305^\circ 03' 16'' \\ &= 31^\circ 02' 44'' \end{aligned}$$

To find length EX (a) such that the area of triangle AXE is equal to 428 954·4 sq. ft.

$$\begin{aligned} \text{Area of triangle } AXE &= \tfrac{1}{2} ax \sin AED \\ a &= \frac{2 \text{ area triangle } AXE}{x \sin AED} \\ &= \frac{2 \times 428\,954{\cdot}4}{1701{\cdot}2 \sin 31^\circ 02' 44''} \\ &= \underline{977{\cdot}84 \text{ ft} \quad (\text{length } EX).} \end{aligned}$$

Exercises 3 (c) (Boundaries)

14. The undernoted bearings and measurements define an irregular boundary line on a mine plan between two points A and B, the latter being a point on a straight line XY, bearing from South to North.

Plot the bearings and measurements to a scale of 1/2 in. = 100 ft, and thereafter lay down a straight line from A to a point on XBY so that the areas to the North and South respectively of that line will be equal.

From A	N 63° 30′ W	185 ft	
	S 45° 00′ W	245 ft	
	S 80° 45′ W	175 ft	
	N 55° 15′ W	250 ft	
	S 60° 30′ W	300 ft	to B

Check your answer by calculation of the respective areas.

(M.Q.B./M)

15. The undernoted traverse was taken along an irregular boundary between two properties:

AB	N 32° 45′ E	464 ft
BC	N 71° 30′ E	308 ft
CD	S 61° 15′ E	528 ft
DE	N 71° 30′ E	212 ft
EF	S 40° 30′ E	248 ft

A lies on a straight boundary fence XY which bears N 7° 30′ W.

Plot the traverse to a scale 1/2400 and thereafter set out a straight line boundary from F to a point G on the fence XAY so that the areas North and South of the line are equal.

What length of fencing will be required?

How far is G from A?

(N.R.C.T. Ans. 1480 ft; AG 515 ft)

3.6 Transposition of Grid

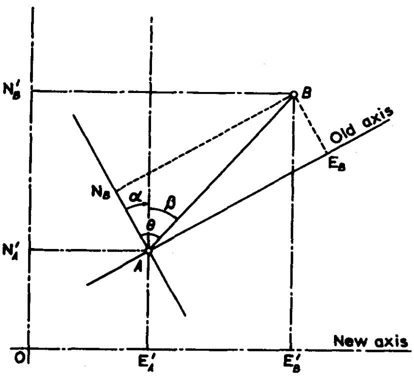


Fig. 3.26 Transposition of grid

Let the line AB (Fig. 3.26) based upon an existing co-ordinate system have a bearing θ and length s.

Then
$$\Delta E_{AB} = s \sin\theta$$
$$\Delta N_{AB} = s \cos\theta$$

The co-ordinate system is now to be changed so that the origin of the new system is 0.

The co-ordinates of the position of 'slew' or rotation
$$A = E'_A N'_A$$
and the axes are rotated clockwise through an angle $+\alpha$ to give a new bearing of AB.

i.e. $\beta = \theta - \alpha$

or $\underline{\alpha = \theta - \beta}$, i.e. Old bearing − New bearing (3.28)

The new co-ordinates of B may now be computed:

$$E'_B = E'_A + s \sin\beta$$
$$= E'_A + s \sin\theta \cos\alpha - s\cos\theta \sin\alpha$$
$$\underline{E'_B = E'_A + \Delta E_{AB} \cos\alpha - \Delta N_{AB} \sin\alpha} \qquad (3.29)$$

Similarly,
$$N'_B = N'_A + s \cos\beta$$

$$= N'_A + s\cos\theta\cos\alpha + s\sin\theta\sin\alpha$$

$$\underline{N'_B = N'_A + \Delta N_{AB}\cos\alpha + \Delta E_{AB}\sin\alpha} \quad (3.30)$$

If a scale factor k is required (e.g. to convert feet into metres), then,

$$E'_B = E'_A + \Delta E'_{AB}$$

$$= E'_A + k[\Delta E_{AB}\cos\alpha - \Delta N_{AB}\sin\alpha] \quad (3.31)$$

and

$$N'_B = N'_A + \Delta N'_{AB}$$

$$= N'_A + k[\Delta N_{AB}\cos\alpha + \Delta E_{AB}\sin\alpha] \quad (3.32)$$

From the above,

$$\Delta E'_{AB} = k[\Delta E_{AB}\cos\alpha - \Delta N_{AB}\sin\alpha]$$

$$= m\,\Delta E_{AB} - n\,\Delta N_{AB} \quad (3.33)$$

$$\Delta N'_{AB} = m\,\Delta N_{AB} + n\,\Delta E_{AB} \quad (3.34)$$

where $m = k\cos\alpha$ and $n = k\sin\alpha$

If the angle of rotation (α) is very small, the equations are simplified as $\cos\alpha \to 0$ and $\sin\alpha \to \alpha$ radians.

$$E'_B = E'_A + k[\Delta E_{AB} - \Delta N_{AB}\,\alpha] \quad (3.35)$$

$$N'_B = N'_A + k[\Delta N_{AB} + \Delta E_{AB}\,\alpha] \quad (3.36)$$

Example 3.16 Transposition of grid

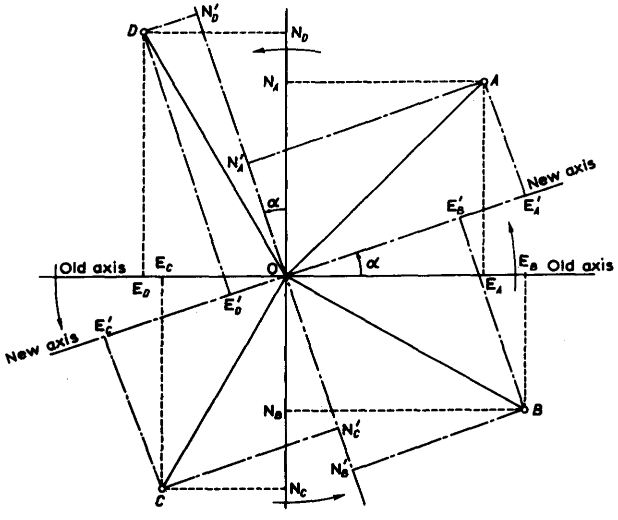


Fig. 3.27

In Fig. 3.27,

					ΔE	ΔN
Let	OA = 045°	i.e.	N 45° E	400	+282·84	+282·84
	OB = 120°		S 60° E	350	+303·10	−175·00
	OC = 210°		S 30° W	350	−175·00	−303·10
	OD = 330°		N 30° W	400	−200·00	+346·40

If the axes are now rotated through −15° the bearings will be increased by +15°.

				$\Delta E'$	$\Delta N'$
∴	OA' = 060°	N 60° E	400	+346·40	+200·00
	OB' = 135°	S 45° E	350	+247·49	−247·49
	OC' = 225°	S 45° W	350	+247·49	−247·49
	OD' = 345°	N 15° W	400	−103·52	+386·36

Applying the transposition of the grid formulae;

equation (3.29) $\quad \Delta E' = \Delta E \cos\alpha - \Delta N \sin\alpha$

equation (3.30) $\quad \Delta N' = \Delta N \cos\alpha + \Delta E \sin\alpha$

	ΔE	ΔN	$\Delta E \cos\alpha$	$\Delta N \sin\alpha$	$\Delta N \cos\alpha$	$\Delta E \sin\alpha$	$\Delta E'$	$\Delta N'$
OA	+282·84	+282·84	+273·20	−73·20	+273·20	−73·20	+346·40	+200·00
OB	+303·10	−175·00	+292·77	+45·29	−169·04	−78·45	+247·48	−247·49
OC	−175·00	−303·10	−169·04	+78·45	−292·77	+45·29	−247·49	−247·48
OD	−200·00	+346·40	−193·19	−89·68	+334·60	+51·76	−103·51	+386·36

N.B. (1) $\cos(-15°) = +0{\cdot}965\,93$

(2) $\sin(-15°) = -0{\cdot}258\,82$

(3) If the point of rotation (slew) had a co-ordinate value ($E'_0 N'_0$) based on the new axes, these values would be added to the partial values, $\Delta E' \Delta N'$ to give the new co-ordinate values.

3.7 The National Grid Reference System

Based on the Davidson Committee's recommendations, all British Ordnance Survey Maps will, on complete revision, be based on the National Grid Reference System with the metre as the unit.

The origin of the 'Modified Transverse Mercator Projection' for the British Isles is

Latitude 49° N
Longitude 2° W

To provide positive co-ordinates for the reference system a 'False Origin' was produced by moving the origin 100 km North and 400 km West.

The basic grid is founded upon a 100 km square; commencing from the false origin which lies to the S.W. of the British Isles, and all squares are referenced by relation to this corner of the square.

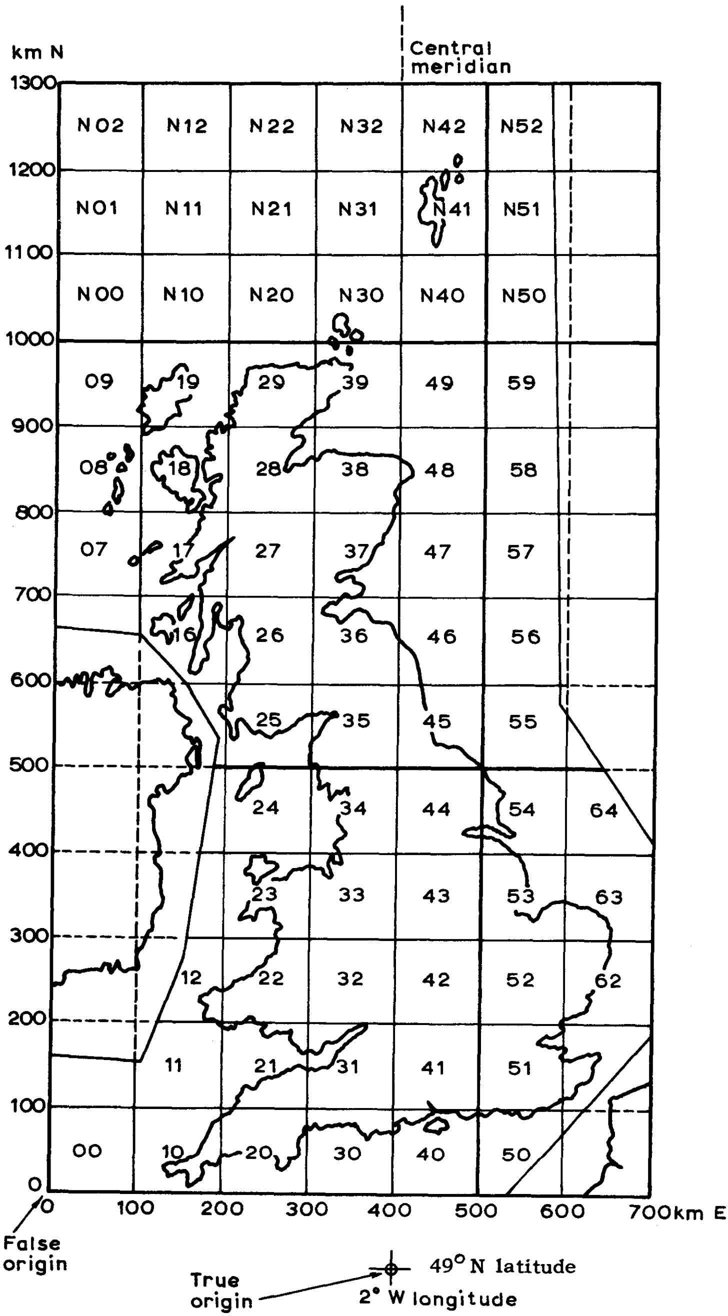


Fig. 3.28 Old O.S. grid reference system

'Eastings are always quoted first.'

Originally the 100 km squares were given a reference based on the number of 100 kilometres East and North from the origin (see Fig. 3.28).

Subsequently, 500 km squares were given prefix letters of S, N and H, and then each square was given a letter of the alphabet (neglecting I). To the right of the large squares the next letter in the alphabet gives the appropriate prefixes, T, O and J (see Fig. 3.29).

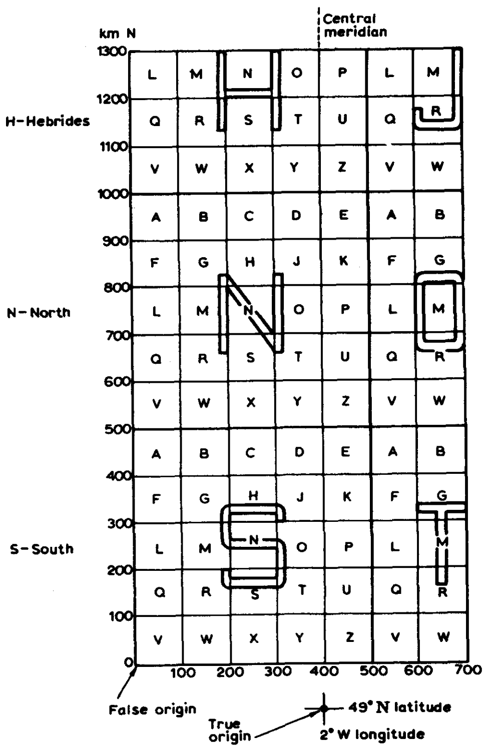


Fig. 3.29 New O.S. grid reference system

Square 32 becomes SO
43 becomes SK
17 becomes NM

The basic reference map is to the scale 1/25 000 (i.e. approximately 2½ inches to 1 mile), Fig. 3.30.

Each map is prefixed by the reference letters followed by two digits representing the reference numbers of the SW corner of the sheet. See example (Fig. 3.30), i.e. SK 54. This shows the relationship between the various scaled maps and the manner in which each sheet is referenced.

A point P in Nottingham Regional College of Technology has the grid co-ordinates E 457 076·32 m, N 340 224·19 m. Its full 'Grid Reference' to the nearest metre is written as SK/5740/076 224 and the sheets on which it will appear are:

Reference	Scale		Sheet size	Grid size
SK 54	1/25 000	($\simeq$ 2½ in. to 1 mile)	10 km	1 km
SK 54 SE	1/10 560	(6 in. to 1 mile)	5 km	1 km
SK 57 40	1/2 500	($\simeq$ 25 in. to 1 mile)	1 km	100 m
SK 57 40 SW	1/1 250	($\simeq$ 50 in. to 1 mile)	500 m	100 m

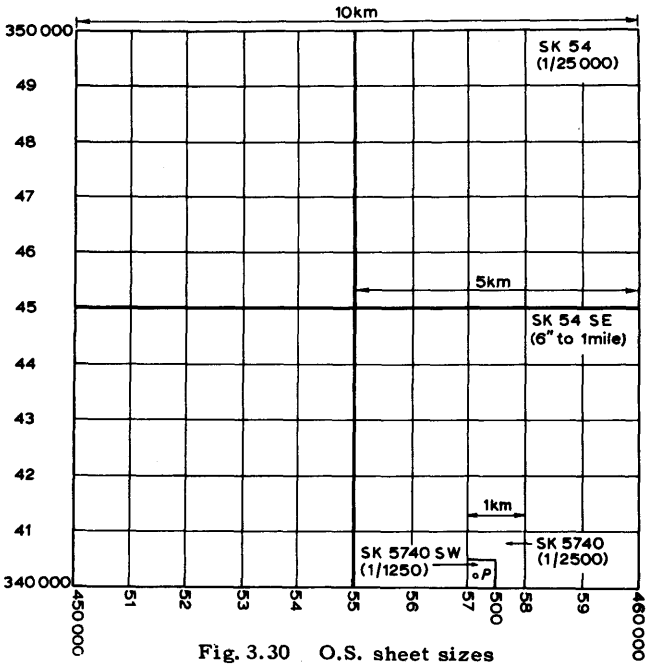


Fig. 3.30 O.S. sheet sizes

Exercises 3(d) (Co-ordinates)

16. The co-ordinates of stations A and B are as follows:

	Latitude	Departure
A	+8257 m	+1321 m
B	+7542 m	− 146 m

Calculate the length and bearing of AB (Ans. 244° 01′; 1632 m)

17. The co-ordinates of two points A and B are given as:

A	N 188·6 m	E 922·4 m
B	S 495·4 m	E 58·6 m

Calculate the co-ordinates of a point P midway between A and B.
(Ans. S 153·4 m, E 490·5 m)

18. The bearings of a traverse have been referred to the magnetic meridian at the initial station A and the total co-ordinates of B, relative to A, are found to be 368 m W, 796 m S.

Calculate (a) the length and magnetic bearing of AB,

(b) the true bearing of AB assuming that the magnetic declination is $13° 10'$ W of true north,

(c) the co-ordinates of B with reference to the true meridian at the initial station.

(Ans. (a) AB mag. bearing $204° 49'$ 877 m
(b) true bearing $191° 39'$;
(c) corrected co-ordinates 177·1 W, 858·9 S)

19. The co-ordinates, in metres of two points, X and Y, are as follows:

X	W 582·47 m	N 1279·80 m
Y	E 1191·85 m	S 755·18 m

Calculate the length and bearing of XY.
(Ans. 2699·92 m; $138° 54' 50''$)

20. Survey station X has a Northing 424·4 ft, Easting 213·7 ft and a height above Ordnance datum of 260·8 ft.

Station Y has a Northing 1728·6 ft, Easting 9263·4 ft and a depth below Ordnance datum 763·2 ft.

Find the length, bearing and inclination of a line joining XY.
(Ans. 9143·3 ft; $081° 47' 54''$; 1 is 8·92)

21. The co-ordinates of four survey stations are given below:

Station	North (ft)	East (ft)
A	718	90
B	822	469
C	164	614
D	210	81

Calculate the co-ordinates of the intersection of the lines AC and BD.
(L.U. Ans. N 520, E 277)

22. Readings of lengths and whole circle bearings from a traverse carried out by a chain and theodolite reading to 1 minute of arc were as follows, after adjusting the angles:

Line	AB	BC	CD	DE
W.C.B.	$0° 00'$	$35° 40'$	$46° 15\frac{1}{2}'$	$156° 13'$
Length (ft)	487·2	538·6	448·9	295·4

Line	EF	FA	AG	GH	HD
W.C.B.	$180° 00'$	$270° 00'$	$64° 58'$	$346° 25'$	$37° 40'$
Length (ft)	963·9	756·2	459·3	590·7	589·0

Taking the direction AB as north, calculate the latitude and departure of each line. If A is taken as origin and the mean co-ordinates of D as obtained by the three routes are taken as correct, find the co-ordinates of the other points by correcting along each line in proportion to chainage (answers are required correct to the nearest 0·1 ft)

(L.U. Ans. D = 1234·7 ft N, 637·6 ft E)

23. The following notes were taken during a theodolite traverse:
Bearing of line AB 14° 48′ 00″

	Angle observed	Length (metres)	
ABC	198° 06′ 30″	AB	245
BCD	284° 01′ 30″	BC	310
CDE	200° 12′ 30″	CD	480
DEF	271° 33′ 30″	DE	709
EFG	268° 01′ 30″	EF	430
		FG	607

Calculate the length and bearing of the line GA.

(Ans. 220·6 m; N 61° 27′ 40″ W)

24. From the following notes, calculate the length and bearing of the line DA:

Line	Bearing	Length
AB	015° 30′	630 m
BC	103° 45′	540 m
CD	270° 00′	227 m

(Ans. 668 m; S 44° 13′ W)

25. The notes of an underground traverse in a level seam are as follows:

Line	Azimuth	Distance (ft)
AB	30° 42′	–
BC	86° 24′	150·6
CD	32° 30′	168·3
DE	315° 06′	45·0

The roadway DE is to be continued on its present bearing to a point F such that F is on the same line as AB produced.

Calculate the lengths of EF and FB.

(M.Q.B./M Ans. EF 88·9 ft; FB 286·2 ft)

26. A shaft is sunk to a certain seam in which the workings to the dip have reached a level DE. It is proposed to deepen the shaft and connect the point E in the dip workings to a point X by a cross-measures drift, dipping at 1 in 200 towards X. The point X is to be

134 ft from the centre of the shaft A and due East from it, AX being level.

The following are the notes of a traverse made in the seam from the centre of the shaft A to the point E.

Line	Azimuth	Distance	Vertical Angle
AB	270° 00′	127	Level
BC	184° 30′	550	Dipping 21°
CD	159° 15′	730	Dipping 18½°
DE	90° 00′	83	Level

Calculate (a) the azimuth and horizontal length of the drift EX and (b) the amount by which it is necessary to deepen the shaft.

(M.Q.B./M Ans. (a) 358° 40′ 1159·6 ft
(b) 434·5 ft)

27. The notes of a traverse between two points A and E in a certain seam are as follows:

Line	Azimuth	Distance (ft)	Angle of Inclination
AB	89°	600	+6°
BC	170°	450	−30°
CD	181°	550	level
DE	280°	355	level

It is proposed to drive a cross-measures drift from a point E to another point F exactly midway between A and B.

Calculate the azimuth and length EF.

(M.Q.B./M Ans. 359° 33′; 867 ft, 888·3 ft inclined)

28. Undernoted are details of a short traverse between the faces of two advancing headings, BA and DE, which are to be driven forward until they meet:

Line	Azimuth	Distance
AB	80°	270·6 ft
BC	180°	488·0 ft
CD	240°	377·0 ft
DE	350°	318·0 ft

Calculate the distance still to be driven in each heading.

(M.Q.B./M Ans. BA + 168·4 ft; DE + 291·5 ft)

29. In order to set out the curve connecting two straights of a road to be constructed, the co-ordinates on the National Grid of I, the point of intersection of the centre lines of the straights produced, are required.

A is a point on the centre line of one straight, the bearing AI being 72° 00′ 00″, and B is a point on the centre line of the other straight, the bearing IB being 49° 26′ 00″.

Using the following data, calculate with full checks the co-ordinates of I.

	Eastings (ft)	Northings (ft)
A	+43 758·32	+52 202·50
B	+45 165·97	+52 874·50

The length AB is 1559·83 ft and the bearing 64° 28′ 50″.

(N.U. Ans. E + 45 309·72 N + 52 706·58)

30. It is proposed to sink a vertical shaft to connect X on a roadway CD in the upper horizon with a roadway GH in the lower horizon which passes under CD. From surveys in the two horizons the following data are compiled:

Upper horizon

Station	Horizontal Angle	Inclination	Inclined Length	Remarks
A				co-ordinates of A
		+1 in 200	854·37	E 6549·10 ft
B	276° 15′ 45″			N 1356·24 ft
		+1 in 400	943·21	Bearing AB
C	88° 19′ 10″			N 30° 14′ 00″ E.
		Level	736·21	
D				

Lower horizon

Station	Horizontal Angle	Inclination	Inclined Length	Remarks
E				Co-ordinates of E
		+1 in 50	326·17	E 7704·08 ft
F	193° 46′ 45″			N 1210·88 ft
		+1 in 20	278·66	Bearing EF
G	83° 03′ 10″			N 54° 59′ 10″ E.
		level	626·10	
H				

Calculate the co-ordinates of X (Ans. E 8005·54 ft, N 1918·79 ft)

31. The surface levels of two shafts X and Y and their depth are respectively as follows:

	Surface Level	Depth
X	820·5 ft A.O.D.	200 yd
Y	535·5 ft A.O.D.	150 yd

The co-ordinates of the centre of the two shafts in fact, are respectively as follows:

	E	N
X	$-778{\cdot}45$	$+2195{\cdot}43$
Y	$+821{\cdot}55$	$+\ 359{\cdot}13$

Calculate the length and gradient of a cross-measures drift to connect the bottom of the shaft.

(Ans. 2439·3 ft (incl.), 2435·6 ft (hor.); 3° 10′, i.e. 1 in 18)

32. The co-ordinates of A are N 25 m E 13 m. From A a line AB runs S 44° 11′ E for 117 m. On the line AB an equilateral triangle ABC is set out with C to the north of AB.

Calculate the co-ordinates of B and C.

(Ans. B E + 94·5 m, N − 55·9 m. C E + 126·4 m, N + 53·7 m)

33. (a) Calculate the gradient (as a percentage) between two points, M and N, which have been co-ordinated and heighted as given below:

Point	Co-ordinates E (ft)	Co-ordinates N (ft)	Height (ft)
M	6206·5	3465·2	212·4
N	5103·2	2146·8	196·6
O	6002·5	2961·4	–

(b) Determine the length (in centimetres) of the line MN when plotted at a scale 1 : 500 (assume 1 ft = 0·3048 m).

(c) Calculate the bearing of the line MO

(R.I.C.S. Ans. 0·92%; 104·8 cm; 202° 03′)

34. From an underground traverse between 2 shaft wires A and D the following partial co-ordinates in feet were obtained:

AB	E 150·632 ft,	S 327·958 ft
BC	E 528·314 ft,	N 82·115 ft
CD	E 26·075 ft,	N 428·862 ft

Transform the above partials to give the total Grid co-ordinates of *station* B given that the Grid co-ordinates of A and D were:

A	E 520 163·462 metres,	N 432 182·684 metres
D	E 520 378·827 metres,	N 432 238·359 metres

(Aide memoire)
$$X = x_1 + K(x - y\theta)$$
$$Y = y_1 + K(y + x\theta)$$

(N.R.C.T.)

Bibliography

MINISTRY OF DEFENCE, *Textbook of Topographical Surveying*, 4th ed. (H.M.S.O.)

GLENDENNING, J., *Principles of Surveying* (Blackie)
CLARK, D., *Plane and Geodetic Surveying*, Vol. 1 (Constable)
HOLLAND, J.L., WARDELL, K. and WEBSTER, A.C., *Surveying*, Coal-mining Series, (Virtue)
A Brief Description of the National Grid and Reference System. O.S. Booklet No. 1/45 (H.M.S.O.)
MIDDLETON and CHADWICK, *Treatise on Surveying*, Vol. 1 (Spon).
SALMON, V.G., *Practical Surveying and Field Work* (Griffin)
BRINKER, R.C. and TAYLOR, W.C., *Elementary Surveying*, 4th ed. (International Textbook Co.)

Appendix

Comparison of scales

Scales in common use with the metric system		Scales in common use with the foot/inch system	
Recommended by BSI	Other Alternative Scales		
1 : 1000000		1 : 1000000	$\frac{1}{16}$ in to 1 mile approx.
1 : 500000	1 : 625000	1 : 625000	$\frac{1}{10}$ in to 1 mile approx.
1 : 200000	1 : 250000	1 : 250000	$\frac{1}{4}$ in to 1 mile approx.
1 : 100000	1 : 125000	1 : 126720	$\frac{1}{2}$ in to 1 mile
1 : 50000	1 : 62500	1 : 63360	1 in to 1 mile
1 : 20000	1 : 25000	1 : 25000	$2\frac{1}{2}$ in to 1 mile approx.
1 : 10000		1 : 10560	6 in to 1 mile
1 : 5000			
1 : 2000	1 : 2500	1 : 2500	1 in to 208.33 ft
1 : 1000	1 : 1250	1 : 1250	1 in to 104.17 ft
1 : 500		1 : 500	1 in to 41.6 ft
		1 : 384	$\frac{1}{32}$ in to 1 ft
1 : 200		1 : 192	$\frac{1}{16}$ in to 1 ft
1 : 100		1 : 96	$\frac{1}{8}$ in to 1 ft
1 : 50		1 : 48	$\frac{1}{4}$ in to 1 ft
1 : 20		1 : 24	$\frac{1}{2}$ in to 1 ft
1 : 10		1 : 12	1 in to 1 ft
1 : 5		1 : 4	3 in to 1 ft

From *Chartered Surveyor*, March, 1968.

4 INSTRUMENTAL OPTICS

4.1 Reflection at Plane Surfaces

4.11 Laws of reflection

(1) The incident ray, the reflected ray, and the normal to the mirror at the point of incidence all lie in the same plane.

(2) The angle of incidence (i) = the angle of reflection (r).

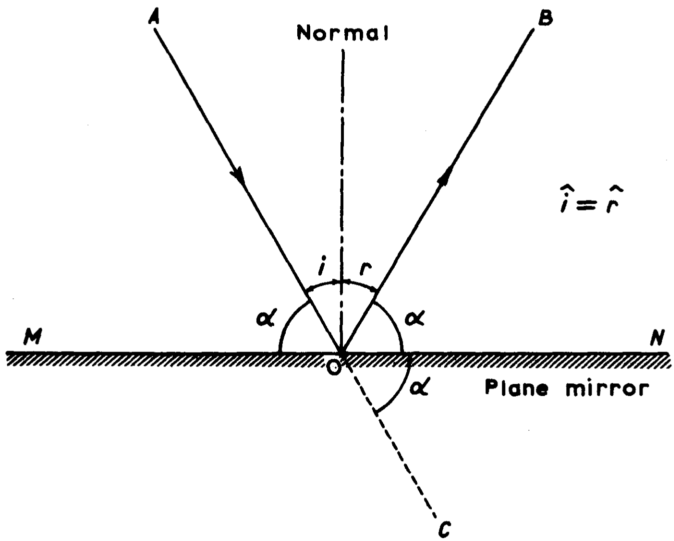


Fig. 4.1

The ray AO, Fig. 4.1, is inclined at α (glancing angle MOA) to the mirror MN. Since $i = r$, angle $BON = MOA = \alpha$. If AO is produced to C,

$$\text{Angle } MOA = NOC = BON = \alpha$$

Thus the deviation of the ray AO is 2α. Therefore *the deviation angle is twice the glancing angle,*

$$\text{i.e.} \quad D = 2\alpha \tag{4.1}$$

4.12 Deviation by successive reflections on two inclined mirrors (Fig. 4.2)

Ray AB is incident on mirror M_1N_1 at a glancing angle α. It is thus deflected by reflection $+2\alpha$.

The reflected ray BC incident upon mirror M_2N_2 at a glancing angle β is deflected by reflection -2β (here clockwise is assumed +ve).

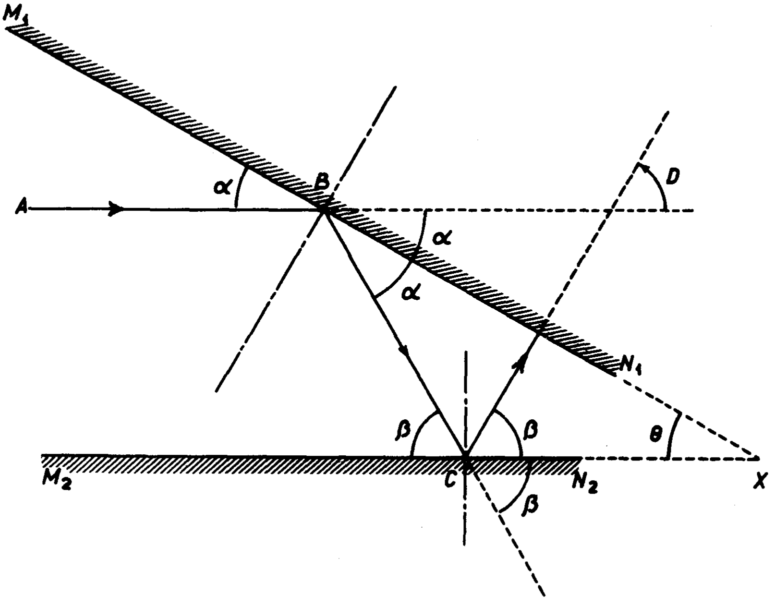


Fig. 4.2

The total deflection D is thus $(2\alpha - 2\beta) = 2(\alpha - \beta)$.

In triangle BCX, $\beta = \alpha + \theta$

$\therefore \quad \theta = \beta - \alpha$

i.e. $D = 2(\alpha - \beta) = 2\theta$ (4.2)

As θ is constant, *the deflection after two successive reflections is constant and equal to twice the angle between the mirrors.*

4.13 The optical square (Fig. 4.3)

This instrument, used for setting out right angles, employs the above principle.

By Eq. (4.2), the deviation of any ray from O_2 incident on mirror M_2 at an angle α to the normal $= 2\theta$, i.e. $2 \times 45° = 90°$.

4.14 Deviation by rotating the mirror (Fig. 4.4)

Let the incident ray AO be constant, with a glancing angle α. The mirror M_1N_1 is then rotated by an anticlockwise angle β to M_2N_2.

When the glancing angle is α the deviation angle is 2α.

After rotation the glancing angle is $(\alpha + \beta)$ and the deviation angle is therefore $2(\alpha + \beta)$.

Thus the reflected ray is rotated by

$$\phi = 2(\alpha + \beta) - 2\alpha = 2\beta \tag{4.3}$$

If the incident ray remains constant the reflected ray deviates by twice the angular rotation of the mirror.

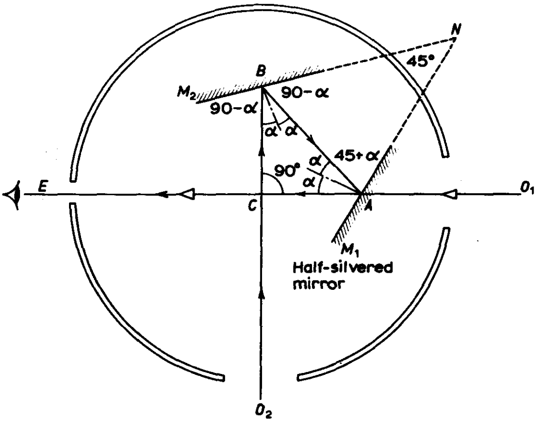


Fig. 4.3 Optical square

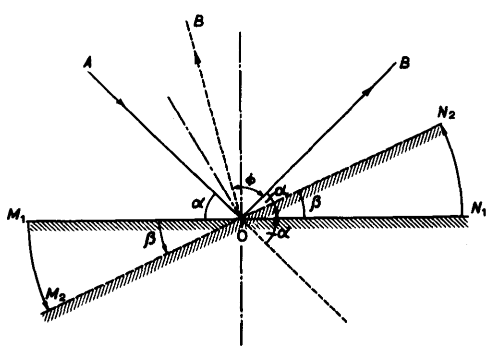


Fig. 4.4

4.15 The sextant

Principles of the sextant (Figs. 4.5 and 4.6)

Mirror M_1, silvered, is connected to a pointer P. As M_1 is rotated the pointer moves along the graduated arc.

Mirror M_2 is only half-silvered and is fixed.

When the reading at P is zero, Fig. 4.5, the image K, reflected from both mirrors, should be seen simultaneously with K through the plain glass part of M_2. With a suitable object K as the horizon, the mirrors should be parallel.

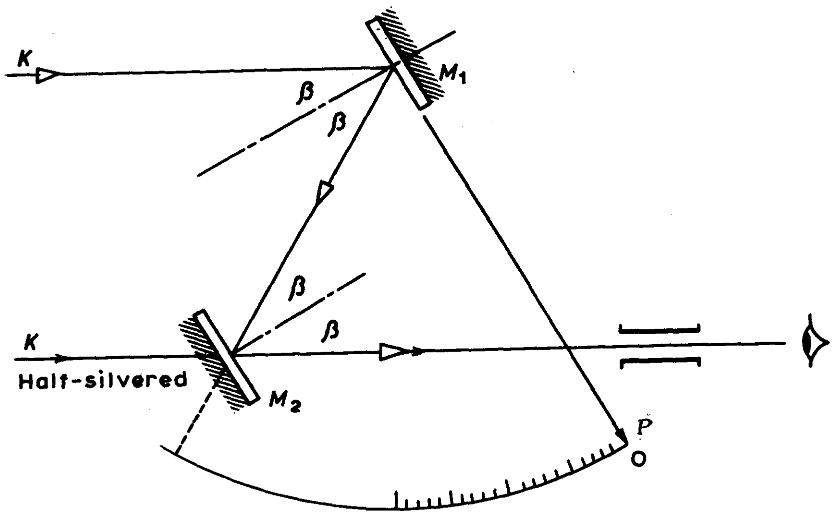


Fig. 4.5 Zero setting on the sextant

When observing an elevated object S, Fig. 4.6, above the horizon K, the mirror M_1 is rotated through angle ϕ until S is simultaneously observed with K. The angle being measured is therefore θ.

From triangle M_1EM_2, $\quad 2\alpha = 2\beta + \theta$

$$\therefore \quad \theta = 2(\alpha - \beta)$$

from triangle M_1QM_2, $\quad 90 + \alpha = 90 + \beta + \phi$

$$\therefore \quad \alpha = \beta + \phi$$

$$\therefore \quad \phi = \alpha - \beta = \tfrac{1}{2}\theta \qquad (4.4)$$

i.e. *the rotation of the mirror is half the angle of elevation.*

When the mirrors are parallel, $\phi = O$ with the index pointer MP at zero on the graduated arm. When the angle θ is being observed, the mirror is turned through angle ϕ, but the recorded value $MP_2 = \theta$, i.e. 2ϕ.

N.B. (1) Horizontal angles are only measured if the objects are at the same height relative to the observer, which means that in most cases the angle measured is in an inclined plane. The horizontal angle may be computed from the equation (see page 107):

$$\cos \text{H.A.} = \frac{\cos\theta - \cos\alpha_1 \cos\alpha_2}{\sin\alpha_1 \sin\alpha_2}$$

where θ = the measured angle in the inclined plane and α_1 and α_2 = vertical angles.

(2) Vertical angles must be measured relative to a true or an artificial horizon.

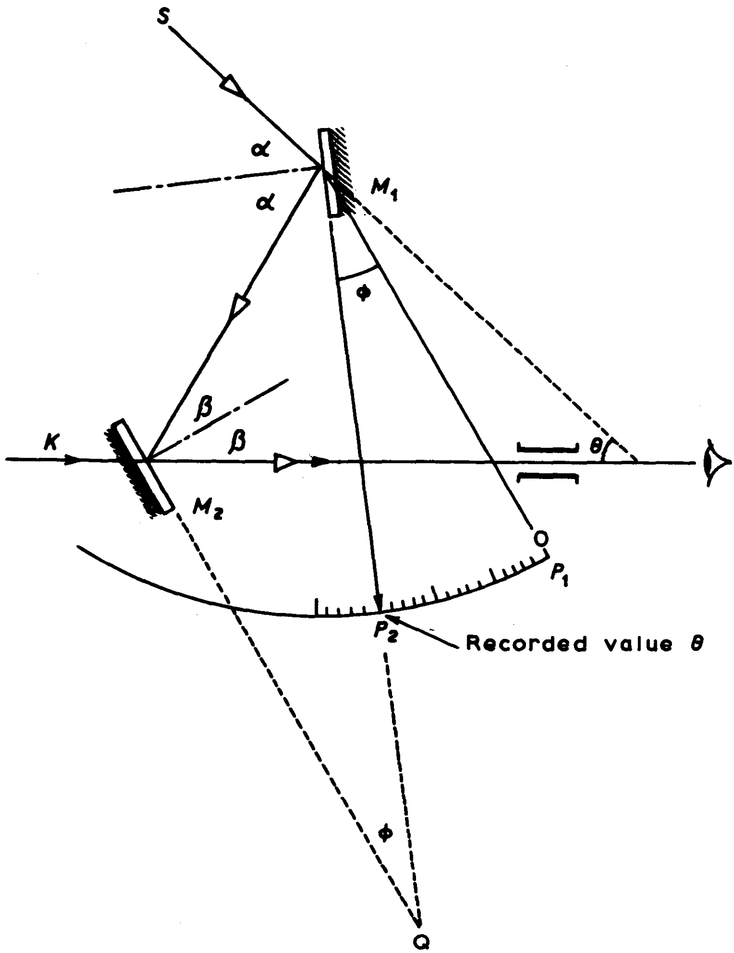


Fig. 4.6 Principles of the sextant

4.16 Use of the true horizon

(a) As the angle of deviation, after two successive reflections, is independent of the angle of incidence on the first mirror, the object will continue to be seen on the horizon no matter how much the observer moves. Once the mirror M_1 has been set, the angle between the mirrors is set, and the observed angle recorded.

This is the main advantage of the sextant as a hand instrument, particularly in marine and aerial navigation where the observer's position is unstable.

(b) If the observer is well above the horizon, a correction $\delta\theta$ is required for the dip of the horizon, Fig. 4.8.

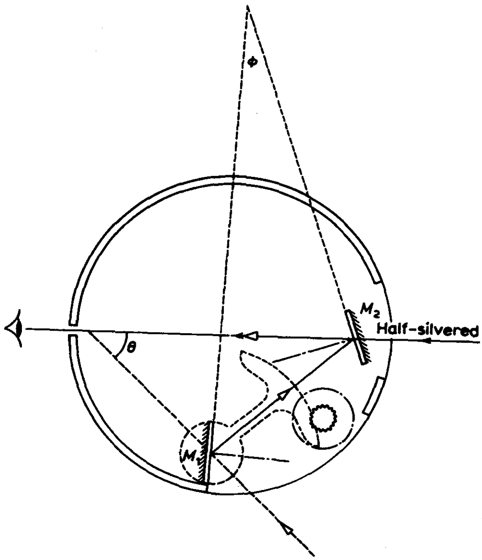


Fig. 4.7 Box sextant

The Nautical Almanac contains tables for the correction factor $\delta\theta$ due to the dip of the horizon based on the equation:

$$\delta\theta = -0{\cdot}97\sqrt{h} \text{ minutes} \tag{4.5}$$

where h = height in feet above sea level,

$$\text{or}\quad \delta\theta = -1{\cdot}756\sqrt{H} \text{ minutes}$$

where H = height in metres above sea level.

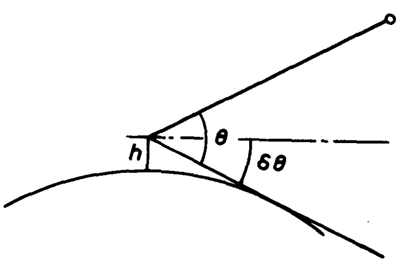


Fig. 4.8 Dip of the horizon

4.17 Artificial horizon (Fig. 4.9)

On land, no true horizon is possible, so an 'artificial horizon' is employed. This consists essentially of a trough of mercury, the surface of which assumes a horizontal plane forming a mirror.

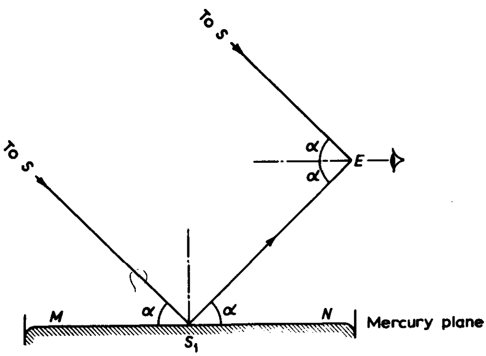


Fig. 4.9 Artificial horizon

The vertical angle observed between the object S and the reflection of the image S_1 in the mercury is twice the angle of altitude (α) required.

$$\text{Observed angle} = S_1ES = 2\alpha$$
$$\text{True altitude} = MS_1S = \alpha$$

Rays SE and SS_1 are assumed parallel due to the distance of S from the instrument.

4.18 Images in plane mirrors

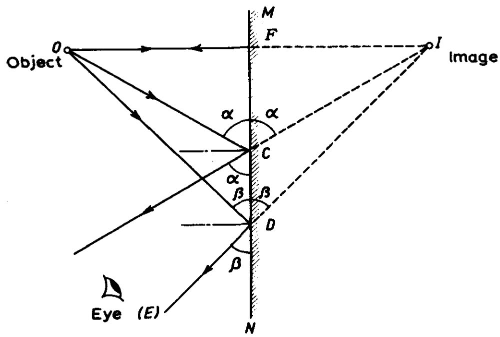


Fig. 4.10 Images in plane mirrors

Object O in front of the mirror is seen at E as though it were situated at I, Fig. 4.10.

From the glancing angles α and β it can be seen that

(a) triangles OFC and ICF are congruent,

(b) triangles OFD and IDF are congruent.

Thus the point I (image) is the same perpendicular distance from the mirror as O (object), i.e. $OF = FI$.

4.19 Virtual and real images

As above, the rays reflected from the mirror appear to pass through I, the image thus being *unreal* or *virtual*. For the image to be real, the object would have to be virtual.

The real test is whether the image can be received on a screen: if it can be – it is real, if not – it is virtual.

4.2 Refraction at Plane Surfaces

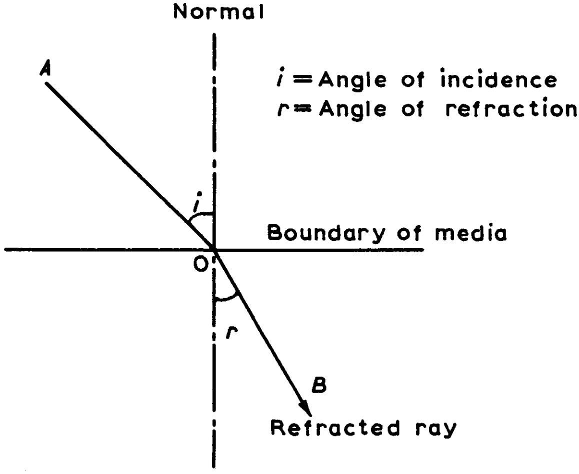


Fig. 4.11

The incident ray AO, meeting the boundary between two media, e.g. air and glass, is refracted to B, Fig. 4.11.

4.21 Laws of refraction

(1) The incident ray, the refracted ray, and the normal to the boundary plane between the two media at the point of incidence all lie in the same plane.

(2) For any two given media the ratio $\frac{\sin i}{\sin r}$ is a constant known as the refractive index (the light assumed to be monochromatic).

Thus $$\text{Refractive Index} = \frac{\sin i}{\sin r} \tag{4.6}$$

4.22 Total internal reflection (Fig. 4.12)

If a ray AO is incident on a glass/air boundary the ray may be refracted or reflected according to the angle of incidence.

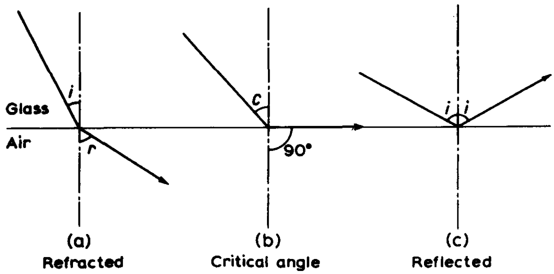


Fig. 4.12

When the angle of refraction is 90°, the critical angle of incidence is reached, i.e.

$$_g\mu_a = \frac{\sin c}{\sin 90^\circ} = \sin c \tag{4.7}$$

For crown glass the refractive index $_a\mu_g \simeq 1{\cdot}5$

$$\therefore \quad \sin c \simeq \frac{1}{1{\cdot}5}$$

$$c \simeq 41^\circ\,30'$$

If the angle of incidence (glass/air) $i > 41^\circ\,30'$, the ray will be internally reflected, and this principle is employed in optical prisms within such surveying instruments as optical squares, reflecting prisms in binoculars, telescopes and optical scale-reading theodolites.

N.B. Total internal reflection can only occur when light travels from one medium to an optically less dense medium, e.g. glass/air.

4.23 Relationships between refractive indices (Fig. 4.13)

(a) *If the refractive index from air to glass is* $_a\mu_g$ *, then the refractive index from glass to air is* $_g\mu_a$

Therefore
$$_g\mu_a = \frac{1}{_a\mu_g} \tag{4.8}$$

e.g., if $_a\mu_g = 1{\cdot}5$ (taking air as 1), then

$$_g\mu_a = \frac{1}{1{\cdot}5} = 0{\cdot}6\dot{6}$$

(b) *Given parallel boundaries of air, glass, air,* then

$$\sin i = \text{constant} \tag{4.9}$$

$$_a\mu_g = \frac{\sin i_a}{\sin i_g}$$

$$_g\mu_a = \frac{\sin i_g}{\sin i_a} = \frac{1}{_a\mu_g}$$

$$\therefore \quad _g\mu_a \sin i_a = {_a\mu_g} \sin i_g = \text{constant}$$

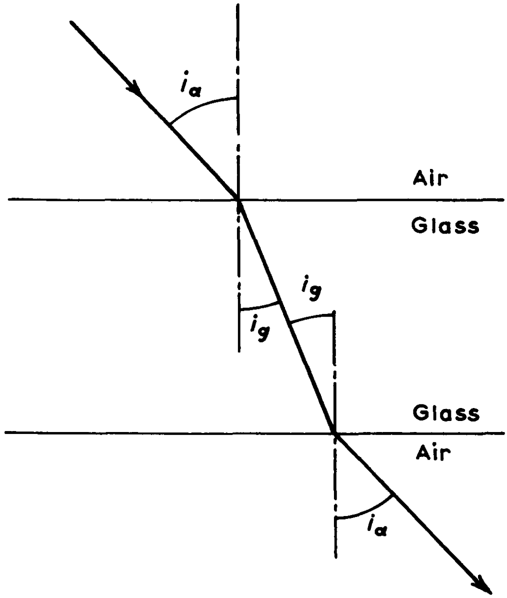


Fig. 4.13

(c) *The emergent ray is parallel to the incident ray when returning to the same medium although there is relative displacement.*

This factor is used in the parallel plate micrometer.

4.24 Refraction through triangular prisms

When the two refractive surfaces are not parallel the ray may be bent twice in the same direction, thus deviating from its former direction by an angle D.

It can be seen from Fig. 4.14 that

$$A = \beta_1 + \beta_2 \tag{4.10}$$

$$\text{and} \quad D = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) \tag{4.11}$$

$$= (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2) \tag{4.12}$$

$$\text{i.e.} \quad D = (\alpha_1 + \alpha_2) - A \tag{4.13}$$

Thus the minimum deviation occurs when $\alpha_1 + \alpha_2 = A$ (4.14)

If A is small, then

$$\alpha = \mu\beta$$

$$\text{and} \quad D = \mu\beta_1 + \mu\beta_2 - A$$

$$= \mu(\beta_1 + \beta_2) - A$$

$$= A(\mu - 1) \tag{4.15}$$

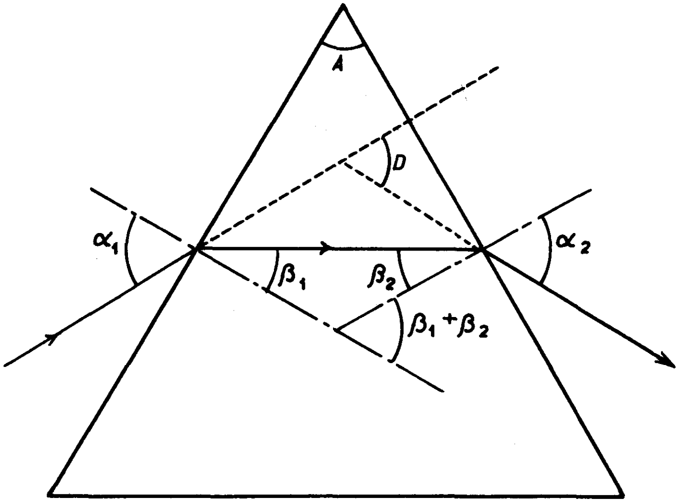


Fig. 4.14 Refraction through a triangular prism

4.25 Instruments using refraction through prisms

The line ranger (Fig. 4.15)

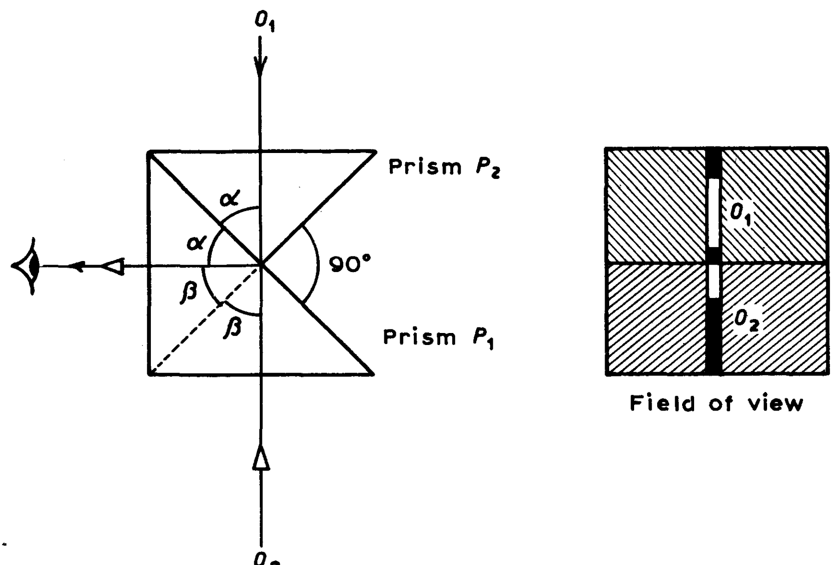


Fig. 4.15 The line ranger

$$\alpha + \beta = 90°$$

$$\therefore \quad 2(\alpha + \beta) = 180°$$

Thus O_1CO_2 is a straight line.

The prism square (Fig. 4.16)

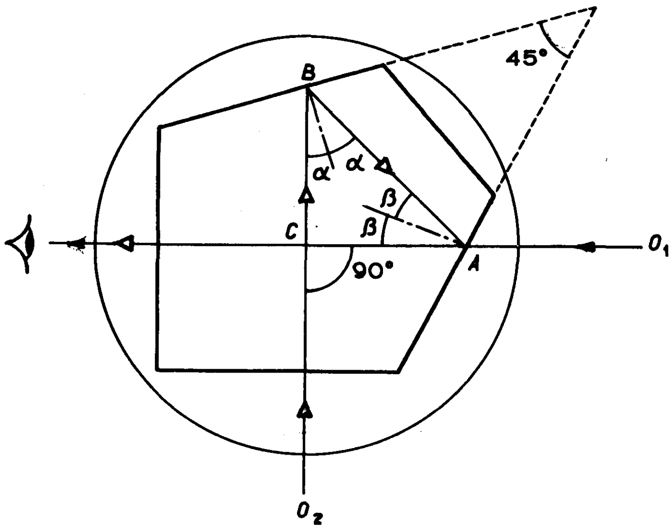


Fig. 4.16 The prism square

This is precisely the same mathematically as the optical square (Fig. 4.3), but light is internally reflected, the incident ray being greater than the critical angle of the glass.

The double prismatic square (Fig. 4.17) combines the advantages of both the above hand instruments.

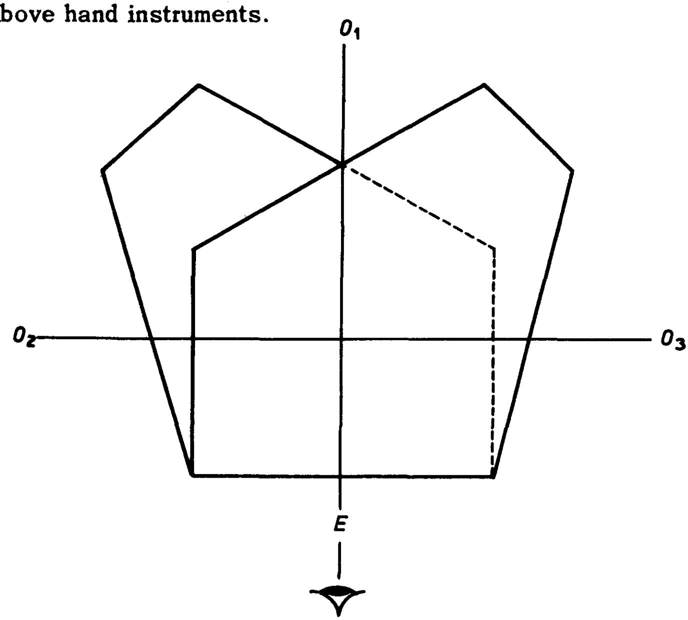


Fig. 4.17

Images O_2 and O_3 are reflected through the prisms. O_1 is seen above and below the prisms.

The parallel plate micrometer (Fig. 4.18)

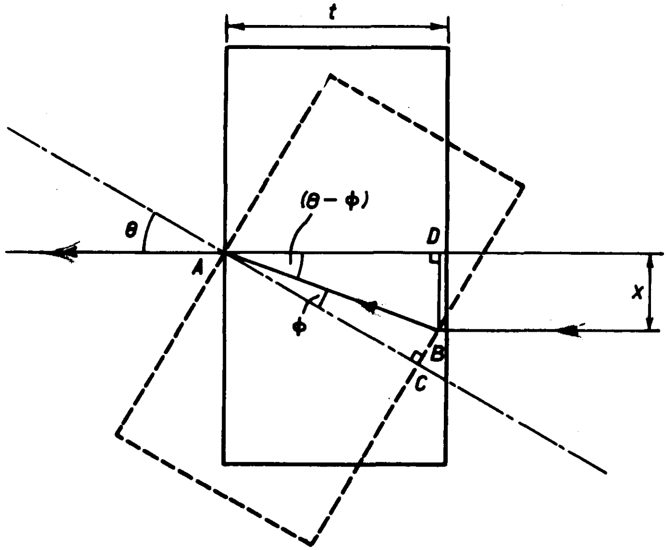


Fig. 4.18 The parallel plate micrometer

A parallel-sided disc of glass of refractive index μ and thickness t is rotated through an angle θ. Light is refracted to produce displacement of the line of sight by an amount x.

$$x = DB = AB \sin(\theta - \phi)$$

$$= \frac{t}{\cos\phi} \times \sin(\theta - \phi)$$

$$= \frac{t(\sin\theta \cos\phi - \cos\theta \sin\phi)}{\cos\phi}$$

$$= t(\sin\theta - \cos\theta \tan\phi)$$

but refractive index

$$\mu = \frac{\sin\theta}{\sin\phi}$$

$$\therefore \quad \sin\phi = \frac{\sin\theta}{\mu} \quad \text{and} \quad \cos\phi = \sqrt{\{1 - \sin^2\phi\}}$$

$$= \sqrt{\left\{1 - \frac{\sin^2\theta}{\mu^2}\right\}}$$

$$= \frac{\sqrt{\{\mu^2 - \sin^2\theta\}}}{\mu}$$

$$\therefore \quad x = t\left[\sin\theta - \frac{\cos\theta \sin\theta}{\sqrt{(\mu^2 - \sin^2\theta)}}\right]$$

$$= t\sin\theta\left[1 - \frac{\cos\theta}{\sqrt{(\mu^2 - \sin^2\theta)}}\right]$$

$$= t\sin\theta\left[1 - \sqrt{\left(\frac{1-\sin^2\theta}{\mu^2 - \sin^2\theta}\right)}\right] \qquad (4.16)$$

If θ is small, then $\sin\theta \simeq \theta_{\text{rad.}}$ and $\sin^2\theta$ may be neglected.

$$\therefore \quad x \simeq t\,\theta\left(1 - \frac{1}{\mu}\right) \qquad (4.17)$$

Example 4.1 A parallel plate micrometer attached to a level is to show a displacement of 0·01 when rotated through 15° on either side of the vertical.

Calculate the thickness of glass required if its refractive index is 1·6.

State also the staff reading to the nearest thousandth of a foot when the micrometer is brought to division 7 in sighting the next lower reading of 4·24, the divisions running 0 to 20 with 10 for the normal position. (L.U.)

Using the formula

$$x = t\sin\theta\left[1 - \sqrt{\left(\frac{1-\sin^2\theta}{\mu^2 - \sin^2\theta}\right)}\right]$$

$$t = \frac{x}{\sin\theta\left[1 - \sqrt{\left\{\frac{(1-\sin\theta)(1+\sin\theta)}{(\mu-\sin\theta)(\mu+\sin\theta)}\right\}}\right]} \text{ in.}$$

$$= \frac{0{\cdot}01 \times 12}{\sin 15^\circ\left[1 - \sqrt{\left\{\frac{(1-\sin 15)(1+\sin 15)}{(1{\cdot}6-\sin 15)(1{\cdot}6+\sin 15)}\right\}}\right]} \text{ in.}$$

$$= \frac{0{\cdot}01 \times 12}{0{\cdot}258\,82 \times 0{\cdot}388\,24} \text{ in.}$$

$$= 1{\cdot}1940 \text{ in.}$$

N.B. If the approximation formula is used, $t = 1{\cdot}222$ in.

The micrometer is geared to the parallel plate and must be correlated. Precise levelling staves are usually graduated in feet and fiftieths of a foot, so the micrometer is also divided into 20 parts, each representing 0·001 ft. (The metric staff requires a metric micrometer).

To avoid confusion, the micrometer should be set to zero before each sight is taken and the micrometer reading is then added to the staff reading as the parallel plate refracts the line of sight to the next lower reading.

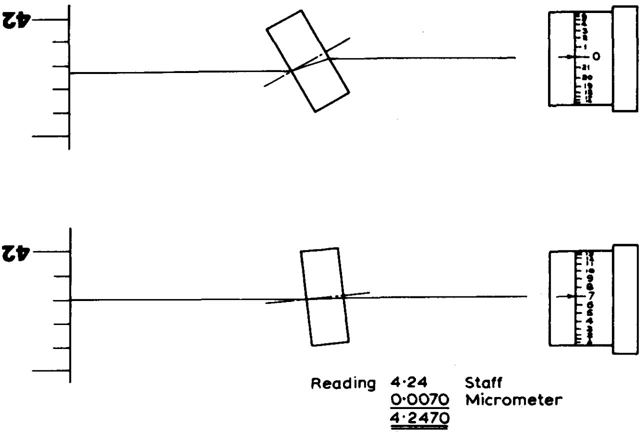


Fig. 4.19 Use of the parallel plate micrometer in precise levelling

Exercises 4(a)

1. Describe the parallel plate micrometer and show how it is used in precise work when attached to a level.

If an attachment of this type is to give a difference of 0·01 of a foot for a rotation of 20°, calculate the required thickness of glass when the refractive index is 1·6.

Describe how the instrument may be graduated to read to 0·001 of a foot for displacements of 0·01 of a foot above and below the mean.

(L.U. Ans. 0·88 in.)

2. Describe the method of operation of a parallel plate micrometer in precise levelling. If the index of refraction from air to glass is 1·6 and the parallel plate prism is 0·6 in. thick, calculate the angular rotation of the prism to give a vertical displacement of the image of 0·001 ft.

(L.U. Ans. 3° 03′ 36″)

4.3 Spherical Mirrors

4.31 Concave or converging mirrors (Fig. 4.20)

A narrow beam of light produces a real principal focus F. P is called the *pole* of the mirror and C is the *centre* of *curvature*. PF is the *focal length* of the mirror.

A ray AB, parallel to the axis, will be reflected to F. BC will be normal to the curve at B, so that

$$\text{Angle } ABC = \text{angle } CBF = \theta.$$

As AB is parallel to PC,

$$\text{Angle } PCB = \text{angle } ABC = \theta$$

$$\therefore \quad BF = FC.$$

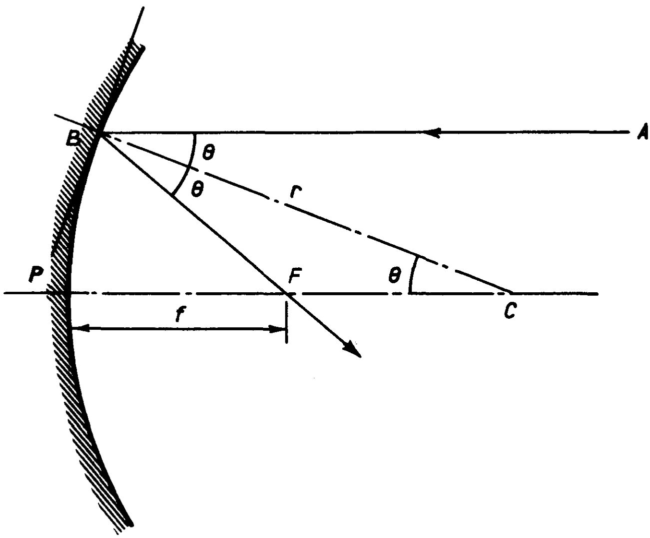


Fig. 4.20 Concave mirror

As the beam is assumed narrow

$$PF \simeq BF \simeq FC$$

$$\therefore \quad PC \simeq 2PF = 2f \simeq r. \tag{4.18}$$

If the beam of light is wide a cusp surface is produced with the apex at the principal focus. The parabolic mirror overcomes this anomaly and is used as a reflector for car headlights, fires, etc, with the light or heat source at the focus.

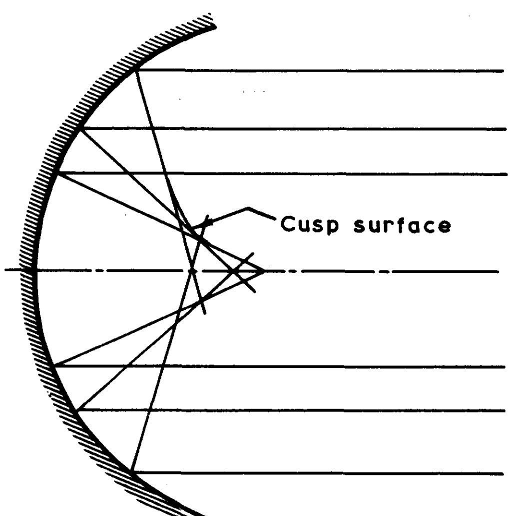


Fig. 4.21 Wide beam on a circular mirror

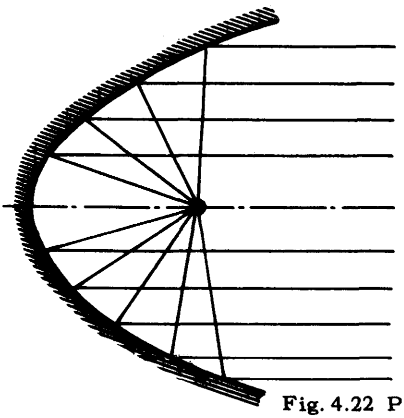


Fig. 4.22 Parabolic mirror

4.32 Convex or diverging mirrors (Fig. 4.23)

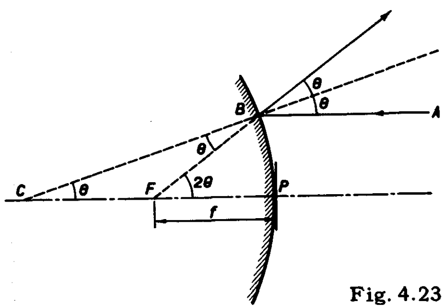


Fig. 4.23 Convex mirror

A narrow beam of light produces a virtual principal point F, being reflected away from the axis.

The angular principles are the same as for a concave mirror and

$$r \simeq 2f$$

4.33 The relationship between object and image in curved mirrors

Assuming a *narrow* beam, the following rays are considered in all cases (Fig. 4.24).

(a) Ray OA, parallel to the principal axis, is reflected to pass through the focus F.

(b) Ray OB, passing through the focus F, is then reflected parallel to the axis.

(c) Ray OD, passing through the centre of curvature C, and thus a line normal to the curve.

N.B. In graphical solutions, it is advantageous to exaggerate the vertical scale, the position of the image remaining in the true position. As the amount of curvature is distorted, it should be represented as a

straight line perpendicular to the axis.

Any two of the above rays produce, at their intersection, the position of the image *I*.

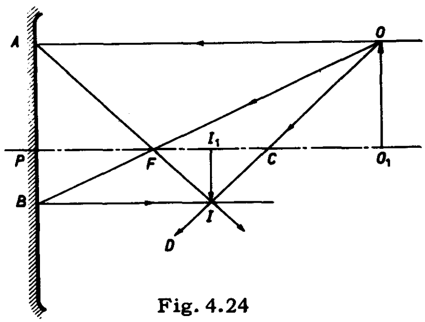


Fig. 4.24

The relationships between object and image for *concave mirrors* are:

(a) When the object is at infinity, the image is small, real, and inverted.

(b) When the object is at the centre of curvature *C*, the image is also at *C*, real, of the same size and inverted.

(c) When the object is between *C* and *F*, the image is real, enlarged and inverted.

(d) When the object is at *F*, the image is at infinity.

(e) When the object is between *F* and *P*, the image is virtual, enlarged and erect.

For *convex mirrors*, in all cases the image is virtual, diminished and erect, Fig. 4.25.

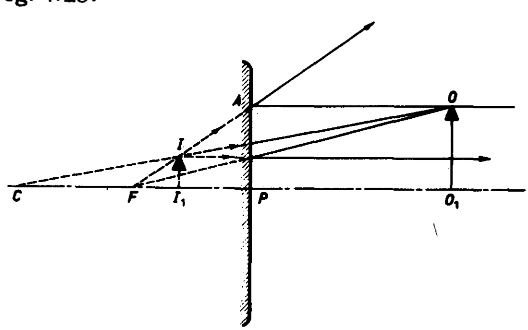


Fig. 4.25

4.34 Sign convention

There are several sign conventions but here the convention *Real-is-positive* is adopted. This has many advantages provided the work is not too advanced.

All real distances are treated as positive values whilst virtual distances are treated as negative values — in all cases distances are measured from the pole.

N.B. In the diagrams real distances are shown as solid lines whilst virtual distances are dotted.

4.35 Derivation of Formulae

Concave mirror (image real), Fig. 4.26

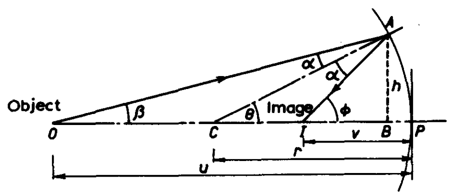


Fig. 4.26

To prove: $\frac{1}{f} = \frac{2}{r} = \frac{1}{u} + \frac{1}{v}$

where f = the focal length of the mirror

r = the radius of curvature

u = the distance of the object from the pole P

v = the distance of the image from the pole P

The ray OA is reflected at A to AI making an equal angle α on either side of the normal AC.

From Fig. 4.26,

$$\theta = \alpha + \beta \qquad \therefore \quad \alpha = \theta - \beta$$

and

$$\begin{aligned} \phi &= 2\alpha + \beta \\ &= 2(\theta - \beta) + \beta \\ &= 2\theta - \beta \end{aligned}$$

i.e. $\phi + \beta = 2\theta.$

As the angles α, β, θ and ϕ are all small, B is closely adjacent to to P.

$$\therefore \quad \phi_{rad} \simeq \sin\phi = \frac{h}{IP} \qquad I \text{ is real so } IP \text{ is +ve.}$$

$$\beta_{rad} \simeq \sin\beta = \frac{h}{OP} \qquad O \text{ is real so } OP \text{ is +ve.}$$

$$\theta_{rad} \simeq \sin\theta = \frac{h}{CP}$$

$$\therefore \quad \frac{h}{IP} + \frac{h}{OP} = \frac{2h}{CP}$$

i.e. $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{2}{r} = \dfrac{1}{f}$ (as $f = \dfrac{r}{2}$) (4.19)

Concave mirror (image virtual), Fig. 4.27

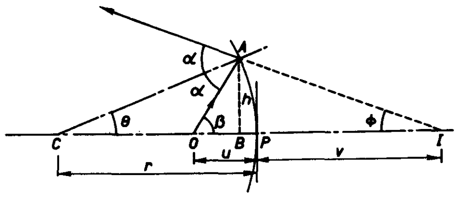


Fig. 4.27

From Fig. 4.27,

$$\theta = \beta - \alpha \qquad \therefore \quad \alpha = \beta - \theta$$

and
$$\phi = 2\alpha - \beta$$
$$= 2(\beta - \theta) - \beta$$
$$= -2\theta + \beta$$
$$\therefore \quad \underline{2\theta = \beta - \phi}$$

As before,
$$\frac{2h}{CP} = \frac{h}{OP} - \frac{h}{IP}$$

i.e.
$$\frac{2}{r} = \frac{1}{u} - \frac{1}{v}$$

but the image is virtual, therefore v is negative.

$$\therefore \quad \frac{2}{r} = \frac{1}{f} = \frac{1}{u} + \frac{1}{v} \qquad (4.19)$$

Convex mirror, Fig. 4.28

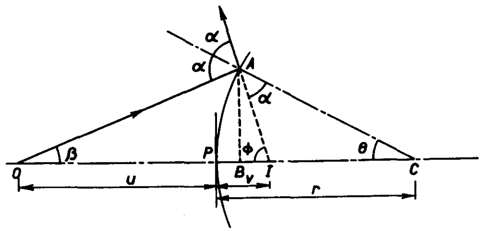


Fig. 4.28

$$\phi = \alpha + \theta \qquad \therefore \quad \alpha = \phi - \theta$$
$$2\alpha = \phi + \beta$$
$$\therefore \quad 2(\phi - \theta) = \phi + \beta$$
i.e.
$$\phi = 2\theta + \beta$$
$$\phi - \beta = 2\theta$$

As before,

$$\phi_{rad} \simeq \sin\phi = \frac{h}{-IP} \qquad (I \text{ is virtual} \quad \therefore IP \text{ is } -\text{ve.})$$

$$\beta_{rad} \simeq \sin\beta = \frac{h}{OP} \qquad (O \text{ is real} \quad \therefore OP \text{ is } +\text{ve}).$$

$$\theta_{rad} \simeq \sin\theta = \frac{h}{-PC} \qquad (C \text{ is virtual} \quad \therefore PC \text{ is } -\text{ve}).$$

Thus

$$\frac{h}{-IP} - \frac{h}{OP} = \frac{2h}{-PC}$$

i.e.

$$\frac{1}{-v} - \frac{1}{u} = \frac{2}{-r}$$

$$\therefore \quad \frac{1}{v} + \frac{1}{u} = \frac{2}{r} = \frac{1}{f} \tag{4.19}$$

Therefore, using the sign convention, the formula is common to both types of mirror in all cases.

4.36 Magnification in spherical mirrors (Fig. 4.29)

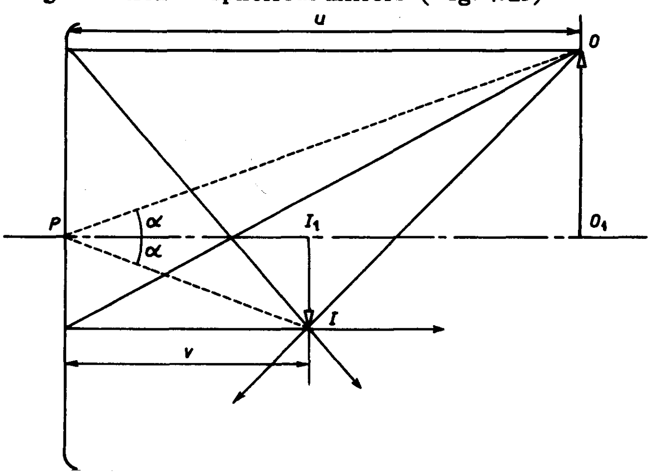


Fig. 4.29 Magnification in spherical mirrors

IB is the image of *OA*.

In the right-angled triangles OPO_1 and IPI_1 the angle α is common, being the angles of incidence and of reflection, and therefore the triangles are similar.

Thus, magnification $= \dfrac{II_1 \text{ (image size)}}{OO_1 \text{ (object size)}} = \dfrac{I_1P\ (v)}{O_1P\ (u)}$

$$\therefore \quad m = \frac{v}{u} \quad \text{neglecting signs} \tag{4.20}$$

Example 4.2 An object 1 in. high is placed on the principal axis 20 in. from a concave mirror which has a radius of curvature of 15 in. Find the position, size and nature of the image.

As the mirror is concave,

$$f = +\frac{15}{2} \text{ in.}$$

the object is real $\quad \therefore \quad u = +20$

Substituting in Eq. (4.19),

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u}$$

$$= \frac{2}{15} - \frac{1}{20} = \frac{1}{12}$$

$$\therefore \quad v = 12 \text{ in.}$$

Thus the image is real (but will be inverted) as v is positive.

Magnification $\quad m = \frac{v}{u} = \frac{12}{20} = 0{\cdot}6$

$\therefore$ Size of image $= 0{\cdot}6$ in.

4.4 Refraction Through Thin Lenses

4.41 Definitions

(a) *Types of lens*

Convex (converging), Fig. 4.30 (a)
Concave (diverging), Fig. 4.30(b)

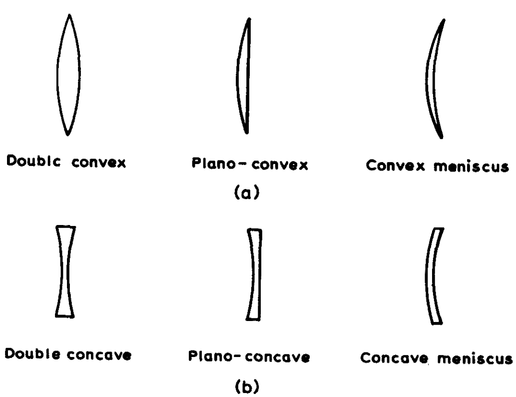


Fig. 4.30 Types of lens

(b) *Focal points* (Fig. 4.31)

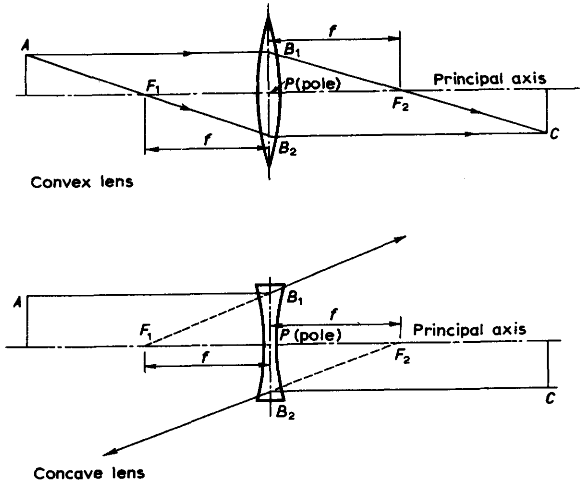


Fig. 4.31 Conjugate foci

4.42 Formation of images (Fig. 4.32)

If a thin lens is assumed to be split into a series of small prisms, any ray incident on the face will be refracted and will deviate by an angle

$$D = A(\mu - 1) \qquad \text{(Eq. 4.15)}$$

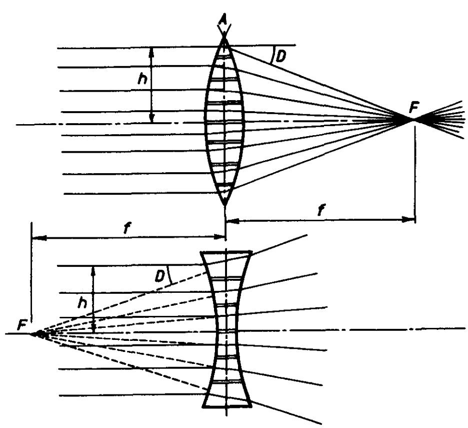


Fig. 4.32 Formation of images

N.B. The deviation angle D is also related to the height h and the focal length f, i.e.

$$D = h/f \qquad (4.21)$$

4.43 The relationship between object and image in a thin lens

The position of the image can be drawn using three rays, Fig. 4.33.

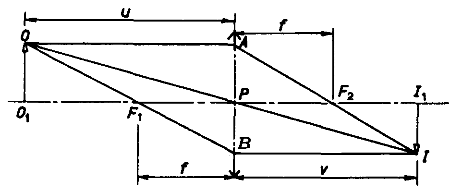


Fig. 4.33

N.B. Two principal foci, F_1 and F_2, exist.

(a) Ray OA parallel to the principal axis is refracted to pass through principal focus F_2.

(b) Ray OB passes through the principal focus F_1 and is then refracted parallel to the principal axis.

(c) Ray OPI passes from object to image through the pole P without refraction.

Convex lens

(a) When the object is at infinity, the image is at the principal focus F_2, real and inverted.

(b) When the object is between infinity and F_1, the image is real and inverted.

(c) When the object is between F_1 and P, the image is virtual, magnified, and erect, i.e. a simple magnifying glass.

Concave lens

The image is always virtual, erect and diminished.

4.44 Derivation of formulae

The real-is-positive sign convention is again adopted, but for convex lenses the real distances and focal lengths are considered positive, whilst for concave lenses the virtual distances and focal lengths are considered negative.

As with mirrors, thin lens formulae depend on small angle approximations.

Convex lens

(a) Image real, Fig. 4.34

$$D = \alpha + \beta$$

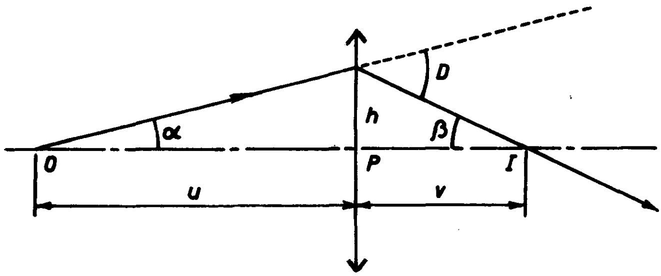


Fig. 4.34

By Eq. (4.21),

$$D = \frac{h}{f}$$

$$\therefore \quad \frac{h}{f} = \frac{h}{u} + \frac{h}{v}$$

$$\text{i.e.} \quad \underline{\frac{1}{f} = \frac{1}{u} + \frac{1}{v}}$$

(b) Image virtual, i.e. object between F and P, Fig. 4.35.

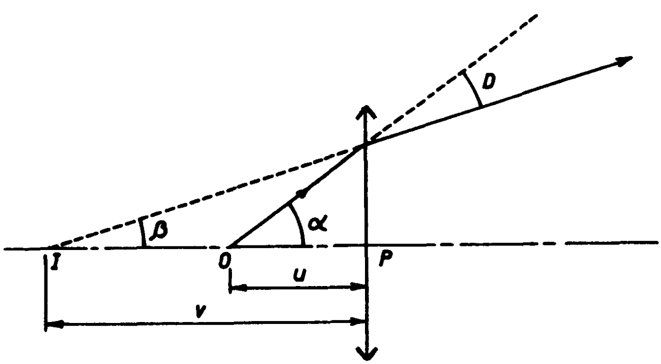


Fig. 4.35

$$D = \alpha - \beta$$

$$\text{i.e.} \quad \frac{h}{f} = \frac{h}{u} - \frac{h}{v}$$

but v is virtual, i.e. negative.

$$\therefore \quad \underline{\frac{1}{f} = \frac{1}{u} + \frac{1}{v}}$$

Concave lens (Fig. 4.36)

$$D = \beta - \alpha$$

i.e. $$\frac{h}{f} = \frac{h}{v} - \frac{h}{u}$$

but v and f are negative, being virtual distances

$$\therefore \quad \underline{\frac{1}{f} = \frac{1}{u} + \frac{1}{v}}$$

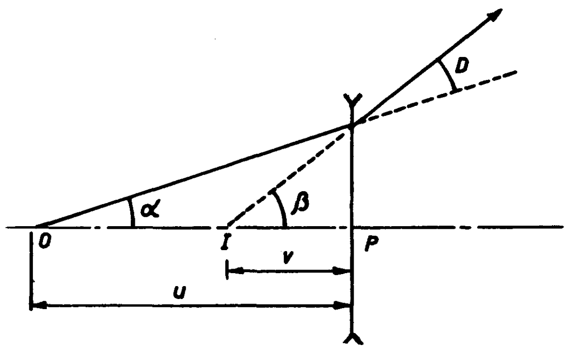


Fig. 4.36

4.45 Magnification in thin lenses (Fig. 4.37)

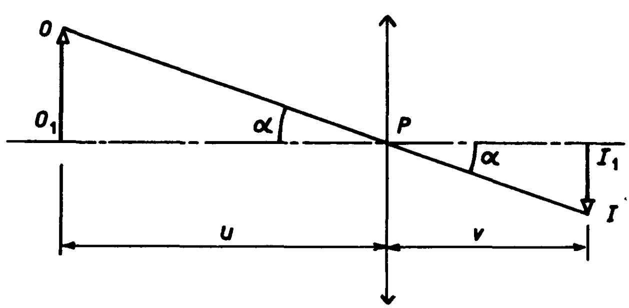


Fig. 4.37

As with spherical mirrors, OPO_1 and IPI_1 are similar right-angled triangles with angle α common.

$$\therefore \quad \text{magnification } m = \frac{II_1}{OO_1} \quad \begin{matrix}\text{(image size)}\\ \text{(object size)}\end{matrix}$$

$$= \frac{I_1P}{O_1P} \quad \begin{matrix}\text{(image distance } v)\\ \text{(object distance } u)\end{matrix}$$

$$= \frac{v}{u} \quad \text{as before}$$

N.B. This should not be confused with angular magnification or magnifying power (M), which is defined as

$$\frac{\text{the angle subtended at the eye by the image}}{\text{the angle subtended at the eye by the object}}$$

For the astronomical telescope, with the image at infinity,

$$M = \frac{\text{focal length of objective}}{\text{focal length of eyepiece}} = \frac{f_o}{f_e} \tag{4.22}$$

4.5 Telescopes

4.51 Kepler's astronomical telescope (Fig. 4.38)

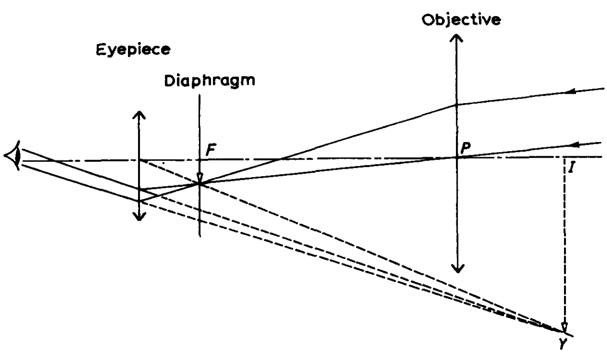


Fig. 4.38 Kepler's atronomical telescope

The telescope is designed to increase the angle subtending distant objects and thus apparently to bring them nearer.

The objective lens, converging and of long focal length, produces an image FX, inverted but real, of the object at infinity.

The eyepiece lens, converging but of short focal length, is placed close to F so as to produce from the real object FX a virtual image IY, magnified but similarly inverted.

4.52 Galileo's telescope (Fig. 4.39)

The eyepiece is concave and produces a virtual, magnified, but erect image IY of the original inverted image XF produced by the objective. As the latter image lies outside the telescope eyepiece, it is unsuitable for surveying purposes where cross hairs are required.

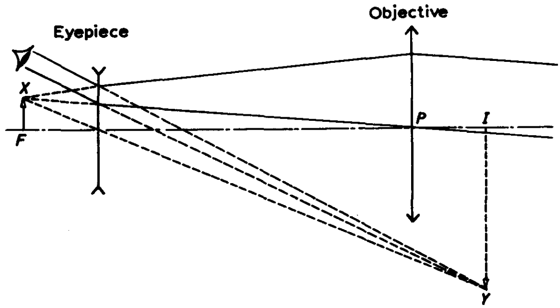


Fig. 4.39 Galileo's telescope

4.53 Eyepieces

Ideally, the eyepieces should reduce *chromatic* and *spherical aberration.*

Lenses of the same material are *achromatic* if their distance apart is equal to the average of their focal lengths, i.e.

$$d = \frac{1}{2}(f_1 + f_2) \tag{4.23}$$

If their distance apart is equal to the differences between their focal lengths, spherical aberration is reduced, i.e.

$$d = f_1 - f_2 \tag{4.24}$$

For surveying purposes the diaphragm must be between the eyepiece and the objective. The most suitable is *Ramsden's* eyepiece, Fig. 4.40.

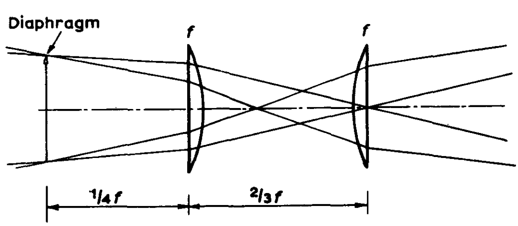


Fig. 4.40 Ramsden's eyepiece

The focal length of each lens is the same, namely f. Neither of the conditions (4.23) or (4.24) is satisfied.

Chromatic	$\frac{1}{2}(f_1 + f_2) = f$	compared with	$2/3\, f$
Spherical	$f_1 - f_2 = 0$	compared with	$2/3\, f$

Huyghen's eyepiece, Fig. 4.41, satisfies the conditions but the focal plane lies between the lenses. It is used in the Galileo telescope.

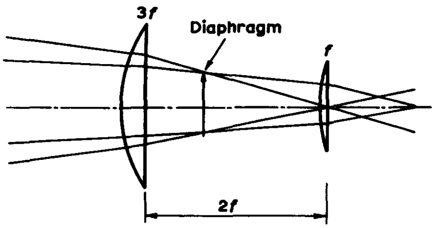


Fig. 4.41 Huyghen's eyepiece

Chromatic condition $\frac{1}{2}(3f + f) = 2f = d$

Spherical condition $3f - f = 2f = d$

Example 4.3 An astronomical telescope consists of two thin lenses 24 in. apart. If the magnifying power is ×12, what are the focal lengths of the two lenses?

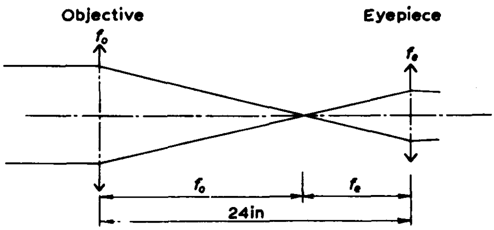


Fig. 4.42

$$\text{magnifying power} = \frac{f_o}{f_e} = 12$$

$$\therefore \quad 12f_e = f_o$$

But $\quad f_o + f_e = 26 \text{ in.}$

$\therefore \quad 12f_e + f_e = 26 \text{ in.}$

$\therefore \quad f_e = \frac{26}{13} = \underline{2 \text{ in.}}$ eyepiece lens

$f_o = 12f_e = \underline{24 \text{ in.}}$ objective lens

4.54 The internal focussing telescope (Fig. 4.43)

The eyepiece and objective are fixed and an internal concave lens is used for focussing.

For the convex lens, by Eq. (4.19)

$$\frac{1}{f_1} = \frac{1}{u_1} + \frac{1}{v_1}$$

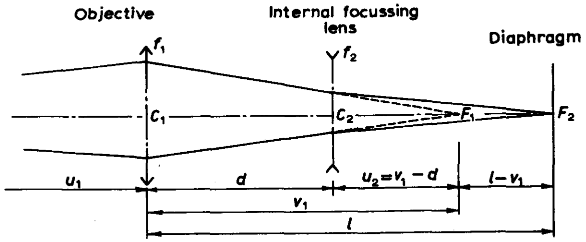


Fig. 4.43 Internal focussing telescope

i.e. $$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1} \quad \text{or} \quad \frac{1}{u_1} = \frac{1}{f_1} - \frac{1}{v_1}$$

For the concave lens,

$$u_2 = -(v_1 - d)$$

$$\therefore \quad \frac{1}{f_2} = \frac{1}{u_2} + \frac{1}{v_2}$$

$$-\frac{1}{f_2} = -\frac{1}{v_1 - d} + \frac{1}{l - d}$$

$$\therefore \quad \frac{1}{f_2} = \frac{1}{v_1 - d} - \frac{1}{l - d} \tag{4.25}$$

An internal focussing telescope has a length l from the objective to the diaphragm. The respective focal lengths of the objective and the internal focussing lens are f_1 and f_2.

To find the distance d of the focussing lens from the objective when the object focussed is u_1 from the objective, Fig. 4.43.

For the objective, $$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1}$$

$$\therefore \quad \frac{1}{v_1} = \frac{u_1 - f_1}{u_1 f_1}$$

For the focussing lens,

$$-\frac{1}{f_2} = -\frac{1}{u_2} + \frac{1}{v_2}$$

i.e. $$\frac{1}{f_2} = \frac{1}{u_2} - \frac{1}{v_2}$$

$$= \frac{1}{v_1 - d} - \frac{1}{l - d}$$

$$\therefore \quad (v_1 - d)(l - d) = f_2(l - d) - f_2(v_1 - d)$$

i.e. $$d^2 - d(l + v_1) + \{ lv_1 - f_2(l - v_1)\} = 0$$

$$d^2 - d(l + v_1) + \{v_1(l + f_2) - f_2 l\} = 0 \tag{4.26}$$

This is a quadratic equation in d and its value will vary according to the distance u_1 of the object from the instrument.

Example 4.4 Describe, with the aid of a sketch, the function of an internal focussing lens in a surveyors' telescope and state the advantages and disadvantages of internal focussing as compared with external focussing.

In a telescope, the object glass of focal length 7 in. is located 9 in. away from the diaphragm. The focussing lens is midway between these when the staff 60 ft away is focussed. Determine the focal length of the focussing lens. (L.U.)

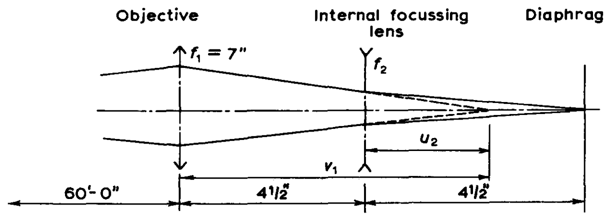


Fig. 4.44

For the convex objective lens,

$$f_1 = 7 \text{ in.}$$

$$u_1 = 60 \times 12 = 720 \text{ in.}$$

Then, by Eq. (4.19),

$$\frac{1}{v_1} = \frac{1}{f_1} - \frac{1}{u_1}$$

$$= \frac{1}{7} - \frac{1}{720}$$

$$= \frac{720 - 7}{720 \times 7} = \frac{713}{5040}$$

For the focussing lens,

$$u_2 = v_1 - 4{\cdot}5 = 7{\cdot}068 - 4{\cdot}5 = 2{\cdot}568$$

$$v_2 = 4{\cdot}5$$

$$\therefore \quad \frac{1}{f_2} = \frac{1}{u_2} + \frac{1}{v_2}$$

$$= -\frac{1}{2{\cdot}568} + \frac{1}{4{\cdot}5}$$

$$= \frac{-4{\cdot}5 + 2{\cdot}568}{11{\cdot}556}$$

$$f_2 = -5{\cdot}98 \text{ in.} \qquad \text{(i.e. the lens is concave)}$$

Example 4.5 In an internally focussing telescope, Fig. 4.43, the objective of focal length 5 in. is 7·5 in. from the diaphragm. If the internal focussing lens is of focal length 10 in., find its distance from the diaphragm when focussed to infinity.

For the objective, $f_1 = 5$ in. and thus the position of F_1 will be 5 in. from C_1.

$$\therefore \qquad C_2F_1 = 5 - d$$

For the internal focussing lens,

$$f_2 = -10$$
$$u_2 = -(5-d)$$
$$v_2 = 7{\cdot}5 - d$$

$$\therefore \qquad \frac{1}{f_2} = \frac{1}{u_2} + \frac{1}{v_2}$$

i.e.
$$-\frac{1}{10} = -\frac{1}{5-d} + \frac{1}{7{\cdot}5-d}$$

$$-(5-d)(7{\cdot}5-d) = -10(7{\cdot}5-d) + 10(5-d)$$

i.e. $$-(37{\cdot}5 - 12{\cdot}5d + d^2) = -75 + 10d + 50 - 10d = -25$$

$$d^2 - 12{\cdot}5d + 12{\cdot}5 = 0$$
$$d = 4{\cdot}235 \text{ in.}$$
$$\therefore \qquad v_2 = 7{\cdot}5 - 4{\cdot}235 = \underline{3{\cdot}265 \text{ in.}}$$

i.e. the internal focussing lens will be 3·265 in. away from the diaphragm when focussed to infinity.

4.55. The tacheometric telescope (external focussing) (Fig. 4.45)

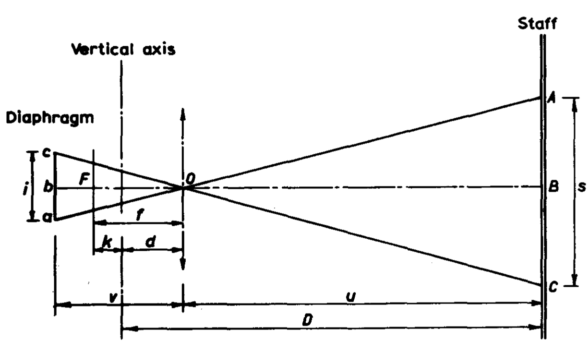


Fig. 4.45 The tacheometric telescope (external focussing)

Let a, b and c represent the three horizontal cross hairs of the diaphragm, ac being a distance i apart and b midway between a and c.

With the telescope in focus, these lines will coincide with the image of the staff observed at A, B and C respectively; the distance $AC = s$ is known as the staff intercept. The line bOB represents the line of collimation of the telescope, with bO and OB conjugate focal lengths of the lens, v and u, respectively. The principal focal length of the lens is FO (f), whilst the vertical axis is a distance k from the principal focus F.

Because the triangles acO and ACO are similar,

$$\frac{AC}{ac} = \frac{OB}{ob} \quad \text{or} \quad \frac{s}{i} = \frac{u}{v} \tag{4.27}$$

Using the lens formula, Eq. (4.19),

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

and multiplying both sides by uf gives,

$$u = f + \frac{uf}{v}$$

Substituting the value of u/v from Eq. (4.27),

$$u = s\frac{f}{i} + f$$

Thus the distance from the vertical axis to the staff is given as

$$\underline{D = s\frac{f}{i} + (f + d)} \tag{4.28}$$

This is the formula which is applied for normal stadia observations with the telescope horizontal and the staff vertical.

The ratio $f/i = M$ is given a convenient value of, say, 100 (occasionally 50), whilst the additive constant $(f + d) = K$ will vary depending upon the instrument.

The formula may thus be simplified as

$$D = M \, . \, s + K \tag{4.29}$$

Example 4.6 The constants M and K for a certain instrument were 100 and 1·5 respectively. Readings taken on to the vertical staff were 3·15, 4·26 and 5·37 ft respectively, the telescope being horizontal. Calculate the horizontal distance from the instrument to the staff.

$$\text{The stadia intercept } s = 5{\cdot}37 - 3{\cdot}15 = 2{\cdot}22 \text{ ft}$$

$$\text{Horizontal distance } D = 100 \times 2{\cdot}22 + 1{\cdot}5$$

$$= \underline{223{\cdot}5 \text{ ft}} \ (68{\cdot}1 \text{ m})$$

If the instrument was set at 103·62 ft A.O.D. and the height to the trunnion axis at 4·83 ft,

then the reduced level of the staff station $= 103{\cdot}62 + 4{\cdot}83 - 4{\cdot}26$

$= \underline{104{\cdot}19\text{ ft}}$ A.O.D.

(31·757 m)

N.B. $4{\cdot}26 - 3{\cdot}15 = 5{\cdot}37 - 4{\cdot}26 = 1{\cdot}11 = \frac{1}{2}s$

If the readings taken on to a metre staff were 0·960, 1·298, 1·636 respectively, then the horizontal distance $= 100 \times (1{\cdot}636 - 0{\cdot}960)$

$$= 67{\cdot}6\text{ m} + 0{\cdot}5\text{ m} = 68{\cdot}1\text{ m}$$

If the instrument was set at 31·583 m A.O.D. and the height of the trunnion axis at 1·472 m,

then the reduced level of the staff station $= 31{\cdot}583 + 1{\cdot}472 - 1{\cdot}298$

$= \underline{31{\cdot}757\text{ m}}$

4.56. The anallatic lens (Fig. 4.46)

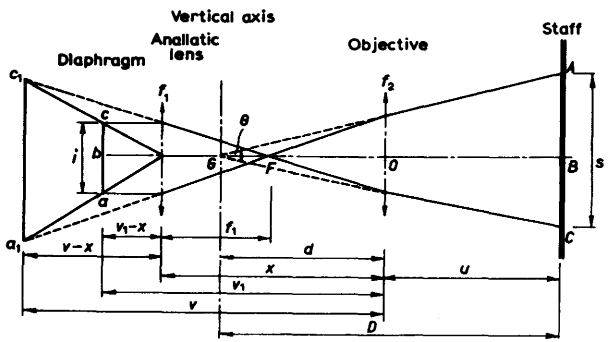


Fig. 4.46 The anallatic lens

In the equation $D = s(f/i) + (f + d)$, the additive factor $(f + d)$ can be eliminated by introducing a convex lens between the objective and the diaphragm.

The basic principles can be seen in Fig. 4.46. The rays from the staff Ad and Ce will for a given distance D always form a constant angle θ intersecting at G. If this fixed point G is made to fall on the vertical axis of the instrument the additive term will be eliminated.

Consider the object lens with the object AC and the image $a_1 c_1$, i.e. neglecting the anallatic lens.

By Eq. (4.19) $$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \qquad (4.30)$$

and by Eq. (4.27) $$\frac{u}{v} = \frac{s}{a_1 c_1} \qquad (4.31)$$

Consider the anallatic lens with the object as $a_1 c_1$ and the image

as ac. Thus the object distance $= v_1 - x$ and
the image distance $= v - x$

Applying the previous equations to this lens,

$$\frac{1}{f_1} = \frac{1}{v_1 - x} - \frac{1}{v - x} \tag{4.32}$$

and
$$\frac{v_1 - x}{v - x} = \frac{ac}{a_1c_1} \tag{4.33}$$

N.B. The object distance is assumed positive but the image distance is negative.

An expression for D can now be found by eliminating v, v_1 and a_1c_1 from these four equations.

From Eq. (4.32)
$$v_1 - x = \frac{f_1(v - x)}{f_1 + v - x}$$

From Eq. (4.33)
$$a_1c_1 = \frac{ac(v - x)}{v_1 - x}$$

Combining these gives
$$a_1c_1 = \frac{ac(f_1 + v - x)}{f_1}$$

but from Eq. (4.30)
$$v = \frac{uf}{u - f}$$

Substituting in the above
$$a_1c_1 = \frac{ac\left(f_1 + \dfrac{uf}{u - f} - x\right)}{f_1}$$

but from Eq. (4.31)
$$a_1c_1 = \frac{sv}{u} = \frac{sf}{u - f}$$

giving
$$\frac{sf}{u - f} = \frac{ac\left(f_1 + \dfrac{uf}{u - f} - x\right)}{f_1}$$

i.e.
$$sff_1 = ac(u - f)\left(f_1 + \frac{uf}{u - f} - x\right)$$

Writing ac as i, the distance apart of the stadia lines
$$sff_1 = i[f_1(u - f) + uf - x(u - f)]$$
$$= i[u(f_1 + f - x) + f(x - f_1)]$$

$\therefore$
$$u = \frac{sff_1}{i(f + f_1 - x)} - \frac{f(x - f_1)}{f + f_1 - x}$$

but
$$D = u + d$$
$$= \frac{sff_1}{i(f + f_1 - x)} - \frac{f(x - f_1)}{f + f_1 - x} + d$$
$$= Ms - \frac{f(x - f_1)}{f + f_1 - x} + d$$

and if
$$d = \frac{f(x - f_1)}{f + f_1 - x} \tag{4.34}$$
$$D = Ms \tag{4.35}$$

where
$$M = \frac{ff_1}{i(f + f_1 - x)} \tag{4.36}$$

a constant factor usually 100.

The manufacturer can therefore choose the lenses where the focal length f_1 is such that $f_1 < x < f$.

Today, this is mainly of academic interest only, as all instruments have internal focussing telescopes, and the tacheometic formula $D = f(s/i) + (f + d)$ is not applicable; nor can the internal focussing be considered anallatic as it is movable.

The variation of the focal length of the objective system is generally considered to be negligible for most practical purposes (see Example 4.7), manufacturers aiming at a low value for K, and in many cases the telescopes are so designed that when focussed at infinity the focussing lens is midway between the objective and the diaphragm. This allows accuracies for horizontal sights of up to 1/1000 for most distances required in this type of work.

Example 4.7 An anallatic telescope is fitted with an object lens of 6 in. focal length. If the stadia lines are 0·06 in. apart and the vertical axis 4 in. from the object lens, calculate the focal length of the anallatic lens and its position relative to the vertical axis if the multiplying constant is 100.

From Eq. (4.34) the distance between objective and axis
$$d = \frac{f(x - f_1)}{f + f_1 - x}$$

When $f = 6$ in.
$$d = \frac{6(x - f_1)}{6 + f_1 - x} = 4$$

Also, from Eq. (4.36),
$$M = \frac{ff_1}{i(f + f_1 - x)}$$

Therefore when $M = 100$, $i = 0{\cdot}06$, $f = 6$,
$$M = \frac{6f_1}{0{\cdot}06(6 + f_1 - x)} = 100$$

Combining these equations,

$$4(6 + f_1 - x) = 6(x - f_1)$$

$$10x = 24 + 10f_1$$

and

$$100 \times 0{\cdot}06(6 + f_1 - x) = 6f_1$$

$$6x = 36 + 0$$

$$\therefore \quad \underline{x = 6\text{ in.}}$$

and

$$\underline{f_1 = 3{\cdot}6\text{ in.}}$$

Thus the focal length of the anallatic lens is 3·6 in. and its position is $(6 - 4) = 2$ in. from the vertical axis.

Example 4.7a An anallatic tacheometer in use on a remote survey was damaged and it was decided to use a glass diaphragm not originally designed for the instrument. The spacing of the outer lines of the new diaphragm was 0·05 in., focal lengths of the object glass and the anallatic lens 3 in., fixed distance between object glass and trunnion axis 3 in., and the anallatic lens could be moved by an adjusting screw between its limiting positions 3 in. and 4 in. from the object glass. In order to make the multiplier 100 it was decided to adjust the position of the anallatic lens, or if this proved inadequate to graduate a special staff for use with the instrument. Make calculations to determine which course was necessary, and if a special staff is required, determine the correct calibration and the additive constant (if any). What is the obvious disadvantage to the use of such a special staff? (L.U.)

From Eq. (4.36),

$$M = \frac{ff_1}{i(f + f_1 - x)}$$

$$\therefore \quad x = f + f_1 - \frac{ff_1}{Mi}$$

$$= 3 + 3 - \frac{3 \times 3}{100 \times 0{\cdot}05}$$

$$= 6 - 1{\cdot}8 = \underline{4{\cdot}2\text{ in.}}$$

i.e. the anallatic lens should be 4·2 in. from the objective. As this is not possible, the lens is set as near as possible to this value, i.e. 4 in.

Then

$$M = \frac{3 \times 3}{0{\cdot}05(3 + 3 - 4)} = \underline{90}$$

The additive factor K from Eq. (4.34)

$$= \frac{f(x - f_1)}{f + f_1 - x} = \frac{3(4 - 3)}{3 + 3 - 4} = \underline{1{\cdot}5\text{ in.}}$$

If the multiplying factor is to be 100, then the staff must be graduated in such a way that in reading 1 foot the actual length on the staff is $12 \times \frac{10}{9}$ in. i.e. $13\frac{1}{3}$ in.

4.57. The tacheometric telescope (internal focussing) (Fig. 4.47)

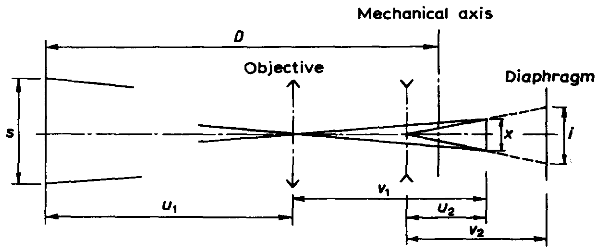


Fig. 4.47 Tacheometric telescope (internal focussing)

To find the spacing of the stadia lines to give a multiplying factor M for a given sight distance:

if i = stadia interval,
and s = stadia intercept;

for the convex lens (objective)

$$\frac{x}{s} = \frac{v_1}{u_1} = m_1 \quad \text{i.e.} \quad x = \frac{sv_1}{u_1} = m_1 s$$

where m_1 is the magnifying power.
for the concave lens (focussing)

$$\frac{i}{x} = \frac{v_2}{u_2} = m_2 \quad \text{i.e.} \quad i = x\frac{v_2}{u_2} = m_2 x$$

$$\therefore \quad i = m_1 m_2 s \tag{4.37}$$

but

$$\text{distance } D = Ms$$

$$\therefore \quad s = \frac{D}{M}$$

$$\therefore \quad i = \frac{D m_1 m_2}{M} \tag{4.38}$$

Example 4.8 An internally focussing telescope has an objective 6 in. from the diaphragm. The respective focal lengths of the objective and the internal focussing lens are 5 in and 10 in. Find the distance apart the stadia lines should be to have a multiplying factor of 100 for an observed distance of 500 ft.

At 500 ft the object will be 500 ft − 6/2 in. from the objective.

i.e. $$u_1 = 500 \times 12 - 3 = 5997 \text{ in}$$

$$v_1 = \frac{5997 \times 5}{5997 - 5} = 5{\cdot}0042 \text{ in.}$$

From Eq. (4.26),

$$d^2 - d(l + v_1) + \{v_1(l + f_2) - f_2 l\} = 0$$

i.e. $$d^2 - d(6 + v_1) + \{16v_1 - 60\} = 0$$

$\therefore$ $$d = \tfrac{1}{2}[(6 + v) \pm \sqrt{\{(6 + v_1)^2 - 64v_1 + 240\}}]$$

$$= \tfrac{1}{2}[(6 + v) \pm \sqrt{(6 - v_1)(46 - v_1)}].$$

i.e. $$d = \tfrac{1}{2}[11{\cdot}0042 \pm \sqrt{(0{\cdot}9958 \times 40{\cdot}9958)}]$$

$$= 2{\cdot}308 \text{ in.}$$

$$v_2 = l - d = 6 - 2{\cdot}308 = 3{\cdot}692 \text{ in}$$

$$u_2 = v_1 - d = 5{\cdot}0042 - 2{\cdot}308 = 2{\cdot}696 \text{ in}$$

From Eq. (4.38),

$$i = \frac{D\,m_1 m_2}{M} = \frac{D\,v_1 v_2}{M\,u_1 u_2}$$

$$= \frac{500 \times 12 \times 5{\cdot}0042 \times 3{\cdot}692}{100 \times 5997 \times 2{\cdot}696}$$

$$= \underline{0{\cdot}06856 \text{ in.}}$$

Example 4.9 What errors will be introduced if the previous instrument is used for distances varying from 50 to 500 ft?

At 50 ft
$$u_1 = 50 \times 12 - 3 = 597 \text{ in.}$$

$$v_1 = \frac{597 \times 5}{597 - 5} = \frac{2985}{592} = 5{\cdot}0422 \text{ in.}$$

Then, from Eq. (4.26),

$$d = \tfrac{1}{2}[(6 + v_1) \pm \sqrt{[(6 - v_1)(46 - v_1)]}$$

$$= \tfrac{1}{2}[11{\cdot}0422 - \sqrt{(0{\cdot}9578 \times 40{\cdot}9578)}]$$

$$= 2{\cdot}389 \text{ in.}$$

$$v_2 = 6 - 2{\cdot}389 = 3{\cdot}611 \text{ in.}$$

$$u_2 = 5{\cdot}042 - 2{\cdot}389 = 2{\cdot}653 \text{ in.}$$

$$\text{The stadia intercept } (s) = 0{\cdot}068\,56 \times \frac{u_1u_2}{v_1v_2}$$

$$= \frac{0{\cdot}068\,56 \times 597 \times 2{\cdot}653}{5{\cdot}042 \times 3{\cdot}611}$$

$$= 5{\cdot}964\,1 \text{ in.}$$

$$= 0{\cdot}497\,0 \text{ ft}$$

The value should be $0{\cdot}500\,0$

$$\text{error} = 0{\cdot}003\,0 \text{ ft}$$

$$\text{representing} \quad \underline{0{\cdot}30 \text{ ft in } 100 \text{ ft}}$$

At	100 ft	error	=	0·27 ft
	200 ft	error	=	0·20 ft
	300 ft	error	=	0·09 ft
	400 ft	error	=	0·01 ft
	500 ft	error	=	0·00 ft

Example 4.10 An internal focussing telescope has an object glass of 8 in. focal length. The distance between the object glass and the diaphragm is 10 in. When the telescope is at infinity focus, the internal focussing lens is exactly midway between the objective and the diaphragm. Determine the focal length of the focussing lens.

At infinity focus the optical centre of the focussing lens lies on the line joining the optical centre of the objective and the cross-hairs, but deviates laterally 0·001 in. from it when the telescope is focussed at 25 ft. Calculate the angular error in seconds due to this cause.

(L.U.)

With the telescope focussed at infinity, $v_1 = f_1$

For the focussing lens,

$$\frac{1}{f_2} = \frac{1}{u_2} - \frac{1}{v_2}$$

$$= \frac{1}{v_1 - d} - \frac{1}{l - d}$$

$$= \frac{1}{f_1 - d} - \frac{1}{l - d}$$

$$= \frac{1}{8 - 5} - \frac{1}{10 - 5} = \frac{2}{15}$$

$$\underline{f_2 = 7{\cdot}5 \text{ in.}} \text{ focal length of focussing lens.}$$

With focus at 25 ft (assuming 25 ft from object lens.)

$$u_1 = 25 \times 12 = 300$$

$$\therefore \quad v_1 = \frac{u_1 f_1}{u_1 - f_1} = \frac{300 \times 8}{300 - 8} = 8{\cdot}219\,2 \text{ in.}$$

From Eq. (4.26),

$$d^2 - d(l + v_1) + \{v_1(l + f_2) - f_2 l\} = 0$$

i.e.

$$d^2 - 18{\cdot}2192d + (143{\cdot}836 - 75) = 0$$

Solving for d, $\underline{d = 5{\cdot}348}$

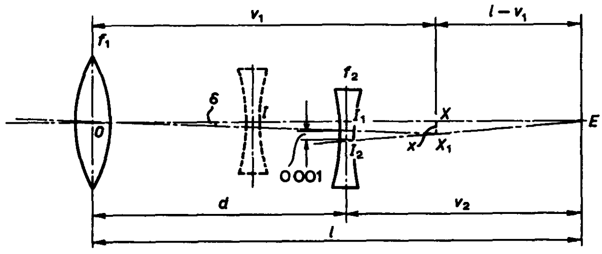


Fig. 4.48

With focus at 25 ft the image would appear at x, neglecting the internal focussing lens, i.e. $OX = v_1$. With the focussing lens moving off line, the line of sight is now EX_1I_2 and all images produced by the objective appear as on this line.

The line of sight through the objective is thus displaced XX_1 in the length v_1.

To calculate XX_1.

$$\frac{XX_1}{I_1I_2} = \frac{XE}{I_1E}$$

i.e.

$$x = \frac{0{\cdot}001 \times (l - v_1)}{l - d}$$

$$= \frac{0{\cdot}001 \times (10 - 8{\cdot}219)}{10 - 5{\cdot}348}$$

$$= \frac{0{\cdot}001\,781}{4{\cdot}552} = 0{\cdot}000\,391 \text{ in.}$$

To calculate the angular error (δ),

$$\tan\delta = \frac{XX_1}{OX} = \frac{x}{v_1}$$

$$\delta = \frac{206\,265 \times 0{\cdot}000\,391}{8{\cdot}219} = \underline{9{\cdot}8 \text{ seconds}}$$

4.6 Instrumental Errors in the Theodolite

4.61 Eccentricity of the horizontal circle

In Fig. 4.49, let O_1 = vertical axis

$$O_2 = \text{Graduated circle axis}$$
$$O_1O_2 = e = \text{eccentricity}$$
$$O_2A_1 \simeq O_1A_2 = r$$

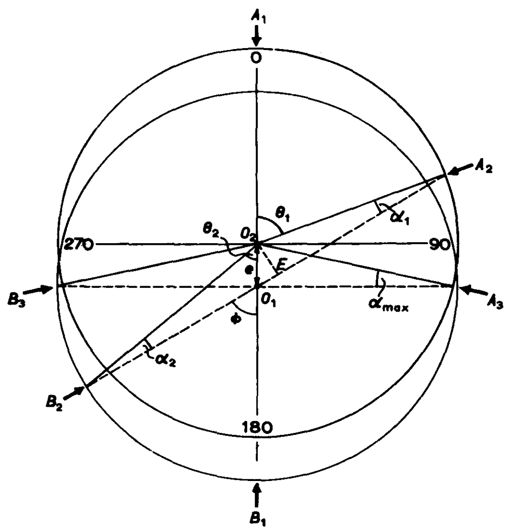


Fig. 4.49

If the graduated circle (Centre O_2) is not concentric with the vertical axis (centre O_1) containing the readers A and B, the recorded value θ will be in error by the angle α.

As the instrument is rotated, the readers will successively occupy positions A_1B_1, A_2B_2, A_3B_3.

$$\alpha_1 = \theta_1 - \phi_1$$
$$= \tan^{-1}\frac{O_2E}{A_2E}$$
$$= \tan^{-1}\frac{e\sin\phi}{r - e\cos\phi} \qquad (4.39)$$
$$\therefore \quad \alpha_1 \simeq \frac{e\sin\phi}{r} \qquad (4.40)$$

Since e is small compared with r and as α is small, $\alpha_{rad} \simeq \tan\alpha$.

Similarly, $$\tan\alpha_2 = -\frac{e\sin\phi}{r + e\cos\phi} \qquad (4.41)$$
$$\alpha_2 \simeq \frac{e\sin\phi}{r} \qquad (4.40)$$

If the readers are 180° apart, $A_1O_1B_1$ is a straight line and the mean of the recorded values θ give the true value of the angle ϕ.

i.e.
$$\phi = \theta_1 - \alpha_1 = \theta_2 + \alpha_2$$
$$\therefore\ 2\phi = \theta_1 + \theta_2 \qquad \text{as } \alpha_1 \simeq \alpha_2$$
$$\phi = \tfrac{1}{2}(\theta_1 + \theta_2) \tag{4.42}$$

N.B. (1) On the line O_1O_2, $\alpha = 0$.

(2) At 90° to this line, α = maximum.

(3) If the instrument has only one reader, the angle should be repeated by transitting the telescope and rotating anticlockwise, thus giving recorded values 180° from original values This is of particular importance with glass arc theodolites in which the graduated circle is of small radius.

To determine the amount of eccentricity and index error on the horizontal circle:

(1) Set index A to 0° and read displacement of index B from 180°, i.e. δ_1.

(2) Set index B to 0° and read displacement of index A from 180°, i.e. δ_2.

(3) Repeat these operations at a constant interval around the plate, i.e. zeros at multiples of 10°.

If the readers A and B are diametrically opposed, let δ_1 = displacement of reader B_1 from 180°, Fig. 4.50.

Index A_1 at 0°.

Index B_1 at 180° − δ_1, i.e. 180 − $(2\alpha + \lambda)$.

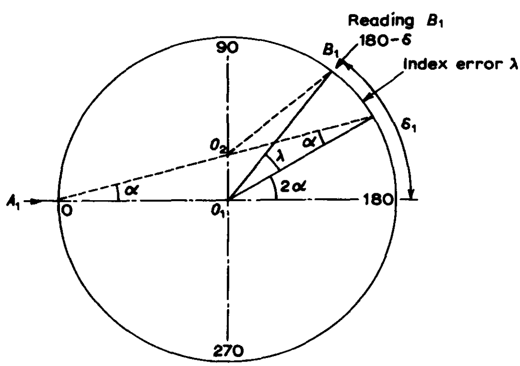


Fig. 4.50

Let δ_2 = displacement of reader A_2 from 180°, Fig. 4.51.

Index B_2 at 0.

Index A_2 at 180 − δ_2, i.e. 180 − $(2\alpha + \lambda)$.

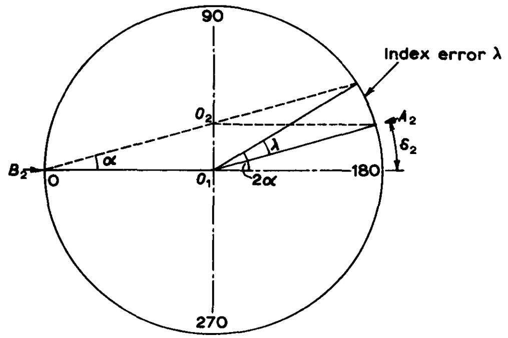


Fig. 4.51

If there is no eccentricity and A and B are 180° apart, then $\delta_1 = \delta_2 = 0$.

If there is eccentricity and A and B are 180° apart, then $\delta_1 = \delta_2 =$ a constant

If there is no eccentricity and A and B are not 180° apart, then $+\delta_1 = -\delta_2$, i.e. equal, but opposite in sign.

If there is eccentricity and A and B are not 180° apart, then δ_1 and δ_2 will vary in magnitude as the zero setting is consecutively changed around the circle of centre O_2, but their difference will remain constant.

A plotting of the values using a different zero for each pair of index settings will give the results shown in Fig. 4.52.

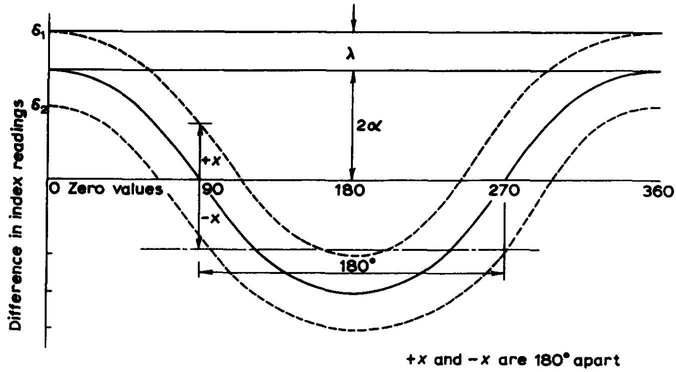


Fig. 4.52

4.62. The line of collimation not perpendicular to the trunnion axis

Let the line of sight make an angle of 90° ± ϵ with the trunnion axis inclined at an angle α, Fig. 4.53.

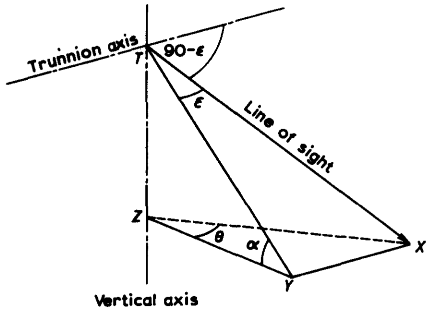


Fig. 4.53 Line of collimation not perpendicular to the trunnion axis

The angular error θ in the horizontal plane due to the error ϵ may be found by reference to Fig. 4.53.

$$\tan\theta = \frac{XY}{YZ}$$

$$\tan\epsilon = \frac{XY}{TY} \quad \text{i.e.} \quad XY = TY\tan\epsilon$$

But $\quad TY = YZ\sec\alpha \quad$ i.e. $\quad YZ = TY\cos\alpha$

$$\therefore \quad \tan\theta = \frac{TY\tan\epsilon}{TY\cos\alpha} = \tan\epsilon\sec\alpha \tag{4.43}$$

If θ and ϵ are small,

then
$$\underline{\theta = \epsilon\sec\alpha} \tag{4.44}$$

If observations are made on the same face to two stations of elevations α_1 and α_2, then the error in the horizontal angle will be

$$\pm(\theta_1 - \theta_2) = \pm\tan^{-1}(\tan\epsilon\sec\alpha_1) - \tan^{-1}(\tan\epsilon\sec\alpha_2) \tag{4.45}$$

$$\pm(\theta_1 - \theta_2) \simeq \pm\epsilon(\sec\alpha_1 - \sec\alpha_2) \tag{4.46}$$

On changing face, the error will be of equal value but opposite in sign. Thus the mean of face left and face right eliminates the error due to collimation in azimuth. The sign of the angle, i.e. elevation of depression, is ignored in the equation.

The extension of a straight line, Fig. 4.54. If this instrument is used to extend a straight line by transitting the telescope, the following conditions prevail

With the axis on the line TQ the line of sight will be OA_1. To observe A, the instrument must be rotated through the angle ϵ to give pointing (1) – the axis will be rotated through the same angle E to T_1Q_1.

On transitting the telescope the line of sight will be $(180° - 2\alpha)$ $A_1OB_1 = AOB_2$. B_2 is thus fixed – pointing (2).

On changing face the process is repeated – pointing (3) – and then

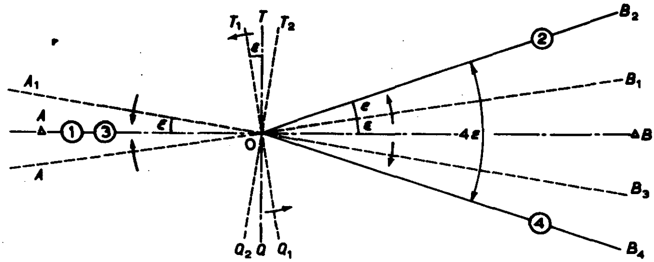


Fig. 4.54

pointing (4) will give position B_4.

The angle $B_2OB_4 = 4\epsilon$, but the mean position B will be the correct extension of the line AO.

The method of adjustment follows the above process, B_2B_4 being measured on a horizontal scale.

The collimation error may be corrected by moving the telescope graticule to read on B_3, i.e. $\frac{1}{4}B_2B_4$.

4.63 The trunnion axis not perpendicular to the vertical axis (Fig. 4.55)

The trunnion (horizontal or transit) axis should be at right angles to the vertical axis; if the plate bubbles are centralised, the trunnion axis will not be horizontal if a trunnion axis error occurs. Thus the line of sight, on transitting, will sweep out a plane inclined to the vertical by an angle equal to the tilt of the trunnion axis.

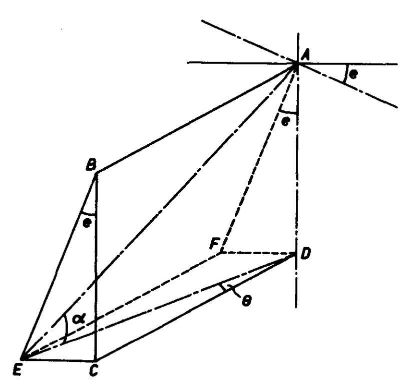


Fig. 4.55 Trunnion axis not perpendicular to the vertical axis

If the instrument is in correct adjustment, the line of sight sweeps out the vertical plane $ABCD$, Fig. 4.55.

If the trunnion axis is tilted by an angle e, the line of sight sweeps

out the inclined plane $ABEF$.

In Fig. 4.55, the line of sight is assumed to be AE. To correct for the tilt of the plane it is necessary to rotate the horizontal bearing of the line of sight by an angle θ, to bring it back to its correct position.

Thus $$\sin\theta = \frac{EC}{ED} = \frac{BC\tan e}{ED}$$

$$= \frac{AD}{ED}\tan e = \tan\alpha\tan e$$

i.e. $$\sin\theta = \tan\alpha\tan e \qquad (4.47)$$

and if θ and e are small, then

$$\theta = e\tan\alpha \qquad (4.48)$$

where θ = correction to the horizontal bearing

e = trunnion axis error

α = angle of inclination of sight

On transitting the telescope, the inclination of the trunnion axis will be in the opposite direction but of equal magnitude. Thus the mean of face left and face right eliminates the error.

Method of adjustment (Fig. 4.56)

(1) Observe a highly elevated target A, e.g. a church spire.

(2) With horizontal plates clamped, depress the telescope to observe a horizontal scale B.

(3) Change face and re-observe A.

(4) As before, depress the telescope to observe the scale at C.

(5) Rotate horizontally to D midway between B and C.

(6) Elevate the telescope to the altitude of A.

(7) Adjust the trunnion axis until A is observed.

(8) On depressing the telescope, D should now be observed.

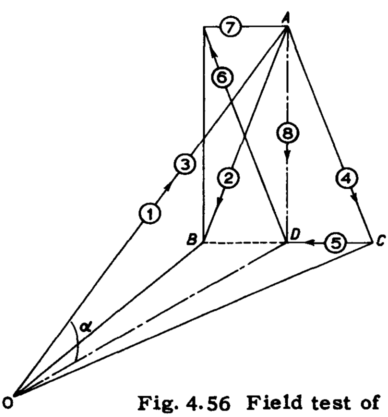


Fig. 4.56 Field test of trunnion axis error

4.64 Vertical axis not truly vertical (Fig. 4.57)

If the instrument is in correct adjustment but the vertical axis is not truly vertical by an angle E, then the horizontal axis will not be truly horizontal by the same angle E.

Thus the error in bearing due to this will be

$$E \tan \alpha \tag{4.49}$$

This is a variable error dependent on the direction of pointing relative to the direction of tilt of the vertical axis, and its effect is *not* eliminated on change of face, as the vertical axis does not change in position or inclination.

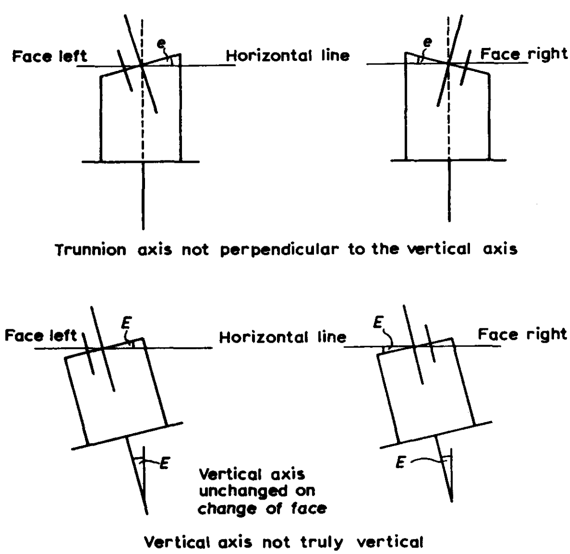


Fig. 4.57

In Fig. 4.58, the true horizontal angle (θ) A_1OB_1 = angle (ϕ) $A_2OB_2 - (c_1)E_1 \tan \alpha_1 + (c_2) E_2 \tan \alpha_2$.

Thus the error in pointing (θ) is dependent on (1) the tilt of the axis E, which itself is dependent on the direction of pointing, varying from maximum (E) when on the line of tilt of the vertical axis to zero when at 90° to this line, and (2) the angle of inclination of the line of sight.

To measure the value of e and E a *striding level* is used, Fig. 4.59.

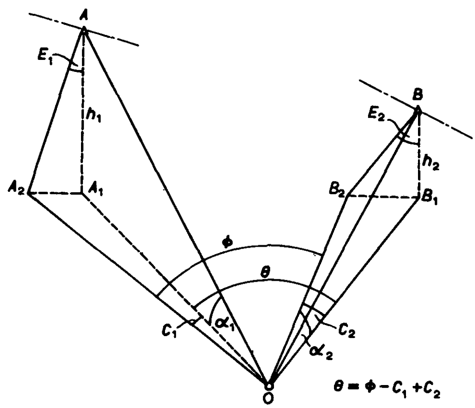


Fig. 4.58

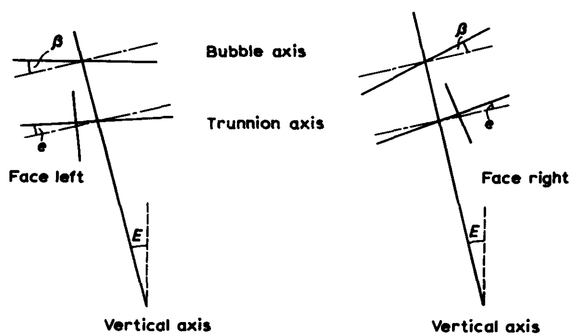


Fig. 4.59

Let the vertical axis be inclined at an angle E

Let the trunnion axis be inclined at an angle e

Let the bubble be out of adjustment by an angle β.

Then *face left*

tilt of the trunnion axis $= E - e$
tilt of the bubble axis $= E - \beta$

face right

tilt of the trunnion axis $= E + e$
tilt of the bubble axis $= E + \beta$

Mean tilt of the trunnion axis $= \frac{1}{2}[(E - e) + (E + e)] \quad = \quad E$ (4.50)

Mean tilt of the bubble axis $= \frac{1}{2}[(E - \beta) + (E + \beta)] \quad = \quad E$ (4.51)

Therefore the mean correction taking all factors into account is

$$E \tan \alpha \tag{4.52}$$

N.B. The value of E is related to the direction of observation and its effective value will vary from maximum to nil. Tilting level readings should be taken for each pointing.

If E is the maximum tilt of the axis in a given direction, then

$$E_1 = E \cos \theta$$

where θ is the angle between pointing and direction of maximum tilt.

Then the bubble recording the tilt does not strictly need to be in adjustment, nor is it necessary to change it end for end as some authors suggest, the mean of face left and face right giving the true value.

If the striding level is graduated from the centre outwards for n pointings and $2n$ readings of the bubble, then the correction to the mean observed direction is given by

$$c = \frac{d}{2n}(\Sigma L - \Sigma R) \tan \alpha \tag{4.53}$$

where c = the correction in seconds
d = the value of one division of the bubble in seconds
ΣL = the sum of the readings of the left-hand end of the bubble
ΣR = the sum of the readings of the right-hand end of the bubble
α = the angle of inclination of sight
n = the number of pointings.

The sign of the correction is positive as stated. Any changes depends upon the sign of $\Sigma L - \Sigma R$ and that of α.

N.B. The greater the change in the value of α the greater the effect on the horizontal angle.

4.65 Vertical circle index error (Fig. 4.60)

When the telescope is horizontal, the altitude bubble should be central and the circle index reading zero (90° or 270° on whole circle reading instruments).

If the true angle of altitude $= \alpha$
the recorded angles of altitude $= \alpha_1$ and α_2
the vertical collimation error $= \phi$
and the circle index error $= \theta$,

$$\text{Recorded value (F.L.)} = \alpha = \alpha_1 - \phi - \theta$$
$$\text{(F.R.)} = \alpha = \alpha_2 + \phi + \theta$$
$$\therefore \quad \alpha = \tfrac{1}{2}(\alpha_1 + \alpha_2) \tag{4.54}$$

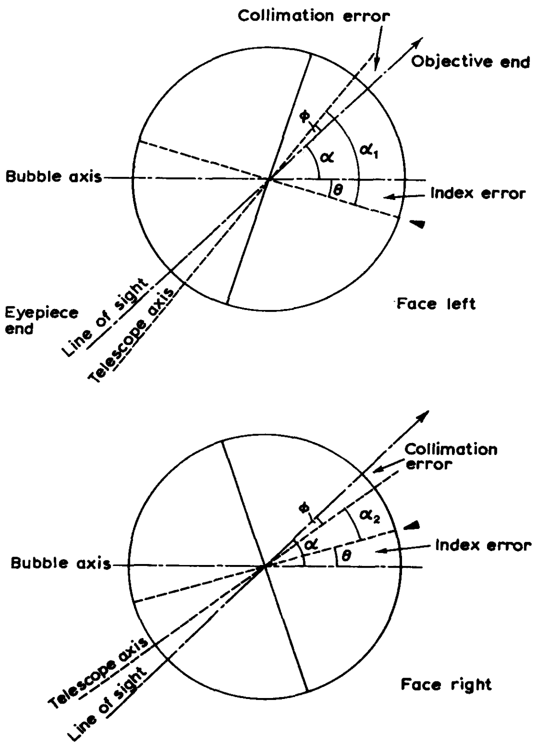


Fig. 4.60

Thus, provided the altitude bubble is centralised for each reading, the mean of face left and face right will give the true angle of altitude.

If the bubble is not centralised then bubble error will occur, and, depending on the recorded displacement of the bubble at the objective and eyepiece ends, the sensitivity will indicate the angular error. As the bubble is rotated, the index is also rotated.

Thus θ will be subjected to an error of $\pm \frac{1}{2}(0 - E)\delta''$, where δ'' = the angular sensitivity of the bubble.

If the objective end of the bubble is higher than the eyepiece end on face left, i.e. $O_L > E_L$, then θ will be decreased by $\frac{1}{2}(O_L - E_L)\delta''$,

i.e. $\quad$ F.L. $\quad \alpha = \alpha_1 - \phi - \{\theta - \frac{1}{2}(O_L - E_L)\delta''\}$

and $\quad$ F.R. $\quad \alpha = \alpha_2 + \phi + \{\theta + \frac{1}{2}(O_R - E_R)\delta''\}$

$$\alpha = \tfrac{1}{2}(\alpha_1 + \alpha_2) + \frac{\delta''}{2}\{(O_L + O_R) - (E_L + E_R)\}$$

$$= \tfrac{1}{2}(\alpha_1 + \alpha_2) + \frac{\delta}{4}(\Sigma O - \Sigma E) \qquad (4.55)$$

To test and adjust the index error

(1) Centralise the altitude bubble and set the telescope to read zero (face left).

(2) Observe a card on a vertical wall – record the line of sight at A.

(3) Transit the telescope and repeat the operation. Record the line of sight at B.

(4) Using the slow motion screw (vertical circle) observe the midpoint of AB. (The line of sight will now be horizontal.)

(5) Bring the reading index to zero and then adjust the bubble to its midpoint.

Example 4.11 *Trunnion axis error.* The following are the readings of the bubble ends A and B of a striding level which was placed on the trunnion axis of a theodolite and then reversed ('Left' indicates the left-hand side of the trunnion axis when looking along the telescope from the eyepiece end with the theodolite face right.)

A on left 11·0, B on right 8·4

B on left 10·8, A on right 8·6

One division of the striding level corresponds to 15″. All adjustments other than the horizontal trunnion axis adjustment of the theodolite being presumed correct, determine the true horizontal angle between P and Q in the following observations (taken with the theodolite face left).

Object	Horizontal circle	Vertical circle
P	158° 20′ 30″	42° 24′
Q	218° 35′ 42″	15° 42′

(L.U.)

By Eq. (4.53),

$$\text{Correction to bearing} = \frac{d}{2n}(\Sigma L - \Sigma R)\tan\alpha$$

$$\text{to } P \quad c = \frac{15}{4}\{(11{\cdot}0 + 10{\cdot}8) - (8{\cdot}4 + 8{\cdot}6)\}\tan 42°\,24'$$

$$= \frac{15 \times 4{\cdot}8}{4}\tan 42°\,24'$$

$$= -18''\tan 42°\,24'' = -16''$$

to Q $\quad c = -18 \tan 15° 42' = -5'$

N.B. The correction would normally be positive when using the general notation, but the face is changed by the definition given in the problem.

$$\begin{aligned} \text{True bearing to } P &= 158°20'30'' - 16'' = 158°20'14'' \\ \text{True bearing to } Q &= 218°35'42'' - 5'' = 218°35'37'' \\ \text{True horizontal angle} &= \underline{60°15'23''} \end{aligned}$$

Example 4.12 In an underground traverse the following mean values were recorded from station B on to stations A and C

Station Observed	Horizontal Angle	Vertical Angle	Striding Level readings L	R
	136° 21′ 32″			
A		−13° 25′ 20″	17·4	5·8
C		+47° 36′ 45″	14·5	2·7

Striding level: 1 division = 10 seconds
bubble graduated 0 to 20

Height of instrument at B 4·63 ft
Height of target at A 3·42 ft
at C 5·15 ft
Ground length AB 256·32 ft
BC 452·84 ft

Calculate the gradient of the line AC

(R.I.C.S.)

Striding level corrections

to A

$$\begin{array}{lr} \text{L} & 17{\cdot}4 \\ \text{R} & 5{\cdot}8 \\ & 2\overline{)23{\cdot}2} \\ & 11{\cdot}6 \end{array}$$

i.e. centre of bubble is 1·6 to left of centre of graduations.

to B

$$\begin{array}{lr} \text{L} & 14{\cdot}5 \\ \text{R} & 2{\cdot}7 \\ & 2\overline{)17{\cdot}2} \\ & 8{\cdot}6 \end{array}$$

i.e. centre of bubble is 1·4 to right of centre of graduations.

The same results may be obtained by using the basic equation (4.53) and transposing the readings as though the graduation were from the centre of the bubble.

i.e. to A 17·4 (L) becomes 7·4 (L)
5·8 (R) becomes 4·2 (R)

$$\frac{\Sigma L - \Sigma R}{2} \quad \frac{7{\cdot}4 - 4{\cdot}2}{2} = +1{\cdot}6$$

to C 14·5 (L) becomes 4·5 (L)
2·7 (R) becomes 7·3 (R)

$$\frac{\Sigma L - \Sigma R}{2} = \frac{4{\cdot}5 - 7{\cdot}3}{2} = -1{\cdot}4$$

Applying these values to Eq. (4.52),

$$\text{Correction to } A = +1{\cdot}6 \times 10 \times \tan(-13^\circ 25') = \underline{-3{\cdot}8''}$$

$$\text{Correction to } C = -1{\cdot}4 \times 10 \times \tan(+47^\circ 37') = \underline{-15{\cdot}3''}$$

$$\text{Total angle correction} = -11{\cdot}5''$$

$$\text{Corrected horizontal angle} = 136^\circ 21' 32'' - 11{\cdot}5'' = \underline{136^\circ 21' 20''}$$

To find true inclination of the ground and true distances (Fig. 4.61)

Line AB

$$\delta\alpha_1 = \sin^{-1} \frac{1{\cdot}21 \cos 13^\circ 25' 20''}{256{\cdot}32}$$

$$\delta\alpha_1 = -0^\circ 15' 47''$$

$$\alpha_1 = \underline{13^\circ 25' 20''}$$

$$\underline{\theta = 13^\circ 09' 33''}$$

Horizontal length (D_1) AB = 256·32 cos 13° 09′ 33″ = 249·59 ft

Vertical difference (H_1) AB = 256·32 sin 13° 09′ 33″ = 58·35 ft

Line BC

$$\delta\alpha_2 = \sin^{-1} \frac{0{\cdot}52 \cos 47^\circ 36' 45''}{452{\cdot}84}$$

$$= -0^\circ 02' 40''$$

$$\alpha_2 = \underline{47^\circ 36' 45''}$$

$$\underline{\theta = 47^\circ 34' 05''}$$

Horizontal length BC = 452·84 cos 47° 34′ 05″ = 305·54 ft

Vertical difference = 452·84 sin 47° 34′ 05″ = 334·23 ft

Difference in height AC = 58·35 + 334·23 = $\underline{392{\cdot}58 \text{ ft}}$

To find the horizontal length AC:

In triangle ABC,

$$\tan \frac{A - C}{2} \quad \frac{305{\cdot}54 - 249{\cdot}59}{305{\cdot}54 + 249{\cdot}59} \tan \frac{(180 - 136^\circ 21' 20'')}{2}$$

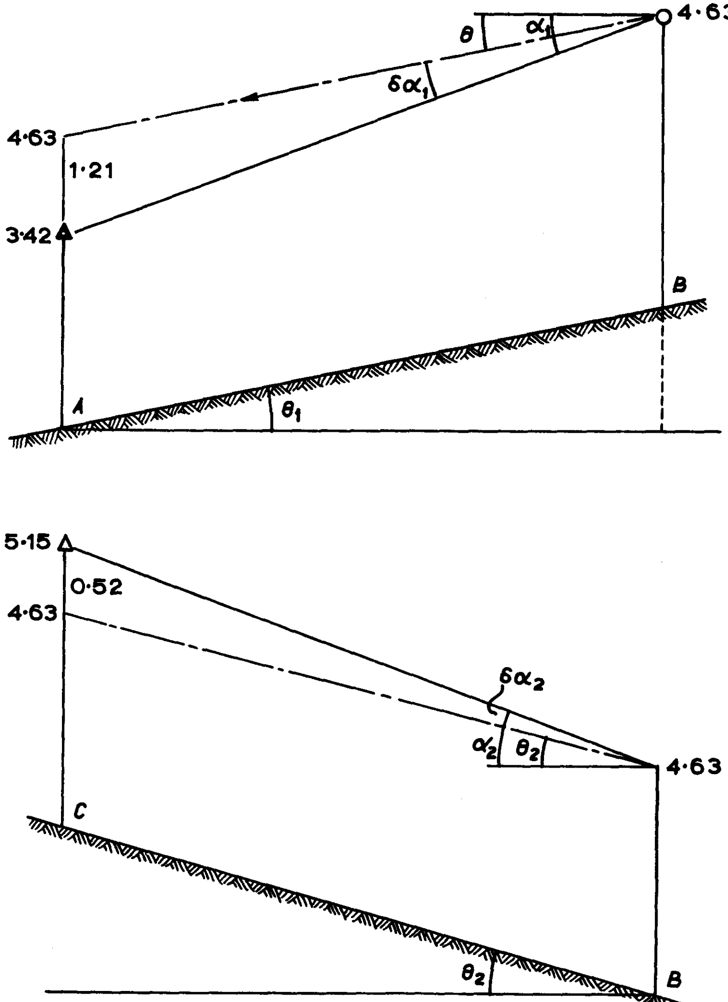


Fig. 4.61

$$\frac{A-C}{2} = 2^\circ 18' 40''$$

$$\frac{A+C}{2} = 21^\circ 49' 20''$$

$$\therefore \quad A = 24^\circ 08' 00''$$

Then $\quad AC = 305{\cdot}54 \sin 136^\circ 21' 20'' \operatorname{cosec} 24^\circ 08' 00'' = 515{\cdot}77$

Gradient $\quad AC = 392{\cdot}58$ ft in $515{\cdot}77$ ft

$$= \underline{1 \text{ in } 1{\cdot}314}$$

Example 4.13 (a) Show that when a pointing is made to an object which has a vertical angle h with a theodolite having its trunnion axis inclined at a small angle i to the horizontal, the error introduced into the horizontal circle reading as a result of the trunnion axis tilt is $i \tan h$.

(b) The observations set out below have been taken at a station P with a theodolite, both circles of which have two index marks. On face left, the vertical circle nominally records 90° minus the angle of elevation. The plate bubble is mounted parallel to the trunnion axis and is graduated with the zero of the scale at the centre of the tube one division represents 20″.

The intersection of the telescope cross-hairs was set on signals A and B on both faces of the theodolite. The means of the readings of the circle and the plate bubble readings were:

Signal	Face	Horizontal Circle	Vertical Circle	Midpoint of Bubble
A	left	116° 39′ 15″	90° 00′ 15″	1·0 division towards circle
	right	346° 39′ 29″	270° 00′ 17″	1·0 division towards circle
B	left	301° 18′ 36″	80° 03′ 52″	central
	right	121° 18′ 30″	279° 56′ 38″	2·0 division towards circle

The vertical axis was then rotated so that the horizontal circle reading with the telescope in the face left position was 256° 40′; the reading of the midpoint of the bubble was then 0·4 division away from the circle.

If the effect of collimation error c on a horizontal circle reading is $c \sec h$, calculate the collimation error, the tilt of the trunnion axis and the index error of the theodolite, the altitude of the vertical axis when the above observations were taken, and the value of the horizontal angle APB. (N.U.)

At A (Fig. 4.62). As the bubble reading is equal and opposite, on change of face the horizontal plate is horizontal at 90° to the line of sight. The bubble is out of adjustment by 1 division = −20″ F.L.
At 90° *to A* the corrected bubble reading gives

$$\text{F.L.} \quad 0{\cdot}4 + 1{\cdot}0 = +1{\cdot}4 \text{ div.} = +28''$$

The horizontal plate is thus inclined at 28″ as is the vertical axis, in the direction A.

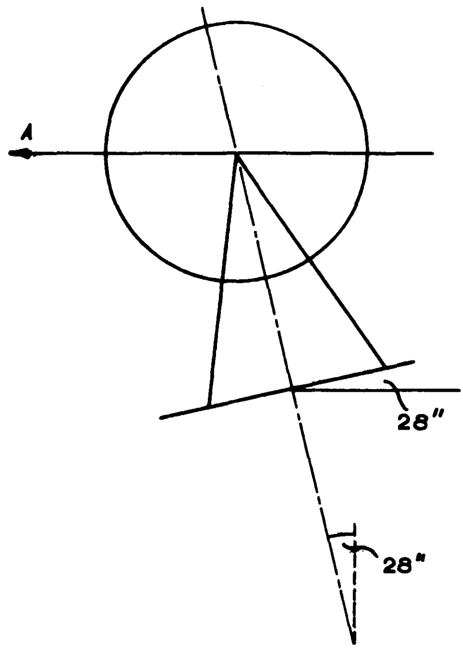


Fig. 4.62

At B (Fig. 4.63). The corrected bubble readings give

$$\left.\begin{array}{lll} \text{F.L.} & 0{\cdot}0 + 1{\cdot}0 = & +1{\cdot}0 \\ \text{F.R} & 2{\cdot}0 - 1{\cdot}0 = & +1{\cdot}0 \end{array}\right\} \quad \text{i.e. } +20''$$

N.B. This value may be checked (see p. 414)

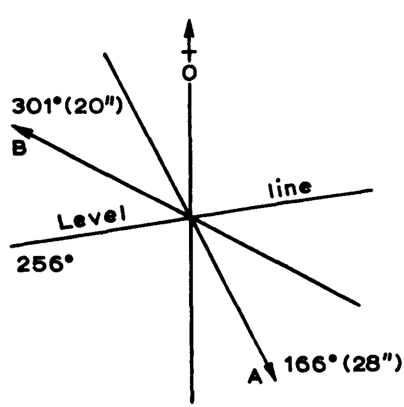


Fig. 4.63

$$
\begin{aligned}
\text{Apparent dip} &= \text{full dip cosine angle between} \\
&= 28'' \cos(301 - 256) \\
&= 28'' \cos 45^\circ \\
&= \underline{20''}
\end{aligned}
$$

The effect of instrumental errors in pointings (Fig. 4.64)

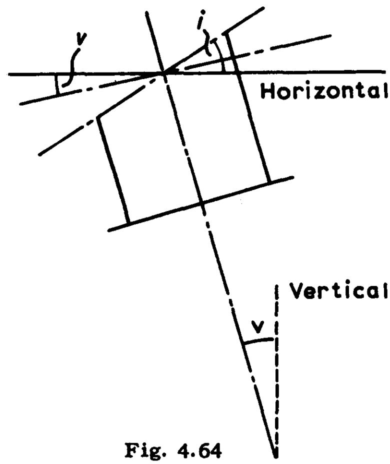


Fig. 4.64

Horizontal

Collimation $\theta_c = \pm c \sec h$ (if $h = 0$ $\theta_c = c$); the mean of faces left and right gives the correct value.

Trunnion Axis $\theta_i = \pm i \tan h$ (if $h = 0$ $\theta_i = 0$); the mean of faces left and right give the correct value.

Vertical Axis $\theta_v = v \tan h$ (if $h = 0$ $\theta_v = 0$); the sign is dependent on the inclination of the axis. (F.L. inclination towards circle, with $+h$, θ_v is $-$ve

Vertical

Index error $\delta h = \frac{1}{2}\{h_l - h_r\}$; the mean of faces left and right gives the correct value

Application to given values:

At A (F.L.) $166^\circ 39' 15'' + c \sec h - v \tan h + i \tan h$

i.e. $166^\circ 39' 15'' + c$

(F.R.) $346^\circ 39' 29'' - c$

F.L. must equal F.R.

$$
\begin{aligned}
\therefore\ 166^\circ 39' 15'' + c &= 346^\circ 39' 29'' - 180^\circ - c \\
\therefore\ 2c &= +14'' \\
\underline{c} &\underline{= +\ 7''} \qquad \text{(collimation error)}
\end{aligned}
$$

At B

(F.L.) $301°18'36'' + 7 \sec 9°57'' - 20 \tan 9°57' + i \tan 9°57''$

i.e. $301°18'36'' + 7{\cdot}1 - 3{\cdot}5 + 0{\cdot}175\,i$

(F.R.) $121°18'30'' - 7{\cdot}1 - 3{\cdot}5 - 0{\cdot}175\,i$

F.L. must equal F.R.

$$\therefore\ 301°18'36'' + 7{\cdot}1 - 3{\cdot}5 + 0{\cdot}175\,i = 121°18'30'' + 180 - 7{\cdot}1 - 3{\cdot}5 - 0{\cdot}175\,i$$

$$\text{i.e.}\quad 0{\cdot}35\,i = -20{\cdot}2''$$

$$i = \underline{-58''} \quad (\theta_i = -10{\cdot}1'')$$

(trunnion axis error)

Corrected readings

F.L.	A	$166°39'15'' + 7''$	=	$166°39'22''$
	B	$301°18'36'' + 7{\cdot}1'' - 3{\cdot}5'' - 10{\cdot}1$	=	$301°18'29{\cdot}5''$
		Angle APB	=	$134°39'07{\cdot}5''$
F.R.	A	$346°39'29'' - 7''$	=	$346°39'22''$
	B	$121°18'30'' - 7{\cdot}1 - 3{\cdot}5 + 10{\cdot}1$	=	$121°18'29{\cdot}5''$
		Angle APB	=	$134°39'07{\cdot}5''$

Vertical angles

At A: $\delta h = \frac{1}{2}\{(90 - 90°00'15'') - (270°00'17'' - 270)\} = -16''$

At B: $\delta h = \frac{1}{2}\{(90 - 80°03'52'') - (279°56'38'' - 270)\} = -15''$

N.B. The discrepancy is assumed to be an observational error.

4.7 The Auxiliary Telescope

This is used where steep sights are involved and in two possible forms:

(1) Side telescope (2) Top telescope

4.71 Side telescope

There are two methods of using this form of telescope: (a) in adjustment and (b) out of adjustment with the main telescope.

Adjustment

(a) *Alignment* (Fig. 4.65). Observe a point A with the main telescope. Turn in azimuth to observe with the side telescope without altering the vertical circle. Raise or lower the side telescope until the horizontal cross-hair coincides with the target A. The horizontal hairs are now in the same plane.

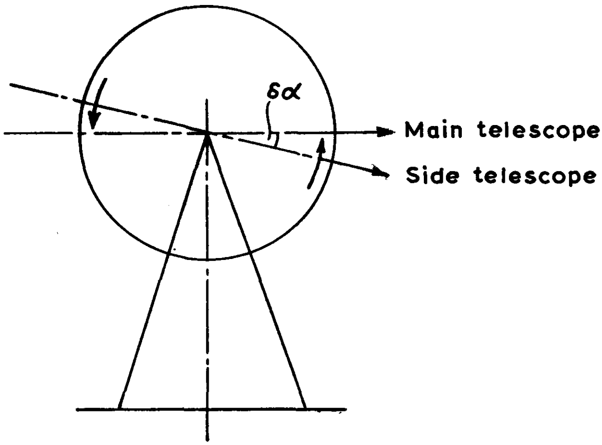


Fig. 4.65 Alignment of telescopes

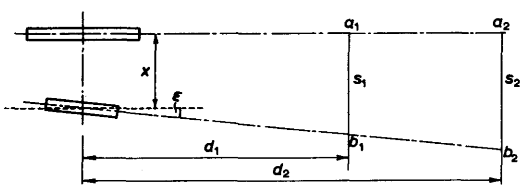


Fig. 4.66

(b) *Parallel lines of sight* (Fig. 4.66). If x is the eccentricity of the telescope at the instrument this should be constant between the lines of sight (see Fig. 4.66).

At a distance d_1 from the instrument, a scale set horizontally may be read as a_1b_1 giving an intercept s_1, and then at d_2 readings a_2b_2 give intercept s_2.

If the lines of sight are parallel,

$$s_1 = s_2 = x$$

If not, the angle of convergence/divergence ϵ is given as

$$\epsilon = \tan^{-1}\frac{s_2 - s_1}{d_2 - d_1} \tag{4.56}$$

If $s_2 > s_1$, the angle is $+ve$, i.e. diverging

If $s_2 < s_1$, the angle is $-ve$, i.e. converging.

The amount of eccentricity x can be obtained from the same readings.

$$\frac{d_1}{d_2 - d_1} = \frac{s_1 - x}{s_2 - s_1}$$

i.e.
$$x = \frac{s_1(d_2 - d_1) - d_1(s_2 - s_1)}{(d_2 - d_1)}$$

$$= \frac{s_1 d_2 - s_2 d_1}{d_2 - d_1} \quad (4.57)$$

By making the intercept $s_2 = x$, the collimation of the auxiliary telescope can be adjusted to give parallelism of the lines of sight.

Observations with the side telescope

(a) *Vertical Angles.* If the alignment is adjusted, then the true vertical angle will be observed.

If an angular error of $\delta\alpha$ exists between the main and the side telescope, then the mean of face left and face right observations is required, i.e.

$$\text{F.L.} = \alpha_1 + \delta\alpha = \alpha$$
$$\text{F.R.} = \alpha_2 - \delta\alpha = \alpha$$
$$\therefore \quad \alpha = \tfrac{1}{2}(\alpha_1 + \alpha_2) \quad (4.58)$$

(b) *Horizontal Angles* (Fig. 4.67)

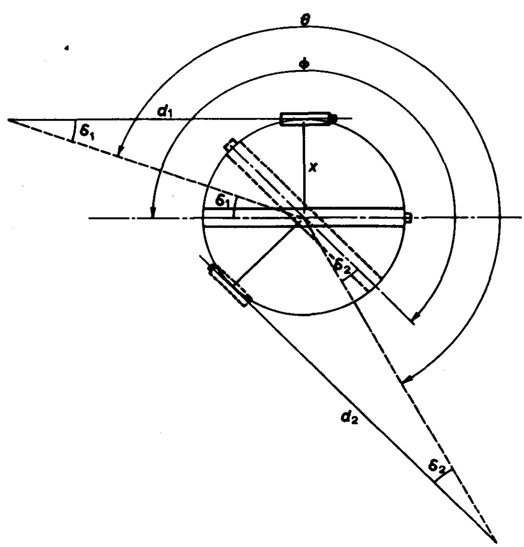


Fig. 4.67 Horizontal angles with the side telescope

If θ = the true horizontal angle;
ϕ = the recorded horizontal angle;
δ_1 and δ_2 = errors due to eccentricity,

then $\theta = \phi_1 - \delta_1 + \delta_2$ say F.L.

$\theta = \phi_2 + \delta_1 - \delta_2$ F.R.

$$\text{i.e.}\quad \theta = \tfrac{1}{2}(\phi_1 + \phi_2) \tag{4.59}$$

Example 4.14 In testing the eccentricity of a side telescope, readings were taken on to levelling staves placed horizontally at X and Y 100 and 200 ft respectively from the instrument.

Readings at X	5·42 ft	5·01 ft
at Y	3·29 ft	2·79 ft

Calculate (a) the collimation error (ϵ), (b) the eccentricity (x).
(R.I.C.S./M)

From the readings, $s_1 = 5{\cdot}42 - 5{\cdot}01 = 0{\cdot}41$

$s_2 = 3{\cdot}29 - 2{\cdot}79 = 0{\cdot}50$

Then, by Eq. (4.56),

$$\epsilon = \tan^{-1}\frac{s_2 - s_1}{d_2 - d_1}$$

$$\epsilon'' = \frac{206\,265 \times (0{\cdot}50 - 0{\cdot}41)}{200 - 100}$$

$$= 185{\cdot}6'' = \underline{03'\ 06''}$$

by Eq. (4.57)

$$x = \frac{s_1 d_2 - s_2 d_1}{d_2 - d_1}$$

$$= \frac{0{\cdot}41 \times 200 - 0{\cdot}50 \times 100}{200 - 100}$$

$$= \frac{82 - 50}{100} = \underline{0{\cdot}32\text{ ft}}$$

Based on the metric system the question becomes:

In testing the eccentricity of a side telescope, readings were taken on to levelling staves placed horizontally at X and Y, 30·48 m and 60·96 m respectively from the instrument.

Readings at X	1·652 m	1·527 m
at Y	1·003 m	0·850 m

Calculate (a) the collimation error (ϵ), (b) the eccentricity (x)

$s_1 = 1{\cdot}652 - 1{\cdot}527 = 0{\cdot}125$ m

$s_2 = 1{\cdot}003 - 0{\cdot}850 = 0{\cdot}153$ m

Then

$$\epsilon = \frac{206\,265 \times (0{\cdot}153 - 0{\cdot}125)}{60{\cdot}96 - 30{\cdot}48}$$

$$= 189{\cdot}5'' = 03'\,10''$$

and $$x = \frac{0 \cdot 125 \times 60 \cdot 96 - 0 \cdot 153 \times 30 \cdot 48}{30 \cdot 48}$$

$$= 0 \cdot 097 \text{ m } (0 \cdot 32 \text{ ft})$$

The effect of eccentricity x and collimation error ϵ

In Fig. 4.68, assuming small angles,

$$\text{Angle } AOB = \delta''$$

$$= \frac{206\,265\,x}{d} \qquad (4.60)$$

$$\text{Angle } BSC = \epsilon''$$

$$= \frac{206\,265\,y}{d} \qquad (4.61)$$

$$\text{Angle } AOC = e$$

$$= \frac{206\,265(x+y)}{d} \qquad (4.62)$$

$$e = \delta + \epsilon \qquad (4.63)$$

$$\therefore \text{ Angle } BOC = \text{Angle } BSC$$

$$= \epsilon$$

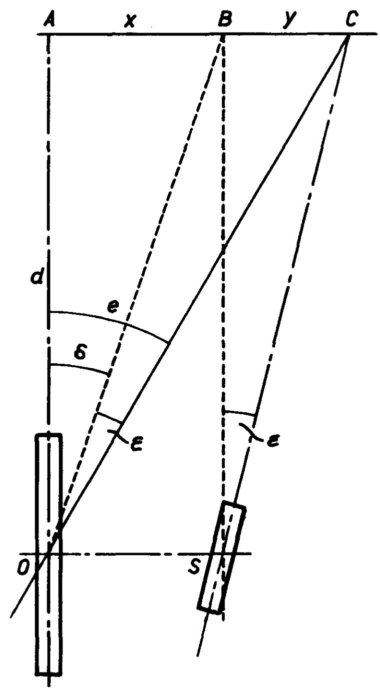


Fig. 4.68

As the eccentricity x is constant, the angle (δ) is dependent upon the length of sight d.

As the collimation angle ϵ is constant, it has the same effect as the collimation error in the main telescope. It affects the horizontal angle by $\epsilon \sec \alpha$, where α is the vertical angle.

Assuming the targets are at different altitudes, the true horizontal angle θ, Fig. 4.67, is given as

$$\theta = \phi_1 - (\delta_1 + \epsilon \sec \alpha_1) + (\delta_2 + \epsilon \sec \alpha_2) \quad \text{say F.L.} \quad (4.64)$$

Also $$\theta = \phi_2 + (\delta_2 + \epsilon \sec \alpha_1) - (\delta_2 + \epsilon \sec \alpha_2) \quad \text{F.R.} \quad (4.65)$$

$$\therefore \quad \theta = \tfrac{1}{2}(\phi_1 + \phi_2) \qquad (4.66)$$

Thus the mean of the face left and face right values eliminates errors from all the above sources.

Example 4.15 Using the instrument of Example 4.14, the following data were recorded:

($\epsilon = 03'\,06''$ $x = 0{\cdot}32$ ft)

Station set at	Station observed	Horizontal circle	Vertical circle	Remarks
B	A	$0°\,05'\,20''$	$+30°\,26'$	Horizontal lengths
	C	$124°\,10'\,40''$	$-10°\,14'$	$AB = 100'$ $BC = 300'$
				side telescope on right.

Calculate the true horizontal angle ABC

(R.I.C.S./M)

By Eq. (4.64),

True horizontal angle (θ) $= \phi - (\delta_1 + \epsilon \sec \alpha_1) + (\delta_2 + \epsilon \sec \alpha_2)$

$$\delta_1 = \frac{206\,265 \times 0{\cdot}32}{100} = 66'' = 1'\,06''$$

$$\delta_2 = \frac{206\,265 \times 0{\cdot}32}{300} = 22'' = 0'\,22''$$

$$\epsilon \sec \alpha_1 = 186'' \sec 30°\,26' = 215{\cdot}7'', \quad \text{say } 216'' = 3'\,36''$$

$$\epsilon \sec \alpha_2 = 186'' \sec 10°\,14' = 189{\cdot}0'' = 3'\,09''$$

$$\phi = 124°\,10'\,40'' - 0°\,05'\,20'' = 124°\,05'\,20''$$

$$\therefore \quad \theta = 124°\,05'\,20'' - (1'06'' + 3'36'') + (0'\,22'' + 3'\,09'')$$

$$= 124°\,04'\,09''$$

4.72 Top telescope

In this position the instrument can be used to measure horizontal angles only if it is in correct adjustment, as it is not possible to change face.

Adjustment

(a) *Alignment.* The adjustment is similar to that of the side telescope but observations are required by both telescopes on to a plumb line to ensure that the cross-hairs are in the same plane.

(b) *Parallel lines of sight* (Fig. 4.69). Here readings are taken on vertical staves with the vertical circle reading zero.

The calculations are the same as for the side telescope:

$$\epsilon' = \tan^{-1} \frac{s_2 - s_1}{d_2 - d_1} \tag{4.67}$$

and

$$x' = \frac{s_1 d_2 - s_2 d_1}{d_2 - d_1} \tag{4.68}$$

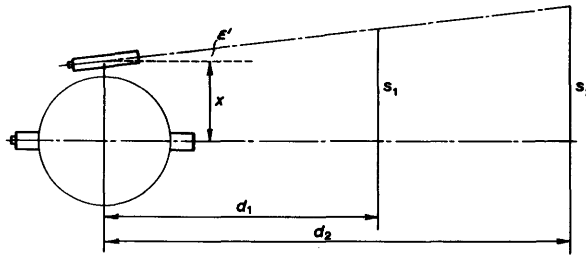


Fig. 4.69

Measurement of Vertical Angles (Fig. 4.70)

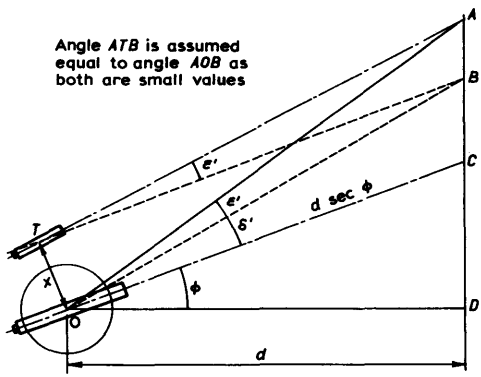


Fig. 4.70 Measurement of vertical angles with the top telescope

The angular error (δ') due to eccentricity is given as

$$\delta' = \tan^{-1} \frac{x}{d \sec \phi} \tag{4.69}$$

where x = eccentricity

d = horizontal distance

ϕ = recorded vertical angle

If ϵ' is the error due to collimation then the true vertical angle (θ) is given as

$$\pm\theta = \pm(\phi + \delta' + \epsilon') \tag{4.70}$$

assuming δ' and ϵ' small.

4.8 Angular Error due to Defective Centring of the Theodolite

The angular error depends on the following, Fig. 4.72:

(a) linear displacement x,

(b) direction of the instrument B_1 relative to the station B

(c) length of lines a and c.

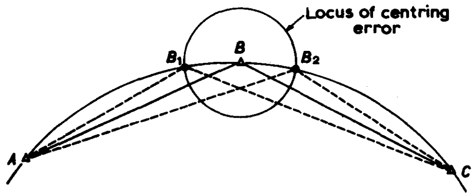


Fig. 4.71 Minimum error due to defective centring of the theodolite

The instrument may be set on the circumference of the circle of radius x.

No error will occur if the instrument is set up at B_1 or B_2 (Fig. 4.71), where A, B_1, B, B_2 and C lie on the arc of a circle.

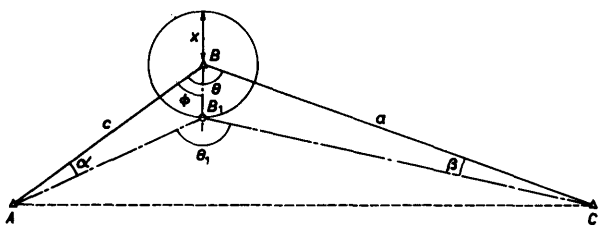


Fig. 4.72 The effects of centring errors

In Fig. 4.72, let the instrument be set at B_1 instead of B.

$\therefore$ Angle θ_1 is measured instead of θ

i.e. $\theta = \theta_1 - (\alpha + \beta)$

Assume the misplumbing x to be in a direction ϕ relative to the line AB.

In triangle ABB_1,

$$\sin\alpha = \frac{x \sin\phi}{AB_1} \tag{4.71}$$

As the angle α is small,

$$\alpha'' = \frac{206265\, x \sin\phi}{c} \tag{4.72}$$

Similarly,

$$\beta'' = \frac{206265\, x \sin(\theta - \phi)}{a} \tag{4.73}$$

$\therefore$ Total error E $= \alpha + \beta = 206265\, x\left[\dfrac{\sin\phi}{c} + \dfrac{\sin(\theta-\phi)}{a}\right]$ (4.74)

For maximum and minimum values,

$$\frac{dE}{d\phi} = 206265\, x\left[\frac{\cos\phi}{c} - \frac{\cos(\theta-\phi)}{a}\right] = 0$$

i.e.
$$\frac{\cos\phi}{c} = \frac{\cos(\theta - \phi)}{a}$$

$$\cos\phi = \frac{c}{a}(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$\div \sin\phi$
$$\cot\phi = \frac{c}{a}(\cos\theta\cot\phi + \sin\theta)$$

$$\cot\phi\left(1 - \frac{c\cos\theta}{a}\right) = \frac{c\sin\theta}{a}$$

$$\therefore \quad \underline{\cot\phi = \frac{c\sin\theta}{a - c\cos\theta}} \tag{4.75}$$

N.B. (1) If $\phi = 90°$, $\theta = 0$ or $180°$

(2) If $a \gg c$, then $\phi \to 90°$, i.e. the maximum error exists when ϕ tends towards 90° relative to the shorter line.

(3) If $a = c$, $\phi = \theta/2$.

Professor Briggs proves that the probable error in the measured angle is

$$e = \pm\frac{2x}{\pi}\sqrt{\left(\frac{1}{a^2} + \frac{1}{c^2} - \frac{2\cos\theta}{a+c}\right)} \tag{4.76}$$

Example 4.16 The centring error in setting up the theodolite at station B in an underground traverse survey is $\pm 0{\cdot}2$ in. Compute the maximum and minimum errors in the measurement of the clockwise angle ABC induced by the centring error if the magnitude of the angle is approximately 120° and the length of the lines AB and BC is approximately 80·1 and 79·8 ft respectively.

(R.I.C.S.)

(1) The minimum error as before will be nil.

(2) The maximum error on the bisection of the angle ABC is $AB \simeq BC$.

i.e.
$$\phi = \frac{\theta}{2} = 60° \qquad a \simeq c \simeq 80\text{ ft}$$

$$x = \frac{0{\cdot}2}{12} = 0{\cdot}0167\text{ ft}$$

$$\therefore \quad E = 206\,265 \times 0{\cdot}016\,7\left[\frac{\sin 60}{80} + \frac{\sin(120-60)}{80}\right]$$

$$= 206\,265 \times 0{\cdot}016\,7 \times 2\sin 60/80$$

$$= 74\text{ seconds} \quad \text{i.e. } \underline{1'\,14''}$$

By Professor Briggs' equation, the probable error

$$e = \pm\frac{2 \times 0{\cdot}016\,7}{3{\cdot}141\,6}\sqrt{\left(\frac{1}{80^2} + \frac{1}{80^2} - \frac{2\cos 120}{80+80}\right)}$$

$$= \pm 35'' \quad \text{i.e.} \simeq 1/2 \text{ max error.}$$

4.9 The Vernier

This device for determining the decimal parts of a graduated scale may be of two types:

(1) Direct reading

(2) Retrograde

both of which may be single or double.

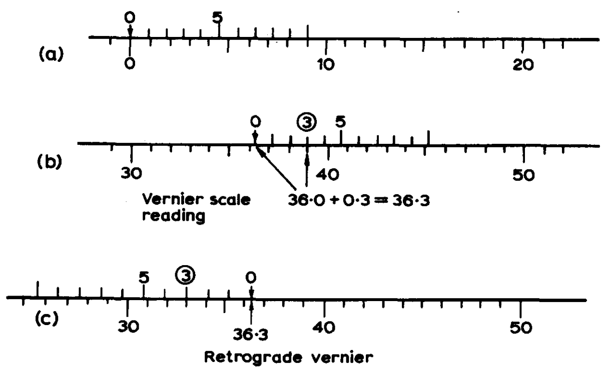


Fig. 4.73 Verniers

4.91 Direct reading vernier

Let d = the smallest value on the main scale

v = the smallest value on the vernier scale

n = number of spaces on the vernier

n vernier spaces occupy $(n-1)$ main scale spaces

i.e. $$nv = (n-1)d$$

$$v = \frac{(n-1)d}{n} \tag{4.77}$$

Therefore the least count of the reading system is given by:

$$d - v = d - \frac{d(n-1)}{n}$$

$$= d\left(1 - \frac{n-1}{n}\right)$$

$$d - v = d\left(\frac{n-n+1}{n}\right) = \frac{d}{n} \tag{4.78}$$

Thus the vernier enables the main scale to be read to $\frac{1}{n}$th of 1 division.

Example 4.17 If the main scale value $d = \frac{1}{10}''$ and the number of spaces on the vernier $(n) = 10$, the vernier will read to $1/10 \times 1/10 = 1/100$ in.

4.92 Retrograde vernier

In this type, n vernier division occupy $(n + 1)$ main scale divisions,

i.e.
$$nv = (n+1)d$$
$$v = d\left(\frac{n+1}{n}\right) \qquad (4.79)$$

$$\text{The least count} = v - d = d\left(\frac{n+1}{n}\right) - d$$
$$= d\left(\frac{n+1}{n} - 1\right)$$
$$v - d = \frac{d}{n} \quad \text{as before} \qquad (4.80)$$

4.93 Special forms used in vernier theodolites

In order to provide a better break down of the graduations, the vernier may be extended in such a way that n vernier spaces occupy $(mn - 1)$ spaces on the main scale. (m is frequently 2.)

$$nv = (mn-1)d$$
$$v = d\left(\frac{mn-1}{n}\right) \qquad (4.81)$$

$$\text{The least count} = md - v = md - d\left(\frac{mn-1}{n}\right)$$
$$= d\left(m - \frac{mn-1}{n}\right)$$
$$md - v = \frac{d}{n} \quad \text{as before} \qquad (4.82)$$

4.94 Geometrical construction of the vernier scale

In Fig. 4.74 (a) the main scale and vernier zeros are coincident.

For the direct reading vernier 10 divisions on the vernier must occupy 9 divisions on the main scale. Therefore

(1) Set off a random line OR of 10 units.

(2) Join R to V i.e. the end of the random line R to the end of the vernier V.

(3) Parallel through each of the graduated lines or the random line

to cut the main scale so that 1 division of the vernier = 0·9 divisions of the main scale.

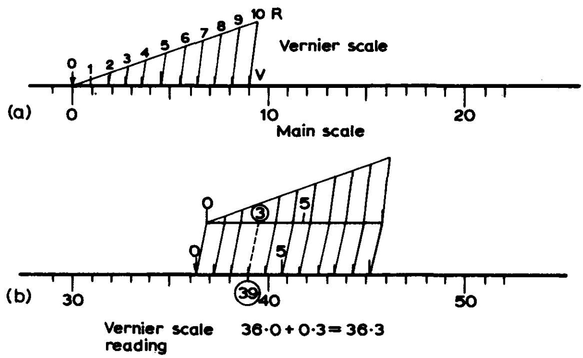


Fig. 4.74 Construction of a direct reading vernier

To construct a vernier to a given reading

In Fig. 4.74(b) the vernier is required to read 36·3. It is thus required to coincide at the 3rd division, i.e. $3 \times 0{\cdot}9 = 2{\cdot}7$ main scale division beyond the vernier index.

Therefore coincidence will occur at $(36{\cdot}3 + 2{\cdot}7) = 39{\cdot}0$ on the main scale and 3 on the vernier scale.

The vernier is constructed as above in the vicinity of the point of coincidence. The appropriate vernier coincidence line (i.e. 3rd) is joined to the main scale coincidence line (i.e. 39·0) and lines drawn parallel as before will produce the appropriate position of the vernier on the main scale.

In the case of the retrograde vernier, Fig. 4.75, 10 divisions on the vernier equals 11 divisions on the main scale, and therefore the point of coincidence of 3 on the vernier with the main scale value is

$$36{\cdot}3 - (3 \times 1{\cdot}1) = 36{\cdot}3 - 3{\cdot}3$$
$$= \underline{33{\cdot}0}$$

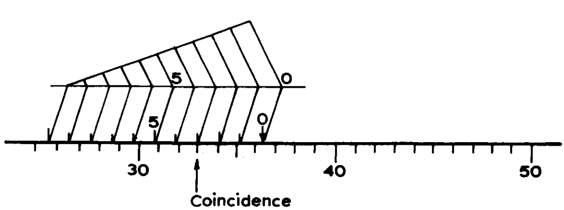


Fig. 4.75 Construction of a retrograde vernier

Example 4.18 Show how to construct the following verniers:

(1) To read to 10″ on a limb divided to 10 minutes.

(2) To read to 20″ on a limb divided to 15 minutes.

(3) The arc of a sextant is divided to 10 minutes. If 119 of these divisions are taken as the length of the vernier, into how many divisions must the vernier be divided in order to read to (a) 5 seconds (b) 10 seconds? (I.C.E.)

(1) The least count of the vernier is given by Eq. (4.77) as d/n

$$\therefore \quad 10'' = \frac{10 \times 60}{n}$$

$$\therefore \quad n = \frac{600}{10} = \underline{60}$$

Therefore the number of spaces on the vernier is 60 and the number of spaces on the main scale is 59.

(2) Similarly, $20'' = \dfrac{15 \times 60}{n}$

$$\therefore \quad n = \frac{15 \times 60}{20} = \underline{45}$$

i.e. the number of spaces in the vernier is 45 and the number of spaces on the main scale is 44.

(3) The number of divisions 119 is not required, and the calculation is exactly as above.

(a) $$n = \frac{10 \times 60}{5} = \underline{120}$$

(b) $$n = \frac{10 \times 60}{10} = \underline{60}$$

Exercises 4 (b)

3. The eccentricity of the line of collimation of a theodolite telescope in relation to the azimuth axis is 1/40th of an inch. What will be the difference, attributable to this defect, between face right and face left measurement of an angle if the lengths of the drafts adjacent to the instrument are 20 ft and 120 ft respectively?

(M.Q.B./S Ans. 17·9″)

4. A horizontal angle is to be measured having one sight elevated to 32° 15′ whilst the other is horizontal. If the vertical axis is inclined at 40″ to the vertical, what will be the error in the recorded value?

(Ans. 25″)

5. In the measurement of a horizontal angle the mean angle of elevation of the backsight is 22° 12′ whilst the foresight is a depression

of 37° 10′. If the lack of verticality of the vertical axis causes the horizontal axis to be inclined at 50″ and 40″ respectively in the same direction, what will be the error in the recorded value of the horizontal angle as the mean of face left and right observations?

(Ans. 51″)

6. In a theodolite telescope the line of sight is not perpendicular to the horizontal axis in but in error by 5 minutes. In measuring a horizontal angle on one face, the backsight is elevated at 33° 34′ whilst the foresight is horizontal. What error is recorded in the measured angle?

(Ans. 60″)

7. The instrument above is used for producing a level line AB by transitting the telescope, setting out C_1, and then by changing face the whole operation is repeated to give C_2. If the foresight distance is 100 ft, what will be the distance between the face left and face right positions, i.e. C_1C_2?

(Ans. 6·98 in.)

8. Describe with the aid of a sketch, the function of an internal focussing lens in a surveyors telescope and state the advantages and disadvantages of internal focussing as compared to external focussing.

In a telescope, the object glass of focal length 6 in. is located 8 in. from the diaphragm. The focussing lens is midway between these when a staff 80 ft away is focussed. Determine the focal length of the focussing lens.

(L.U. Ans. 4·154 in.)

9. In testing the trunnion axis of a vernier theodolite, the instrument was set up at '0', 100 ft from the base of the vertical wall of a tall building where a well-defined point A was observed on face left at a vertical angle of 36° 52′. On lowering the telescope horizontally with the horizontal plate clamped, a mark was placed at B on the wall. On changing face, the whole operation was repeated and a second position C was fixed.

If the distance BC measured 0·145 ft, calculate the inclination of the trunnion axis.

(Ans. 3′ 20″)

10. The following readings were taken on fine sighting marks at B and C from a theodolite station A.

Instrument At	To	Vertical Angle	F.R.S.R. Vernier A	F.R.S.R. Vernier B	F.L.S.L. Vernier A	F.L.S.L. Vernier B
A	B	72° 30′	292° 26′ 30″	112° 26′ 30″	23° 36′ 24″	203° 36′ 24″
	C	−10° 24′	52° 39′ 36″	232° 39′ 36″	143° 50′ 10″	323° 50′ 10″
	B	72° 30′	292° 26′ 30″	112° 26′ 30″	23° 36′ 24″	203° 36′ 24″

Calculate the value of the horizontal collimation error assuming

this to be the only error in the theodolite and state whether it is to the right or left of the line perpendicular to the trunnion axis when the instrument is face left.

Describe as briefly as possible how you would adjust the theodolite to eliminate this error.

(L.U. Ans. 8·66″ left)

11. The focal length of object glass and anallatic lens are 5 in and $4\frac{1}{2}$ in. respectively. The stadia interval was 0·1 in.

A field test with vertical staffing yielded the following:

Inst Station	Staff Station	Staff Intercept	Vertical Angle	Measured Horizontal Distance (ft)
P	Q	2·30	$+7° 24'$	224·7
	R	6·11	$-4° 42'$	602·3

Find the distance between the object glass and anallatic lens. How far, and in what direction, must the latter be moved so that the multiplying constant of the instrument is to be 100 exactly?

(L.U. Ans. 7·25 in.; 0·02 in. away from objective)

12. An object is 20 ft from a convex lens of focal length 6 in. On the far side of this lens a concave lens of focal length 3 in. is placed. Their principal axes are on the line of the object, and 3 in. apart. Determine the position, magnification and nature of the image formed.

(Ans. Virtual image 43 in. away from the concave lens towards object; magnification 0·28)

13. A compound lens consists of two thin lenses, one convex, the other concave, each of focal length 6 in. and placed 3 in apart with their principal axes common. Find the position of the principal focus of the combination when the light is incident first on (a) the convex lens and (b) the concave lens.

(Ans. (a) real image 6 in. from concave lens away from object,
(b) real image 18 in. from convex lens away from object)

14. Construct accurately a 30 second vernier showing a reading of $124° 23' 30''$ on a main scale divided to 20 minutes. A straight line may be used to represent a sufficient length of the arc to a scale of 0·1 in. to 20 min.

(N.R.C.T.)

15. (a) Explain the function of a vernier.

(b) Construct a vernier reading 4·57 in. on a main scale divided to 1/10 in.

(c) A theodolite is fitted with a vernier in which 30 vernier divisions are equal to $14° 30'$ on a main scale divided to 30 minutes. Is the vernier direct or retrograde, and what is its least count?

(N.R.T.C. Ans. direct; 1 min.)

Bibliography

THOMAS, W.N., *Surveying* (Edward Arnold)
SANDOVER, J.A., *Plane Surveying* (Edward Arnold)
GLENDENNING, J., *Principles and Use of Surveying Instruments* (Blackie)
CLARK, D., *Plane and Geodetic Surveying*, Vol. 1 (Constable)
HIGGINS, A.L., *Elementary Surveying* (Longmans)
TAYLOR, W.E., *Theodolite Design and Construction* (Cooke, Troughton & Simms)
CURTIN, W. and LANE, R.F., *Concise Practical Surveying* (English Universities Press)
SHEPPARD, J.S. *Theodolite Errors and Adjustments* (Stanley)
The Theodolite and Its Application English version by A.H. Ward, F.R.I.C.S. (Wild (Heerbrugg))
JAMESON, A.H. *Advanced Surveying* (Pitman)
HIGGINS, A.L. *Higher Surveying* (Macmillan)
SMIRNOFF, M.V., *Measurements for Engineering and Other Surveys* (Prentice – Hall)
BANNISTER, A. and RAYMOND, S., *Surveying* (Pitman)
DIBDIN, F.S.H., *Essentials of Light* (Macmillan)
NELKON, M., *Light and Sound* (Heinemann)
STANLEY, W.W., *Introduction to Mine Surveying* (Stanford University Press)

5 LEVELLING

5.1 Definitions

Levelling is the process concerned with the determination of the differences in elevation of two or more points between each other or relative to some given datum.

A *Datum* may be purely arbitrary but for many purposes it is taken as the mean sea level (M.S.L.) or Ordnance Datum (O.D.).

A *Level Surface* can be defined as a plane, tangential to the earth's surface at any given point. The plane is assumed to be perpendicular to the direction of gravity which for most practical purposes is taken as the direction assumed by a plumb-bob.

A *level line*, Fig. 5.1, is a line on which all points are equidistant from the centre of gravity. Therefore, it is curved and (assuming the earth to be a sphere) it is circular. For more precise determinations the geoidal shape of the earth must be taken into consideration.

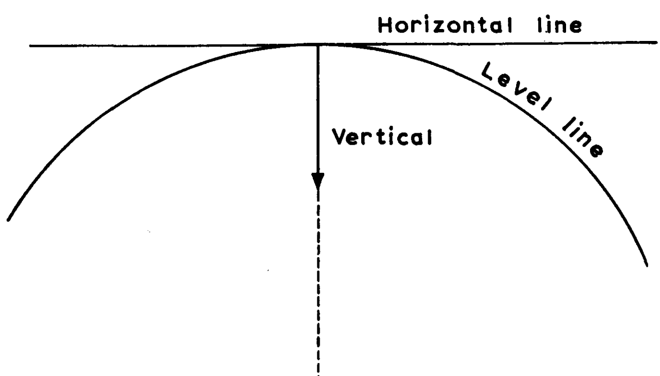


Fig. 5.1

A *Horizontal Line*, Fig. 5.1, is tangential to a level line and is taken, neglecting refraction, as the line of collimation of a perfectly adjusted levelling instrument. (As the lengths of sights in levelling are usually less than 450 ft, level and horizontal lines are assumed to be the same – see §5.6 Curvature and Refraction.)

The *Line of Collimation* is the imaginary line joining the intersection of the main lines of the diaphragm to the optical centre of the object-glass.

Mean Sea Level. This is the level datum line taken as the

reference plane. In the British Isles the Ordnance Survey originally accepted the derived mean sea level value for Liverpool. This has been superseded by a value based upon Newlyn in Cornwall.

Bench Mark (⩚) (B.M.). This is a mark fixed by the Ordnance Survey and cut in stable constructions such as houses or walls. The reduced level of the horizontal bar of the mark is recorded on O.S. maps and plans.

Temporary Bench Mark (T.B.M.). Any mark fixed by the observer for reference purposes.

Backsight (B.S.) is the first sight taken after the setting up of the instrument. Initially it is usually made to some form of bench mark.

Foresight (F.S.) is the last sight taken before moving the instrument.

Intermediate Sight (I.S.) is any other sight taken.

N.B. During the process of levelling the instrument and staff are never moved together, i.e. whilst the instrument is set the staff may be moved, but when the observations at one setting are completed the staff is held at a selected stable point and the instrument is moved forward. The staff station here is known as a *Change Point* (C.P.).

5.2 Principles

Let the staff readings be a, b, c etc.

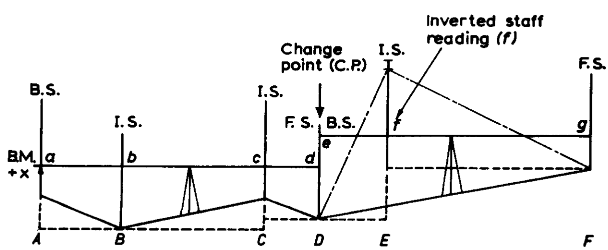


Fig. 5.2

In Fig. 5.2,

Difference in level	A to $B = a-b$	$a < b$	$\therefore$ −ve	i.e. fall
	B to $C = b-c$	$b > c$	$\therefore$ +ve	i.e. rise
	C to $D = c-d$	$c < d$	$\therefore$ −ve	i.e. fall
	D to $E = e-(-f)$	inverted staff	$\therefore$ +ve	i.e. rise
	E to $F = -f-g$		−ve	i.e. fall

$$A \text{ to } F = (a-b)+(b-c)+(c-d)+(e+f)+(-f-g)$$
$$= a-d+e-g$$

$$= (a+e) - (d+g)$$
$$= \Sigma \text{B.S.} - \Sigma \text{F.S.}$$
$$\Sigma \text{ rises} = (b-c) + (e+f)$$
$$\Sigma \text{ falls} = (b-a) + (d-c) + (f+g)$$
$$\Sigma \text{ rises} - \Sigma \text{ falls} = (a+e) - (d+g) = \Sigma \text{B.S.} - \Sigma \text{F.S.} \qquad (5.1)$$

The difference in level between the start and finish

$$= \Sigma \text{B.S.} - \Sigma \text{F.S.} = \Sigma \text{ rises} - \Sigma \text{ falls} \qquad (5.2)$$

N.B. (1) Intermediate values have no effect on the final results, and thus reading errors at intermediate points are not shown up.

(2) Where the staff is inverted, the readings are treated as negative values and indicated in booking by a bracket or an asterisk.

5.3 Booking of Readings

5.31 Method 1, rise and fall

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level
a					x
	b			$b-a$	$x-(b-a)$
	c		$b-c$		$x-(b-a)+(b-c)$
e		d		$d-c$	$x-(b-a)+(b-c)-(d-c)$
	$[f]$		$e-(-f)$		$x-(b-a)+(b-c)-(d-c)+(e+f)$
		g		$f+g$	$x-(b-a)+(b-c)-(d-c)+(e+f)-(f+g)$
$a+e$ $-(d+g)$		$d+g$	$(b-c)+(e+f)$	$(b-a)+(d-c)+(f+g)$	

Example 5.1 Given the following readings:

$a = 2{\cdot}06 \qquad e = 7{\cdot}41$

$b = 5{\cdot}13 \qquad f = -6{\cdot}84$

$c = 3{\cdot}28 \qquad g = 3{\cdot}25$

$d = 3{\cdot}97$

N.B. (1) The difference between adjacent readings from the same instrument position gives rise or fall according to the sign + or −.

(2) At the change point B.S. and F.S. are recorded on the same line.

(3) Check ΣB.S. − ΣF.S. = ΣRise − ΣFall before working out reduced levels. Difference between reduced levels at start and finish must equal ΣB.S. − ΣF.S.

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level	Remarks
2·06					100·00	St. *A*. B.M. 100·00 A.O.D.
	5·13			3·07	96·93	St. *B*.
	3·28		1·85		98·78	St. *C*.
7·41		3·97		0·69	98·09	St. *D*. C.P.
	[6·84]		14·25		112·34	St. *E*. Inverted staff on girder
		3·25		10·09	102·25	St. *F*.
9·47		7·22	16·10	13·85	102·25	
−7·22			−13·85		100·00	
+2·25			+ 2·25		+2·25	

5.32 Method 2, height of collimation

B.S.	I.S.	F.S.	Height of Collimation	Reduced Level
a			$x+a$	x
	b			$x+a-b$
	c			$x+a-c$
e		d	$x+a-d+e$	$x+a-d$
	$[f]$			$x+a-d+e-(-f)$
		g		$x+a-d+e-g$
$a+e$	$b+c-f$	$d+g$		$6x+5a-b-c-3d+2e+f-g$

Arithmetical Check

Σ Height of each collimation

$\times$ no. of applications $= 3(x+a)+2(x+a-d+e)$

$= 5x+5a-2d+2e$

Σ Reduced levels − first $= 5x+5a-b-c-3d+2e+f-g$

Σ I.S. $= +b+c \quad -f$

Σ F.S. $= +d \quad +g$

$5x+5a \quad -2d+2e$

Thus the full arithmetical check is given as:

Σ Reduced levels less the first + Σ I.S. + Σ F.S. should equal
Σ Height of each collimation $\times$ no. of applications (5.3)

Using the values in Example 5.1,

B.S.	I.S.	F.S.	Height of Collimation	Reduced Level	Remarks
2·06			102·06	100·00	St. *A* B.M. + 100·00 A.O.D.
	5·13			96·93	St. *B*
	3·28			98·78	St. *C*
7·41		3·97	105·50	98·09	St. *D* C.P.
	[6·84]			112·34	St. *E* Inverted staff on girder
		3·25		102·25	St. *F*
9·47	8·41	7·22		508·39	
7·22	−6·84				
2·25	1·57				

Check $508{\cdot}39 + 1{\cdot}57 + 7{\cdot}22 = 517{\cdot}18$

$$102{\cdot}06 \times 3 = 306{\cdot}18$$
$$105{\cdot}50 \times 2 = \underline{211{\cdot}00} = 517{\cdot}18$$

N.B. (1) The height of collimation = the reduced level of the B.S. + the B.S. reading.

(2) The reduced level of any station = height of collimation − reading at that station.

(3) Whilst ΣB.S. − ΣF.S. = the difference in the reduced level of start and finish this does not give a complete check on the intermediate values; an arithmetical error can be made without being noticed.

(4) The full arithmetical check is needed to ensure there is no arithmetical error.

On the metric system the bookings would appear thus:
(for less accurate work the third decimal place may be omitted)

B.S.	I.S.	F.S.	Rise	Fall	Height of Collimation	Reduced Level	Remarks
0·628					31·108	30·480	St. *A* 30·480m A.O.D.
	1·564			0·936		29·544	
	1·000		0·564			30·108	
2·259		1·210		0·210	32·157	29·898	
	[2·085]		4·344			34·242	
		0·991		3·076		31·166	
2·887	2·564	2·201	4·908	4·222		154·958	
	2·085						
2·201	0·479		4·222				
0·686			0·686			0·686	

Check on collimation $154{\cdot}958 + 0{\cdot}479 + 2{\cdot}201 = \underline{157{\cdot}638}$

$$31{\cdot}108 \times 3 = 93{\cdot}324$$
$$32{\cdot}157 \times 2 = \underline{64{\cdot}314}$$
$$\underline{157{\cdot}638}$$

Example 5.2 Using the height of collimation calculate the respective levels of floor and roof at each staff station relative to the floor level at *A* which is 20 ft above an assumed datum. It is important that a complete arithmetical check on the results should be shown. Note that the staff readings enclosed by brackets thus (3·43) were taken with the staff reversed.

B.S.	I.S.	F.S.	Height of Collimation	Reduced Level	Horizontal Distance (ft)	Remarks
2·47			22·47	20·00	0	Floor at *A*
	(3·43)			25·90	0	Roof at *A*
	3·96			18·51	50	Floor at *B*
	(2·07)			24·54	50	Roof at *B*
	4·17			18·30	100	Floor at *C*
	(1·22)			23·69	100	Roof at *C*
	3·54			18·93	150	Floor at *D*
(4·11)		(2·73)	21·09	25·20	150	Roof at *D*
	1·96			19·13	200	Floor at *E*
	(5·31)			26·40	200	Roof at *E*
	2·85			18·24	250	Floor at *F*
	(3·09)			24·18	250	Roof at *F*
	4·58			16·51	300	Floor at *G*
(3·56)		(1·16)	18·69	22·25	300	Roof at *G*
	2·22			16·47	350	Floor at *H*
	(4·67)			23·36	350	Roof at *H*
	1·15			17·54	400	Floor at *I*
		(6·07)		24·76	400	Roof at *I*
+2·47 −7·67	+24·43 −19·79	−9·96		363·91		
−5·20 −(−9·96)	+ 4·64					
+4·76				4·76		

Checks

1. ΣB.S. − ΣF.S. = 4·76 = diff. in level *A*−*I*
2. (a) Σ Reduced levels except first 363·91

   (b) 22·47 × 7 = 157·29

   21·09 × 6 = 126·54

   18·69 × 4 = 74·76

   358·59

$$\begin{array}{llrr} \text{(c)} & \Sigma\text{F.S.} + \Sigma\text{I.S.} = & -9{\cdot}96 & \\ & & +4{\cdot}64 & \\ \cline{3-3} & & -5{\cdot}32 & 5{\cdot}32 \\ \cline{4-4} & & & 363{\cdot}91 \end{array}$$ Checks with (a)

N.B. Inverted staff readings must always be treated as negative values.

Example 5.3 The following readings were taken with a level and a 14 ft staff. Draw up a level book page and reduce the levels by

(a) the rise and fall method,

(b) the height of collimation method.

2·24, 3·64, 6·03, 11·15, (12·72 and 1·48) C.P., 4·61, 6·22, 8·78, B.M. (102·12 ft A.O.D.), 11·41, (13·25 and 6·02) C.P., 2·13, 5·60, 12·21.

What error would occur in the final level if the staff had been wrongly extended and a plain gap of 0·04 had occurred at the 5 ft section joint?

(L.U.)

B.S.	I.S.	F.S.	Rise	Fall	Height of Collimation	Reduced Level	Remarks
2·24					122·14	119·90	
	3·64			1·40		118·50	
	6·03			2·39		116·11	
	11·15			5·12		110·99	
1·48		12·72		1·57	110·90	109·42	
	4·61			3·13		106·29	
	6·22			1·61		104·68	
	8·78			2·56		102·12	B.M. 102·12 A.O.D.
	11·41			2·63		99·49	
6·02		13·25		1·84	103·67	97·65	
	2·13		3·89			101·54	
	5·60			3·47		98·07	
		12·21		6·61		91·46	
9·74		38·18	3·89	32·23		28·44	
38·18			32·23				
28·44			28·44				

A combined booking is shown for convenience.

If a 0·04 ft gap occurred at the 5 ft section all readings > 5 ft will be 0·04 ft too small.

The final level value will only be affected by the B.S. and F.S. readings after the reduced level of the datum, i.e. 102·12, although the I.S. 8·78 would need to be treated for booking purposes as a B.S.

$$\text{i.e.}\quad \Sigma\text{B.S.} = 8{\cdot}78 + 6{\cdot}02 = 14{\cdot}80$$
$$\Sigma\text{F.S.} = 13{\cdot}25 + 12{\cdot}21 = 25{\cdot}46$$
$$\text{Difference} = -\ 10{\cdot}66$$
$$\therefore\quad \text{Final level} = 102{\cdot}12 - 10{\cdot}66 = \underline{91{\cdot}46}$$

As all these values are > 5 ft no final error will be created, since all the readings are subjected to equal error.

Therefore the final reduced level is correct, i.e. 91·46 ft A.O.D.

Example 5.4 The following readings were observed with a level:

3·75 (B.M. 112·28), 5·79, 8·42, 12·53, C.P., 4·56, 7·42, 2·18, 1·48, C.P., 12·21, 9·47, 5·31, 2·02, T.B.M.

(a) Reduce the levels by the Rise and Fall method.

(b) Calculate the level of the T.B.M. if the line of collimation was tilted upwards at an angle of 6 min and each backsight length was 300 ft and the foresight length 100 ft.

(c) Calculate the level of the T.B.M. if the staff was not held upright but leaning backwards at 5° to the vertical in all cases.

(L.U.)

(a)

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level	Remarks
3·75					112·28	B.M. 112·28
	5·79			2·04	110·24	
	8·42			2·63	107·61	
4·56		12·53		4·11	103·50	
	7·42			2·86	100·64	
	2·18		5·24		105·88	
12·21		1·42	0·76		106·64	
	9·47		2·74		109·38	
	5·31		4·16		113·54	
		2·02	3·29		116·83	T.B.M.
20·52		15·97	16·19	11·64		
15·97			11·64			
4·55			4·55		4·55	

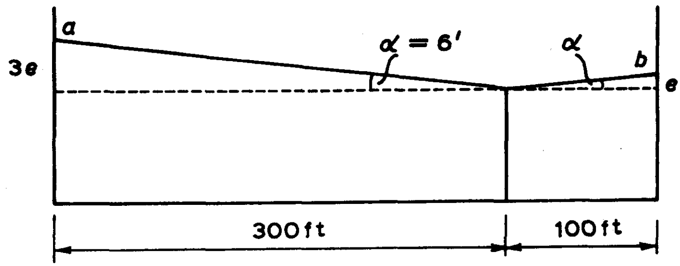


Fig. 5.3

(b) In Fig. 5.3,

$$\text{True difference in level} = (a-3e)-(b-e) = (a-b)-2e$$

$$\text{where } e = 100\tan 06' = 100 \times 06'_{\text{radians}}$$

$$= \frac{100 \times 6 \times 60}{206\,265} = \underline{0{\cdot}175 \text{ per } 100\,\text{ft}}$$

Total length of backsight $= 3 \times 300 = 900$ ft

of foresight $= 3 \times 100 = 300$ ft

Effective difference in length $= 600$ ft

$\therefore$ Error $= 6 \times 0{\cdot}175 = 1{\cdot}050$ ft

i.e. B.S. readings are effectively too large by 1·05 ft.

$\therefore$ True difference in level $= 4{\cdot}55 - 1{\cdot}05 = 3{\cdot}50$ ft

$\therefore$ Level of T.B.M. $= 112{\cdot}28 + 3{\cdot}50 = 115{\cdot}78$ ft A.O.D.

(c) If the staff was not held vertical the readings would be too large, the value depending on the staff reading.

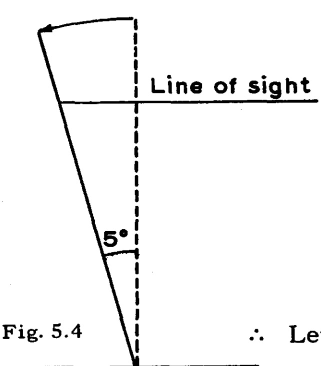


Fig. 5.4

True reading $=$ observed reading $\times \cos 5°$

Apparent difference in level ΣB.S. $-$ ΣF.S. $= 4{\cdot}55$

True difference in level $= \Sigma\text{B.S.}\cos 5° - \Sigma\text{F.S.}\cos 5°$

$$= (\Sigma\text{B.S.} - \Sigma\text{F.S.})\cos 5°$$

$$= 4{\cdot}55 \cos 5° = \underline{4{\cdot}53}$$

$\therefore$ Level of T.B.M. $= 112{\cdot}28 + 4{\cdot}53 = \underline{116{\cdot}81\,\text{ft}}$

Example 5.5 *Missing values in booking.* It has been found necessary to consult the notes of a dumpy levelling carried out some years ago.

Whilst various staff readings, rises and falls and reduced levels are undecipherable, sufficient data remain from which all the missing values can be calculated.

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level A.O.D.	Distance	Remarks
2·36					121·36	0	B.M. on Watson House
				1·94		100	
4·05		7·29				200	
				4·46		300	
	4·31					400	
	6·93					500	
					113·32	600	
		7·79		0·63		715	Peg 36
	3·22		1·58			800	
					112·01	900	
	6·53					1000	
					113·53	1100	
		5·86				1200	
				3·10		1286	B.M. on boundary wall
14·96				21·18			

Calculate the missing values and show the conventional arithmetical checks on your results.

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level A.O.D.	Distance	Remarks
2·36					121·36	0	B.M. on Watson House
	(a) 4·30			1·94	119·42	100	
4·05		7·29		(b) 2·99	116·43	200	
	(c) 8·51			4·46	111·97	300	
	4·31		(d) 4·20		116·17	400	
	6·93			(e) 2·62	113·55	500	
	(f) 7·16			0·23	113·32	600	
(g) 4·80		7·79		0·63	112·69	715	Peg 36
	3·22		1·58		114·27	800	
	(h) 5·48			2·26	112·01	900	
	6·53			(j) 1·05	110·96	1000	
	(k) 3·96		2·57		113·53	1100	
(m) 3·75		5·86		(l) 1·90	111·63	1200	
		(n) 6·85		3·10	108·53	1286	B.M. on boundary wall
14·96		27·79	8·35	21·18			
27·79				8·35			
−12·83				−12·83	−12·83		

Notes:
(a) 4·30 I.S. is deduced from fall 1·94.
(b) Fall 2·99 is obtained from I.S. – F.S., 4·30 – 7·29.
(c) 8·51 as (a).
(d) Rise 4·20 as (b).
(e) Fall 2·62 as (b).
(f) Reduced levels 113·32 – 113·55 gives fall 0·23, which gives staff reading 7·16.
(g) 4·80 B.S. must occur on line of F.S. – deduced value from rise 1·58 with I.S. 3·22.
(h) 5·48 as (f).
(j) 1·05 as normal.
(k) 3·96 as (f).
(l) Fall 1·90 normal.
(m) 3·75 B.S. must occur opposite 5·86 F.S. Value from ΣB.S. 14·96.
(n) 6·85 from fall 3·10.
Checks as usual.

Exercises 5(a) (Booking)

1. The undernoted staff readings were taken successively with a level along an underground roadway.

Staff Readings	Distances from A	Remarks
5·77	0	B.S. to A
2·83	120	I.S.
0·30	240	F.S.
5·54		B.S.
1·41	360	I.S.
3·01	480	F.S.
2·23		B.S.
2·20	600	I.S.
1·62	720	F.S.
7·36		B.S.
5·52	840	I.S.
0·71	960	F.S.
4·99		B.S.
2·25	1080	F.S. to B

Using the Height of Collimation method, calculate the reduced level of each staff station relative to the level of A, which is 6010·37 ft above an assumed datum of 10 000 ft below O.D.

Thereafter check your results by application of the appropriate method used for verifying levelling calculations derived from heights of collimation.

(M.Q.B./S Ans. Reduced level of B 6028·37)

2. A section levelling is made from the bottom of a staple pit A to the bottom of a staple pit B. Each group of the following staff readings relates to a setting of the levelling instrument and the appropriate distances from the staff points are given.

From bottom of staple pit A.

Staff Readings	Distance (ft)
4·63	
2·41	72
0·50	51
7·23	
4·80	32
3·08	26
1·02	45
4·09	
3·22	48
1·98	53
1·47	
3·85	52
6·98	46
1·17	
2·55	106
2·16	
3·64	54
5·27	45

To bottom of staple pit B.

The top of staple pit A is 8440 ft above the assumed datum of 10 000 ft below O.D. and the shaft is 225 ft deep.

(a) Enter the staff readings and distances in level book form, complete the reduced levels and apply the usual checks.

(b) Plot a section on a scale 100 ft to 1 in. for horizontals and 10 ft to 1 in. for verticals.

(c) If the staple pit B is 187·4 ft deep what is the reduced level at the top of the shaft?

(M.Q.B./UM Ans. 8404·85 ft)

3. Reduce the following notes of a levelling made along a railway affected by subsidence. Points A and B are outside the affected area, and the grade was originally constant between them.

Find the original grade of the track, the amount of subsidence at each chain length and the maximum grade in any chain length.

B.S.	I.S.	F.S.	Distance (links)	Remarks
10·15			0	*A*
	8·84		100	
7·58		7·69	200	
	6·65		300	
4·21		5·50	400	
	2·72		500	
8·21		5·55		Not on track
	3·76		600	
	2·38		700	
		1·09	800	*B*

(M.Q.B./M Ans. Grade 1·29 in 100 links;
subsidence +0·02, −0·12, −0·48,
−0·62, −0·42, −0·90
max. grade 1·62 ft in 100 links between 500 − 600 ft)

4. The following staff readings were observed in the given order when levelling up a hillside from a temporary bench mark 135·20 ft A.O.D. With the exception of the staff position immediately after the bench mark, each staff position was higher than the preceding one.

Enter the readings in level book form by both the rise and fall and collimation systems. These may be combined, if desired, into a single form to save copying.

4·62, 8·95, 6·09, 3·19, 12·43, 9·01, 5·24, 1·33, 10·76, 6·60,
2·05, 13·57, 8·74, 3·26, 12·80, 6·33, 11·41, 4·37.

(L.U.)

N.B. There are alternative solutions

5. The undernoted readings, in feet, on a levelling staff were taken along a roadway *AB* with a dumpy level, the staff being held in the first case at a starting point *A* and then at 50 ft intervals: 2·51, 3·49, [2·02], 6·02, 5·00.

The level was then moved forward to another position and further readings taken. These were as follows, the last reading being at *B*: 7·73, 4·52, [6·77], 2·22, 6·65.

The level of *A* is 137·2 ft A.O.D.

Set out the staff readings and complete the bookings.

Calculate the gradient from *A* to *B*.

(Figures in brackets denote inverted staff readings.)

(R.I.C.S./M Ans. 1 in 284)

6. An extract from a level book is given below, in which various bookings are missing. Fill in the missing bookings and re-book and

complete the figures by the Height of Collimation method.

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level
					154·86
	7·62			3·70	
	2·32		5·30		
7·11					147·72
		5·55	1·56		149·28
	7·37		2·00		151·28
	8·72				149·93
				1·61	
		4·24	6·09		154·41

All the figures are assumed correct.

(I.C.E.)

7. The following figures are the staff readings taken in order on a particular scheme, the backsights being shown in italics:

2·67, 7·12, 9·54, 8·63, 10·28, *12·31*, 10·75, 6·23, 7·84, *9·22*, 5·06, 4·18, 2·11.

The first reading was taken on a bench mark 129·80 O.D.

Enter the readings in level book form, check the entries and find the reduced level of the last point.

Comment on your completed reduction.

(L.U./E)

5.4 Field Testing of the Level

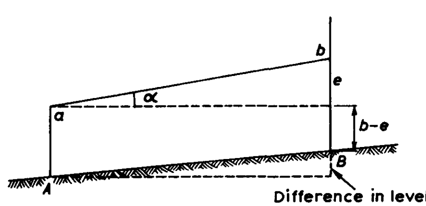
Methods available are (1) by reciprocal levelling, (2) the two-peg methods.

5.41 Reciprocal levelling method

In Fig. 5.5, the instrument is first set at A of height a. The line of sight is assumed to be inclined at an angle of elevation $+\alpha$ giving an error e in the length AB. The reading on the staff at B is b.

$$\text{Difference in level } A - B = a - (b - e) \qquad (5.4)$$

Fig. 5.5



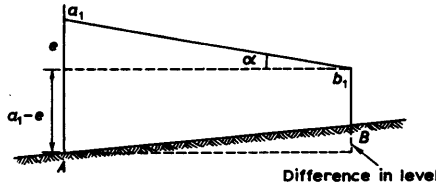


Fig. 5.6

In Fig. 5.6, the instrument is set at B of height b_1 and the reading on the staff at A is a_1.

$$\text{Difference in level } A - B = (a_1 - e) - b_1 \qquad (5.5)$$

Thus, from Eqs (5.4) and (5.5),

$$\underline{\text{Difference in level}} = a - b + e \qquad (5.4)$$

$$= a_1 - b_1 - e \qquad (5.5)$$

Adding (5.4) and (5.5),

$$= \tfrac{1}{2}[(a-b) + (a_1 - b_1)] \qquad (5.6)$$

By subtracting Eqs. (5.4) and (5.5), the <u>error in collimation</u>

$$e = \tfrac{1}{2}[(a_1 - b_1) - (a-b)] \qquad (5.7)$$

Example 5.6 A dumpy level is set up with the eyepiece vertically over a peg A. The height from the top of A to the centre of the eyepiece is measured and found to be 4·62 ft. A level staff is then held on a distant peg B and read. This reading is 2·12 ft. The level is then set over B. The height of the eyepiece above B is 4·47 ft and a reading on A is 6·59 ft.

(1) What is the difference in level between A and B?

(2) Is the collimation of the telescope in adjustment?

(3) If out of adjustment, can the collimation be corrected without moving the level from its position at B? (I.C.E.)

(1) From Eq. (5.6),

$$\text{Difference in level } (A-B) = \tfrac{1}{2}[(4{\cdot}62 - 2{\cdot}12) + (6{\cdot}59 - 4{\cdot}47)]$$

$$= \tfrac{1}{2}[2{\cdot}50 + 2{\cdot}12]$$

$$= \underline{+\,2{\cdot}31\text{ ft}}$$

(2) From Eq. (5.7),

$$\text{Error in collimation } e = \tfrac{1}{2}[2{\cdot}12 - 2{\cdot}50]$$

$$= \underline{-\,0{\cdot}19\text{ ft per length } AB.}$$

(i.e. the line of sight is depressed.)

(3) True staff reading at A (instrument at B) should be

$$6{\cdot}59 - (-0{\cdot}19)$$
$$= 6{\cdot}59 + 0{\cdot}19 = \underline{6{\cdot}78\text{ ft.}}$$

The cross-hairs must be adjusted to provide this reading.

5.42 Two-peg method

In the following field tests the true difference in level is ensured by making backsight and foresight of equal length.

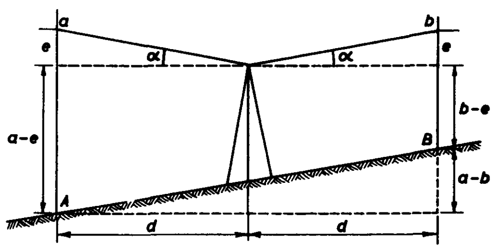


Fig. 5.7

Assuming the line of collimation is elevated by α,

the displacement vertically $= d\tan\alpha$

Thus, if B.S. = F.S., $d\tan\alpha = e$ in each case.

$\therefore$ True difference in level $AB = (a-e)-(b-e)$

$= \underline{a - b}$

Method (*a*). Pegs are inserted at A and B so that the staff reading $a = b$ when the instrument is midway between A and B. The instrument may now be moved to A or B.

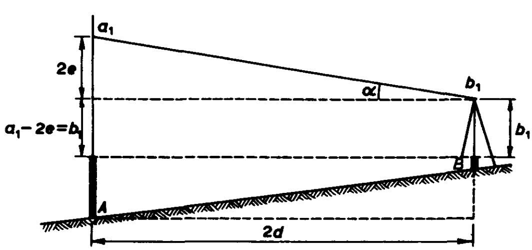


Fig. 5.8

In Fig. 5.8, if the height of the instrument at B is b_1 above peg B, the staff reading at peg A should be b_1 if there is no error, i.e. if $\alpha = 0$.

If the reading is a_1 and the distance $AB = 2d$, then the true read-

ing at A should be

$$b_1 = a_1 - 2e$$

$$\therefore \quad e = \tfrac{1}{2}[a_1 - b_1] \tag{5.8}$$

Method (*b*). The instrument is placed midway between staff positions A and B, Fig. 5.9. Readings taken give the true difference in level = $a - b$.

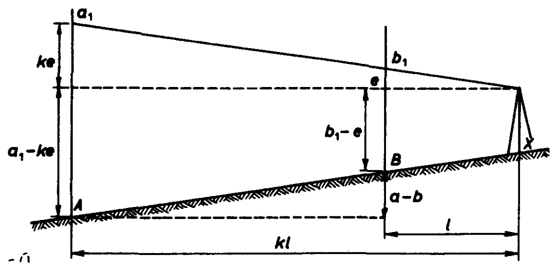


Fig. 5.9

The instrument is now placed at X so that

$$XA = kXB$$

where k is the multiplying factor depending on the ratio of AX/BX.

If $AX = kBX$, then the error at $A = ke$

$$\text{True difference in level} = (a_1 - ke) - (b_1 - e) = a - b$$

$$\therefore \quad a_1 - b_1 - (k-1)e = a - b$$

$$\therefore \quad \text{Error per length } BX = e = \frac{(a_1 - b_1) - (a - b)}{k - 1} \tag{5.9}$$

N.B. (1) If the instrument is placed nearer to A than B, k will be less than 1 and $k - 1$ will be negative (see Example 5.8).

(2) If the instrument is placed at station B, then Eq. (5.9) is modified as follows:

$$(a_1 - E) - b_1 = a - b$$

$$\therefore \quad E = (a_1 - b_1) - (a - b) \tag{5.10}$$

where E is the error in the length AB (see Example 5.9).

Example 5.7 (a) When checking a dumpy level, the following readings were obtained in the two-peg test:

Level set up midway between two staff stations A and B 400 ft apart; staff readings on A 5·75 ft and on B 4·31 ft.

Level set up 40 ft behind B and in line AB; staff reading on B 3·41 ft and on A 4·95 ft.

Complete the calculation and state the amount of instrumental error.

(b) Describe the necessary adjustments to the following types of level, making use of the above readings in each case:

(i) Dumpy level fitted with diaphragm screws and level tube screws.

(ii) A level fitted with level screws and tilting screw.

(a) By Eq. (5.9)

$$e = \frac{(4{\cdot}95 - 3{\cdot}41) - (5{\cdot}75 - 4{\cdot}31)}{11 - 1}$$

$$= \frac{1{\cdot}54 - 1{\cdot}44}{10} = \underline{0{\cdot}01 \text{ per } 40\,\text{ft}}$$

Check

With instrument 40 ft beyond B,

Staff reading at A should be $4{\cdot}95 - (11 \times 0{\cdot}01)$

$= 4{\cdot}95 - 0{\cdot}11 = 4{\cdot}84\,\text{ft}$

Staff reading at B should be $3{\cdot}41 - 0{\cdot}01 = 3{\cdot}40\,\text{ft}$

Difference in level $= 1{\cdot}44$

This agrees with the first readings $5{\cdot}75 - 4{\cdot}31 = 1{\cdot}44$

(b) (i) With a dumpy level the main spirit level should be first adjusted.
The collimation error is then adjusted by means of the diaphragm screws until the staff reading at A from the second setting is 4·84 and this should check with the staff reading at B of 3·40.

(ii) With a tilting level, the circular (pill-box) level should be first adjusted.
The line of sight should be set by the tilting screw until the calculated readings above are obtained.
The main bubble will now be off centre and must be centralised by the level tube screws.

Example 5.8 (a) Describe with the aid of a diagram the basic principles of a tilting level, and state the advantages and disadvantages of this type of level compared with the dumpy level.

(b) The following readings were obtained with a tilting level to two staves A and B 200 ft apart.

Position of Instrument	Reading at A (ft)	Reading at B (ft)
Midway between A and B	5·43	6·12
10 ft from A and 200 ft from B	6·17	6·67

What is the error in the line of sight per 100 ft of distance and how would you adjust the instrument? (R.I.C.S. L/Inter.)

Part (b) illustrates the testing of a level involving a negative angle of inclination of the line of collimation and a fractional k value.

Using Eq. (5.9),

$$e = \frac{(a_1 - b_1) - (a - b)}{k - 1} \quad \text{where} \quad k = \frac{AX}{BX} = \frac{10}{200}$$

$$= \frac{(6{\cdot}17 - 6{\cdot}67) - (5{\cdot}43 - 6{\cdot}12)}{1/20 - 1}$$

$$= \frac{20(-0{\cdot}50 + 0{\cdot}69)}{-19} = -0{\cdot}01 \times 20 = \underline{-0{\cdot}2\,\text{ft per length } BX}$$

i.e. $e = -0{\cdot}2$ per 200 ft $= \underline{-0{\cdot}1 \text{ per } 100\,\text{ft}}$

Check

At X, Reading on A should be $6{\cdot}17 + 0{\cdot}01 = 6{\cdot}18$

Reading on B should be $6{\cdot}67 + 0{\cdot}20 = 6{\cdot}87$

difference in level $= -0{\cdot}69$

Alternative solution from first principles (Fig. 5.10)

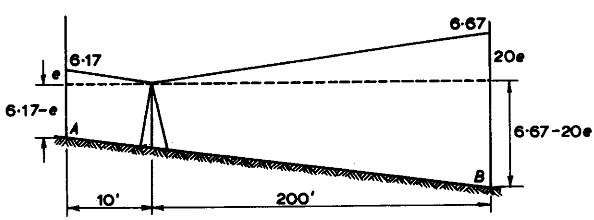


Fig. 5.10

True difference in level $= (6{\cdot}17 - e) - (6{\cdot}67 - 20e) = (5{\cdot}43 - 6{\cdot}12)$

i.e. $19e - 0{\cdot}50 = -0{\cdot}69$

$$e = -\frac{0{\cdot}19}{19}$$

$= -0{\cdot}01$ per 10 ft

$= -0{\cdot}1$ ft per 100 ft

Example 5.9 The following readings taken on to two stations A and B were obtained during a field test of a dumpy level. Suggest what type of error exists in the level and give the magnitude of the error as a percentage. How would you correct it in the field?

B.S.	F.S.	Remarks	
6·21		Staff at station A	$A-B$ 200 ft apart
	5·46	Staff at station B	Instrument midway
4·99		Staff at station A	AB 200 ft apart
	4·30	Staff at station B	Instrument v. near to B

(R.I.C.S./G)

By Eq. (5.10)

$$\begin{aligned} E &= (a_1-b_1)-(a-b) \\ &= (4{\cdot}99-4{\cdot}30)-(6{\cdot}21-5{\cdot}46) \\ &= 0{\cdot}69-0{\cdot}75 \\ &= -0{\cdot}06\text{ ft per }200\text{ ft} \\ &= \underline{-0{\cdot}03\text{ ft per }100\text{ ft.}} \end{aligned}$$

Check

$$\begin{aligned} 4{\cdot}99+0{\cdot}06 &= 5{\cdot}05 \\ 4{\cdot}30+0 &= \underline{4{\cdot}30} \\ & \quad\; 0{\cdot}75\text{ ft} \end{aligned}$$

$$\text{True difference in level} = 6{\cdot}21-5{\cdot}46 = \underline{0{\cdot}75\text{ ft.}}$$

Example 5.10 In levelling up a hillside to establish a T.B.M. (temporary bench mark) the average lengths of ten backsights and ten foresights were 80 ft and 40 ft respectively.

As the reduced level of the T.B.M. of 82·50 ft A.O.D. was in doubt, the level was set up midway between two pegs A and B 200 ft apart, the reading on A being 4·56 and that on B 5·24. When the instrument was moved 40 ft beyond B on the line AB produced, the reading on A was 5·34 and on B 5·88.

Calculate the true value of the reduced level.

True difference in level $A-B = 4{\cdot}56-5{\cdot}24 = -0{\cdot}68$

When set up 40 ft beyond B,

$$\begin{aligned} (5{\cdot}34-6e)-(5{\cdot}88-e) &= -0{\cdot}68 \\ 5{\cdot}34-5{\cdot}88-5e &= -0{\cdot}68 \\ 5e &= 0{\cdot}14 \\ e &= 0{\cdot}028\text{ ft}/40\text{ ft.} \end{aligned}$$

Check

True readings should have been:

at A	$5{\cdot}34 - 0{\cdot}168$	=	$5{\cdot}172$
at B	$5{\cdot}88 - 0{\cdot}028$	=	$5{\cdot}852$
			$-\ 0{\cdot}680$

Error in levelling $= 0{\cdot}028$ ft per 40 ft

Difference in length between B.S. and F.S. per set $= 80 - 40 = 40$ ft

per 10 sets $= 400$ ft

$\therefore$ Error $= 10 \times 0{\cdot}028 = -\ 0{\cdot}28$ ft

$\therefore$ True value of T.B.M. $= 82{\cdot}50 - 0{\cdot}28$

$= \underline{82{\cdot}22\text{ ft}}$ A.O.D.

Exercises 5(b) (Adjustment)

8. Describe how you would adjust a level fitted with tribrach screws, a graduated tilting screw and bubble-tube screws, introducing into your answer the following readings which were taken in a 2 peg test:

Staff stations at A and B 400 ft apart.

Level set up halfway between A and B: staff readings on A $4{\cdot}21$ ft, on B $2{\cdot}82$ ft.

Level set up 40 ft behind B in line AB: staff readings on A $5{\cdot}29$ ft, on B $4{\cdot}00$ ft.

Complete the calculation and show how the result would be used to adjust the level.

(L.U. Ans. Error $-0{\cdot}01$ per 40 ft)

9. A modern dumpy level was set up at a position equidistant from two pegs A and B. The bubble was adjusted to its central position for each reading, as it did not remain quite central when the telescope was moved from A to B. The readings on A and B were $4{\cdot}86$ and $5{\cdot}22$ ft respectively. The instrument was then moved to D, so that the distance DB was about five times the distance DA, and the readings with the bubble central were $5{\cdot}12$ and $5{\cdot}43$ ft respectively. Was the instrument in adjustment?

(I.C.E. Ans. Error $= 0{\cdot}0125$ ft at A from D)

10. The table gives a summary of the readings taken when running a line of levels A, B, C, D. The level used was fitted with stadia hairs and had tacheometric constants of 100 and 0. For all the readings the staff was held vertically.

Reduce the levels shown in the table

Position of Staff	Backsight Top	Backsight Middle	Backsight Bottom	Foresight Top	Foresight Middle	Foresight Bottom
A	6·22	4·37	2·52			
B	4·70	2·94	1·18	11·06	9·38	7·70
C	7·63	5·27	2·91	9·32	7·43	5·54
D				8·17	6·04	3·91

It was suspected that the instrument was out of adjustment and to check this, the following staff readings were taken, using the centre hair of the level diaphragm; *P* and *Q* are 300 ft apart.

Instrument Station	Staff at *P*	Staff at *Q*
Near *P*	4·65	8·29
Near *Q*	2·97	6·17

Find the true reduced level of *D* if the reduced level of *A* is 125·67 ft A.O.D.

(Ans. 115·36 ft A.O.D.)

11. A level was set up on the line of two pegs *A* and *B* and readings were taken to a staff with the bubble central. If *A* and *B* were 150 metres apart, and the readings were 2·763 m and 1·792 m respectively, compute the collimation error. The reduced levels were known to be 27·002 m and 27·995 m respectively.

The level was subsequently used, without adjustment, to level between two points *X* and *Y* situated 1 km apart. The average length of the backsights was 45 m and of the foresights 55 m. What is the error in the difference in level between *X* and *Y*?

(N.R.C.T. Ans. 30·5″ depressed; error + 0·0145 m)

12. The following readings were taken during a 'two-peg' test on a level fitted with stadia, and reading on a vertical staff, the bubble being brought to the centre of its run before each reading.

The points *A*, *B*, *X* and *Y* were in a straight line, *X* being midway between *A* and *B* and *Y* being on the side of *B* remote from *A*.

(A) If the reduced level of *A* is 106·23, find the reduced level of *B*.

(B) Explain what is wrong with the instrument and how you would correct it.

(C) Find what the centre hair readings would have been if the instrument had not been out of adjustment.

Instrument at	Staff at	Staff Readings
X	A	5·56
		4·81
		4·06
X	B	8·19
		7·44
		6·69
Y	A	5·32
		3·72
		2·12
Y	B	6·31
		6·21
		6·11

(R.I.C.S./L/M. Ans. 103·60; 4·74, 7·37, 3·57, 6·20)

13. A level set up in a position 100 ft from peg A and 200 ft from peg B reads 6·28 on a staff held on A and 7·34 on a staff held on B, the bubble having been carefully brought to the centre of its run before each reading. It is known that the reduced levels of the tops of the pegs A and B are 287·32 and 286·35 ft O.D. respectively.

Find (a) the collimation error and

(b) the readings that would have been obtained had there been no collimation error.

(L.U. Ans. (a) 0·09 ft per 100 ft, (b) 6·19; 7·16 ft)

14. P and Q are two points on opposite banks of a river about 100 yds wide. A level with an anallatic telescope with a constant 100 is set up at A on the line QP produced, then at B on the line PQ produced, and the following readings taken on to a graduated staff held vertically at P and Q.

From	To	Staff Readings in ft		
		Upper Stadia	Collimation	Lower Stadia
A	P	5·14	4·67	4·20
	Q	3·27	1·21	below ground
B	P	10·63	8·51	6·39
	Q	5·26	4·73	4·20

What is the true difference in level between P and Q and what is the collimation error of the level expressed in seconds of arc, there being 206 265 seconds in a radian?

(I.C.E. Ans. 3·62 ft; 104″ above horizon)

5.5 Sensitivity of the Bubble Tube

The sensitivity of the bubble tube depends on the radius of curvature (R) and is usually expressed as an angle (θ) per unit division (d) of the bubble scale.

5.51 Field test

Staff readings may be recorded as the position of the bubble is changed by a footscrew or tilting screw. Readings at the eye and objective ends of the bubble may be recorded or alternatively the bubble may be set to the exact scale division.

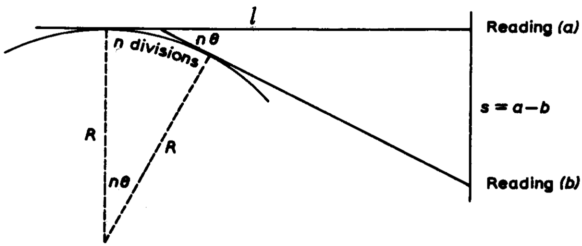


Fig. 5.11

In Fig. 5.11, $$\tan(n\theta) = \frac{s}{l} \qquad (5.11)$$

but θ is very small (usually 1 to 60 seconds).

$$\therefore \quad n\theta_{rad} = \frac{s}{l}$$

$$\theta_{rad} = \frac{s}{nl} \qquad (5.12)$$

$$\theta_{sec} = \frac{206\,265\, s}{nl} \qquad (5.13)$$

where s = difference in staff readings a and b

n = number of divisions the bubble is displaced between readings

l = distance from staff to instrument.

If d = length of 1 division on the bubble tube, then

$$d = R.\theta_{rad}$$

$$\text{i.e.} \quad R = \frac{d}{\theta} \qquad (5.14)$$

$$= \frac{ndl}{s} \qquad (5.15)$$

5.52 $O - E$ correction

If the bubble tube is graduated from the centre then an accurate reading is possible, particularly when seen through a prismatic reader, Fig. 5.12.

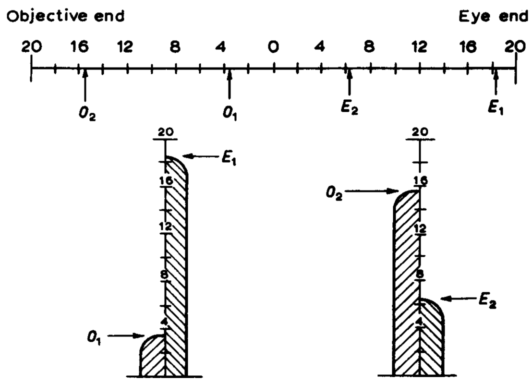


Fig. 5.12

If the readings at the objective end are O_1 and O_2 and those at the eye end E_1 and E_2, then the movement of the bubble in n divisions will equal

$$\frac{(O_1 - E_1) + (O_2 - E_2)}{2} \tag{5.16}$$

or $$\frac{(O_1 + O_2) - (E_1 + E_2)}{2}$$

The length of the bubble will be $\quad O + E \quad$ (5.17)

The displacement of the bubble will be $\quad \dfrac{O - E}{2} \quad$ (5.18)

If $O > E$ the telescope is elevated,
$O < E$ the telescope is depressed.

5.53 Bubble scale correction

With a geodetic level, the bubble is generally very sensitive, say 1 division = 1 second.

Instead of attempting to line up the prismatically viewed ends of

the bubble, their relative positions are read on the scale provided and observed in the eyepiece at the time of the staff reading.

The correction to the middle levelling hair is thus required.

By Eq. (5.13),

$$\theta_{sec} = \frac{206\,265\, s}{nl}$$

Transposing gives
$$e = \frac{nl\theta}{206\,265} \tag{5.19}$$

where e = the error in the staff reading

θ = the sensitivity of the bubble tube in seconds

n = the number of divisions displaced

l = length of sight.

Example 5.11 Find the radius of curvature of the bubble tube attached to a level and the angular value of each 2 mm division from the following readings taken to a staff 200 ft from the instrument.
(2 mm = 0·006 56 ft).

Staff Readings		3·510	3·742
Bubble Readings	Eye End	18·3	6·4
	Objective End	3·4	15·3

By Eq. (5.16),
$$n = \tfrac{1}{2}[(18{\cdot}3 - 3{\cdot}4) + (15{\cdot}3 - 6{\cdot}4)]$$
$$= \tfrac{1}{2}[14{\cdot}9 + 8{\cdot}9]$$
$$= \underline{11{\cdot}9} \text{ divisions.}$$

By Eq. (5.13),
$$\theta = \frac{206\,265\, s}{nl}$$
$$= \frac{206\,265 \times (3{\cdot}742 - 3{\cdot}510)}{11{\cdot}9 \times 200}$$
$$= \underline{20} \text{ sec}$$

By Eq. (5.15),
$$R = \frac{ndl}{s}$$
$$= \frac{0{\cdot}006\,56 \times 200 \times 11{\cdot}9}{0{\cdot}232}$$
$$= \underline{67{\cdot}3} \text{ ft}$$

In the metric system the above readings would be given as:

Staff readings 1·070 m 1·141 m

Distance between staff and level = 60·96 m

Then, by Eq. (5.13), $\theta = \dfrac{206\,265 \times (1{\cdot}141 - 1{\cdot}070)}{11{\cdot}9 \times 60{\cdot}96} = \underline{20 \text{ sec}}$

by Eq. (5.15), $R = \dfrac{0{\cdot}002 \times 60{\cdot}96 \times 11{\cdot}9}{0{\cdot}071} = \underline{20{\cdot}43\,\text{m}}$

Example 5.12 The following readings were taken through the eye-piece during precise levelling. What should be the true middle hair reading of the bubble value if 1 division is 1 second. The stadia constant of the level is $\times 100$.

Stadia Readings			Bubble Scale Readings	
Top	Middle	Bottom	*E*	*O*
6·371 6	5·507 4	4·643 1	10·6	8·48

By Eq. (5.18), $n = \dfrac{O - E}{2}$

$= \dfrac{8{\cdot}4 - 10{\cdot}6}{2}$

$= \underline{-\ 1{\cdot}1}$

Then by Eq. (5.19), $e = \dfrac{nl\theta}{206\,265}$

$= \dfrac{-\ 1{\cdot}1 \times 100(6{\cdot}371\,6 - 4{\cdot}643\,1) \times 1''}{206\,265}$

$= \dfrac{110 \times 1{\cdot}728\,5}{206\,265}$

$= -\ 0{\cdot}000\,9$

$\therefore$ True middle reading should be $5{\cdot}507\,4 + 0{\cdot}000\,9$

$= \underline{5{\cdot}508\,3}$

Exercises 5(c) (Sensitivity)

15. A level is set with the telescope perpendicular to two footscrews at a distance of 100 ft from a staff.

The graduations on the bubble were found to be 0·1 in. apart and after moving the bubble through 3 divisions the staff readings differed by 0·029 ft.

Find the sensitivity of the spirit bubble tube and its radius of curvature.

(Ans. $\theta \simeq 20$ seconds; $R = 86{\cdot}2$ ft)

16. State what is meant by the term 'sensitivity' when applied to a spirit level, and discuss briefly the factors which influence the choice

of spirit level of sensitivity appropriate for the levelling instrument of specified precision.

The spirit level attached to a levelling instrument contains a bubble which moves 1/10 in. per 20 seconds change in the inclination of the axis of the spirit level tube. Calculate the radius of curvature of the spirit level tube.

(M.Q.B./S Ans. 85·94 ft)

5.54 Gradient screws (tilting mechanism)

On some instruments the tilting screw is graduated as shown in Fig. 5.13.

The vertical scale indicates the number of complete revolutions whilst the horizontal scale indicates the fraction of a revolution.

The positive and negative tilt of the telescope are usually shown in black and red respectively and these must be correlated with similar colours on the horizontal scale.

The gradient of the line of sight is given as 1 in x.

$$\text{where } 1/x = nr \tag{5.20}$$

where n = numbers of revs.

r = the ratio of 1 rev (frequently 1/1000).

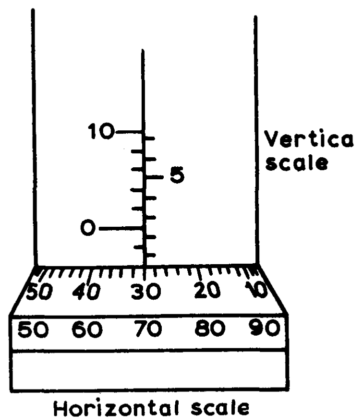


Fig. 5.13

Using the gradient screw, it is also possible to obtain the approximate distance by taking staff readings.

$$\text{If gradient} = \frac{s}{L} = nr$$

$$\text{then } L = \frac{s}{nr} \tag{5.21}$$

where s = staff intercept

L = length of sight

Example 5.13 Staff reading (a) = 6·32

(b) = 6·84

Number of revs (n) = 6·35

Gradient ratio (r) = 1/1 000

K

$$\text{Then} \quad L = \frac{(6{\cdot}84 - 6{\cdot}32) \times 1\,000}{6{\cdot}35}$$
$$= \frac{520}{6{\cdot}35}$$
$$= \underline{\underline{81{\cdot}88\text{ ft}}}$$

5.6 The Effect of the Earth's Curvature and Atmospheric Refraction

5.61 The earth's curvature

Over long distances the effect of the earth's curvature becomes significant.

Let the error due to the earth's curvature $E = AC$.

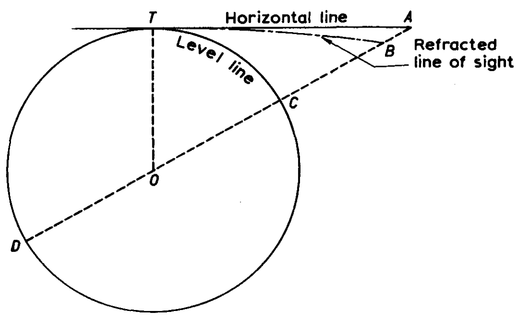


Fig. 5.14

In Fig. 5.14,

$$AC.AD = TA^2 \qquad \text{(intersecting chord and tangent)}$$
$$\therefore \quad AC = \frac{TA^2}{AD}$$
$$= \frac{L^2}{2R + AC} \qquad (\text{where } L = \text{length of sight } \simeq TA)$$
$$\text{i.e.} \quad E \simeq \frac{L^2}{2R} \qquad (\text{as } E \text{ is small compared with } R) \quad (5.22)$$

Alternatively, by Pythagoras,

$$AO^2 = OT^2 + AT^2$$
$$\text{i.e.} \quad (E+R)^2 = R^2 + L^2$$
$$E^2 + 2RE + R^2 = R^2 + L^2$$
$$E = \frac{L^2}{2R + E}$$
$$\simeq \frac{L^2}{2R} \quad \text{as above.}$$

As R, the radius of the earth, is $\simeq$ 3 960 miles ($\simeq$ 6 370 km).

$$E \simeq \frac{(5\,280\,L)^2}{2 \times 5\,280 \times 3\,960}$$

$$= \frac{5\,280\,L^2}{7\,920}$$

$$= 0{\cdot}667\,L^2 \text{ ft.} \qquad (5.23)$$

where L = length of sight in miles;

or metric values give $E = 0{\cdot}078\,5\,L^2$ metres (where L = length in km)

5.62 Atmospheric refraction

Due to variation in the density of the earth's atmosphere, affected by atmospheric pressure and temperature, a horizontal ray of light TA is refracted to give the bent line TB.

If the coefficient of refraction m is defined as the multiplying factor applied to the angle TOA (subtended at the centre) to give the angle ATB,

$$\text{Angle of refraction } ATB = m\,T\widehat{O}A$$

$$= 2m\,A\widehat{T}C$$

As the angles are small

$$AB : AC :: A\widehat{T}B : A\widehat{T}C$$

$$\text{Then} \quad AB = \frac{2m\,A\widehat{T}C}{A\widehat{T}C} \times AC$$

$$= \underline{2m\,AC} \qquad (5.24)$$

The value of m varies with time, geographical position, atmospheric pressure and temperature. A mean value is frequently taken as 0·07.

$$\therefore \quad AB = 0{\cdot}14\,AC$$

$$\text{Error due to refraction} \simeq \tfrac{1}{7} AC \qquad (5.25)$$

$$\simeq \frac{0{\cdot}667\,L^2}{7} = 0{\cdot}095\,L^2 \qquad (5.26)$$

5.63 The combined effect of curvature and refraction

$$\text{The net effect } e = BC$$

$$= AC - AB$$

$$= \frac{L^2}{2R} - 2m\frac{L^2}{2R}$$

$$= \underline{\frac{L^2}{2R}[1 - 2m]} \qquad (5.27)$$

If m is taken as 0·07

$$\begin{aligned} \text{then} \quad e &= 0{\cdot}667\,L^2\,(1 - 0{\cdot}14) \\ &= 0{\cdot}667\,L^2 \times 0{\cdot}86 \\ &= \underline{0{\cdot}574\,L^2\,\text{ft}} \qquad (5.28) \\ \text{or metric value} \quad e_m &= 0{\cdot}0673\,L^2\ \text{metres} \end{aligned}$$

Alternatively, taking refraction as 1/7 of the curvature error,

$$\begin{aligned} e &= \tfrac{6}{7} \times 0{\cdot}667\,L^2 \\ &= 0{\cdot}572\,L^2 \simeq \underline{0{\cdot}57\,L^2} \end{aligned}$$

Example 5.14 *Effect of curvature*

1. What difference will exist between horizontal and level lines at the following distances?

(a) 1 mile	(d) 100 miles
(b) 220 yards	(e) 1 km
(c) 5 miles	(f) 160 km

(a) $E_a = 0{\cdot}667\,L^2\,\text{ft}$ (L in miles)

$= 0{\cdot}667\,\text{ft} = 8{\cdot}004$ inches

(b) $E_b \propto L^2$

thus $E_b = 0{\cdot}667\left(\frac{1}{8}\right)^2 = \dfrac{0{\cdot}667}{64} = \underline{0{\cdot}0104\,\text{ft}} = \underline{0{\cdot}1248\ \text{inches}}$

(c) $E_c = 0{\cdot}667 \times 5^2 = 25 \times 0{\cdot}667$

$= \dfrac{66{\cdot}7}{4} = \underline{16{\cdot}675\,\text{ft}}$

(d) $E_d = 0{\cdot}667 \times 100^2 = 0{\cdot}667 \times 1\,000$

$= \underline{6\,670\,\text{ft}}$

(e) $E_e = 0{\cdot}0785 \times 1^2 = 0{\cdot}0785\,\text{m}$

(f) $E_f = 0{\cdot}0785 \times 160^2 = 2\,009{\cdot}6\,\text{m}$

In ordinary precise levelling it is essential that the lengths of the backsight and foresight be equal to eliminate instrumental error. This is also required to counteract the error due to curvature and refraction, as this error should be the same in both directions providing the climatic conditions remain constant. To minimise the effect of climatic change the length of sights should be kept below 150 ft.

In precise surveys, where the length of sight is greater than this value and climatic change is possible, e.g. crossing a river or ravine, 'reciprocal levelling' is employed.

Exercises 5(d) (Curvature and refraction)

17. Derive the expression for the combined curvature and refraction correction used in levelling practice.

If the sensitivity of the bubble tube of a level is 20 seconds of arc per division, at what distance does the combined curvature and refraction correction become numerically equal to the error induced by dislevelment of one division of the level tube.

(R.I.C.S. D/M Ans. 4 742·4 ft)

18. A geodetic levelling instrument which is known to be in adjustment is used to obtain the difference in level between two stations A and B which are 2 430 ft apart. The instrument is set 20 ft from B on the line AB produced

If A is 1·290 ft above B, what should be the reading on the staff at A if the reading on the staff at B is 4·055 ft.

(M.Q.B./S Ans. 2·886 ft)

5.64 Intervisibility

The earth's curvature and the effect of atmospheric refraction affect the maximum length of sight, Fig. 5.15.

$$h_1 = 0{\cdot}57\,d^2$$
$$h_2 = 0{\cdot}57\,(D-d)^2$$

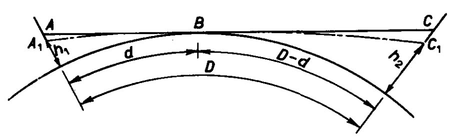


Fig. 5.15

This will give the minimum height h_2 at C which can be observed from A height h_1 assuming the ray grazes the surface of the earth or sea.

With intervening ground at B

In Fig. 5.16,

let the height of $AA_1 = h_1$

$$\text{of } BB_3 = h_2 = BB_1 + B_1B_2 + B_2B_3$$
$$= 5\,280\,d\tan\alpha + h_1 + 0{\cdot}57\,d^2 \quad (5.29)$$
$$\text{of } CC_3 = h_3$$
$$= 5\,280\,D\tan\beta + h_1 + 0{\cdot}57\,D^2 \quad (5.30)$$

If C is to be visible from A then $\alpha \leqslant \beta$.

If $\alpha = \beta$, then

$$\tan\alpha = \frac{h_2 - h_1 - 0{\cdot}57\,d^2}{5\,280\,d} = \frac{h_3 - h_1 - 0{\cdot}57\,D^2}{5\,280\,D} \quad (5.31)$$

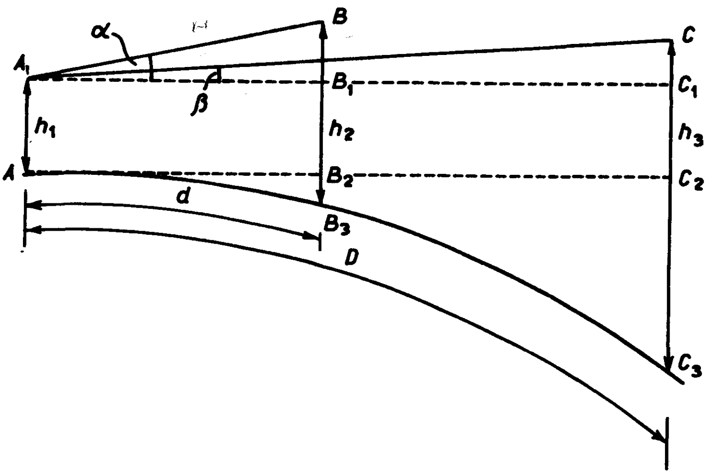


Fig. 5.16

Thus the minimum height

$$h_2 = \frac{d}{D}(h_3 - h_1 - 0{\cdot}57\,D^2) + h_1 + 0{\cdot}57\,d^2$$

$$= \frac{dh_3}{D} + (D - d)\left(\frac{h_1}{D} - 0{\cdot}57\,d\right) \qquad (5.32)$$

Clendinning quotes the formula as

$$h_2 = \frac{dh_3}{D} + \frac{h_1}{D}(D - d) - Kd(D - d)\operatorname{cosec}^2 Z \qquad (5.33)$$

where $K \simeq 0{\cdot}57$

Z is the zenith angle of observation.

Over large distances $Z \simeq 90°$ $\therefore$ $\operatorname{cosec}^2 Z \simeq 1$.

Example 5.15

If $h_1 = 2\,300$ ft (at A), $d = 46$ miles
$h_2 = 1\,050$ ft (at B), $D = 84$ miles
$h_3 = 1\,800$ ft (at C).

Can C be seen from A?

By Eq. (5.32),

$$h_2 = \frac{1\,800 \times 46}{84} + (84 - 46)\left(\frac{2\,300}{84} - 0{\cdot}57 \times 46\right)$$

$$= 985{\cdot}7 + 44{\cdot}1$$

$$= 1\,029{\cdot}8\text{ ft}$$

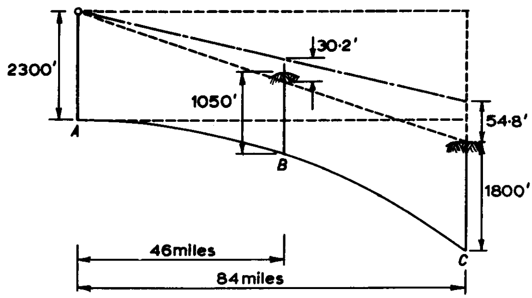


Fig. 5.17

The station C cannot be seen from A as h_2 is $> 1\,029{\cdot}8$.

If the line of sight is not to be nearer than 10 ft to the surface at B, then it would be necessary to erect a tower at C of such a height that the line of sight would be 10 ft above B,

$$\text{i.e. so that its height } h = (1\,050 + 10 - 1\,029{\cdot}8) \times \frac{84}{46}$$

$$= 30{\cdot}2 \times 1{\cdot}826$$

$$\simeq \underline{54{\cdot}8\,\text{ft}}$$

Exercises 5(e) (Intervisibility)

19. Two ships A and B are 20 miles (32·18 km) apart. If the observer at A is 20 ft (6·096 m) above sea level, what should be the height of the mast of B above the sea for it to be seen at A?

(Ans. 113·0 ft (34·5 m))

20. As part of a minor triangulation a station A was selected at 708·63 ft A.O.D. Resection has been difficult in the area and as an additional check it is required to observe a triangulation station C 35 miles away (reduced level 325·75 ft A.O.D.). If there is an intermediate hill at B, 15 miles from A (spot height shown on map near B 370 ft A.O.D.), will it be possible to observe station C from A assuming that the ray should be 10 ft above B?

(N.R.C.T. Ans. Instrument + target should be $\simeq$ 20 ft)

21. (a) Discuss the effects of curvature and refraction on long sights as met with in triangulation, deriving a compounded equation for their correction.

(b) A colliery headgear at A, ground level 452·48 ft A.O.D., is 145 ft to the observing platform.

It is required to observe a triangulation station C, reduced level 412·68 ft A.O.D., which is 15 miles from A, but it is thought that intervening ground at B approximately 500 ft A.O.D. and 5 miles from A

will prevent the line of sight.

Assuming that the ray should not be nearer than 10 ft to the ground at any point, will the observation be possible?

If not, what height should the target be at C?

(R.I.C.S./M Ans. > 5 ft)

22. Describe the effect of earth's curvature and refraction on long sights. Show how these effects can be cancelled by taking reciprocal observations.

Two beacons A and B are 60 miles apart and are respectively 120 ft and 1 200 ft above mean sea level. At C, which is in the line AB and is 15 miles from B, the ground level is 548 ft above mean sea level.

Find by how much, if at all, B should be raised so that the line of sight from A to B should pass 10 ft above the ground at C. The mean radius of the earth may be taken as 3 960 miles.

(L.U. Ans. +17 ft)

5.65 Trigonometrical levelling

For plane surveying purposes where the length of sight is limited to say 10 miles the foregoing principles can be applied, Fig. 5.18.

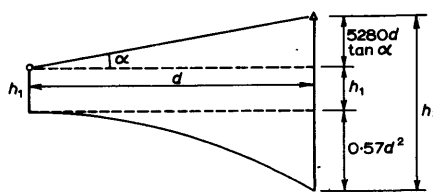


Fig. 5.18

$$\text{The difference in elevation} = h_2 - h_1 = 5\,280\, d \tan\alpha + 0{\cdot}57\, d^2 \tag{5.34}$$

where d = distance in miles

α = angle of elevation.

If the distance D is given in feet, then

$$h_2 - h_1 = D\tan\alpha + 0{\cdot}57\left(\frac{D}{5\,280}\right)^2$$

$$= D\tan\alpha + 2{\cdot}04 \times 10^{-8} D^2 \tag{5.35}$$

N.B. It is considered advisable in trigonometrical levelling, and in normal geometrical levelling over long distances, to observe in both directions, simultaneously where possible, in order to eliminate the effects of curvature and refraction, as well as instrumental errors. This is known as reciprocal levelling.

Example 5.16 The reduced level of the observation station A is 350·36 ft A.O.D. From A, instrument height 4·31 ft, the angle of elevation is $5°30'$ to station B, target height 6·44 ft. If the computed distance AB is 35 680·1 ft what is the reduced level of B?

Reduced level of B = Reduced level of A + difference in elevation + instrument height − target height

By Eq. (5.35),

$$\text{Difference in elevation} = 35\,680{\cdot}1 \tan 5°30' + 2{\cdot}04 \times 10^{-8} \times 35\,680{\cdot}1^2$$
$$= 3\,435{\cdot}60 + 25{\cdot}97$$
$$= 3\,461{\cdot}57\text{ ft}$$

$$\text{Reduced level of } B = 3\,461{\cdot}57 + 4{\cdot}31 - 6{\cdot}44$$
$$= \underline{3\,459{\cdot}44\text{ ft A.O.D.}}$$

Based on metric values the problem becomes:

The reduced level of the observation station A is 106·790 m A.O.D. From A, instrument height 1·314 m, the angle of elevation is $5°30'$ to station B, target height 1·963 m. If the computed distance AB is 10·875 3 km, what is the reduced level of B?

$$\text{Difference in elevation} = 10\,875{\cdot}3 \tan 5°30' + 0{\cdot}067\,3\,(10{\cdot}875\,3^2)$$
$$= 1\,047{\cdot}172 + 7{\cdot}960$$
$$= 1\,055{\cdot}132\text{ m}$$

$$\text{Reduced level of } B = 1\,055{\cdot}132 + 1{\cdot}314 - 1{\cdot}963$$
$$= 1\,054{\cdot}483\text{ m } (3\,459{\cdot}59\text{ ft}) \text{ A.O.D.}$$

5.7 Reciprocal Levelling

Corrections for curvature and refraction are only approximations as they depend on the observer's position, the shape of the geoid and atmospheric conditions.

To eliminate the need for corrections a system of Reciprocal Levelling is adopted for long sights.

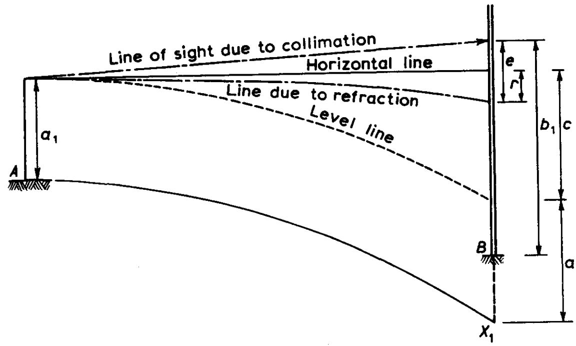
In Fig. 5.19,

$$\text{Difference in level from } A = d = BX_1 = a_1 + c + e - r - b_1$$
$$= \underline{(a_1 - b_1) + (c - r) + e}$$

$$\text{Also from } B \quad d = AX_2 = -(b_2 + c + e - r - a_2)$$
$$= \underline{(a_2 - b_2) - (c - r) - e}$$

$$\text{By adding,} \quad 2d = (a_1 - b_1) + (a_2 - b_2)$$

$$d = \underline{\tfrac{1}{2}[(a_1 - b_1) + (a_2 - b_2)]} \qquad (5.36)$$



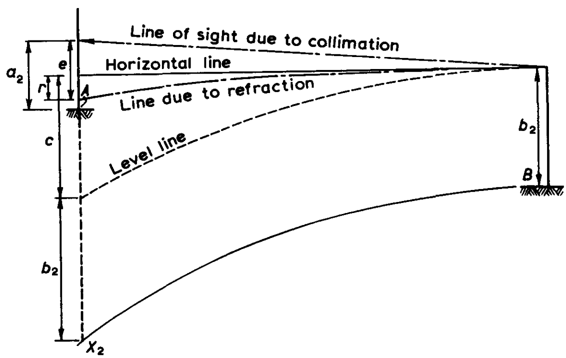


Fig. 5.19

$$\text{Subtracting,} \quad 2(c - r + e) = [(a_2 - b_2) - (a_1 - b_1)]$$

$$\therefore \quad \text{Total error} \quad (c - r) + e = \tfrac{1}{2}[(a_2 - b_2) - (a_1 - b_1)] \tag{5.37}$$

By calculating the error due to refraction and curvature, Eq. (5.28), for $(c-r)$ the collimation error e may be derived. (See §5.4.)

Example 5.17 (a) Obtain from first principles an expression giving the combined correction for earth's curvature and atmospheric refraction in levelling, assuming that the earth is a sphere of 7 920 miles diameter.

(b) Reciprocal levelling between two points Y and Z 2 400 ft apart on opposite sides of a river gave the following results:

Instrument at	Height of instrument	Staff at	Staff reading
Y	4·80	Z	5·54
Z	4·71	Y	3·25

Determine the difference in level between Y and Z and the amount of any collimation error in the instrument. (I.C.E.)

(a) By Eq. (5.28), $c \simeq 0{\cdot}57\,d^2$

(b) By Eq. (5.36),

$$\text{Difference in level} = \tfrac{1}{2}[(a_1 - b_1) + (a_2 - b_2)]$$
$$= \tfrac{1}{2}[(4{\cdot}80 - 5{\cdot}54) + (3{\cdot}25 - 4{\cdot}71)]$$
$$= \tfrac{1}{2}[-0{\cdot}74 - 1{\cdot}46]$$
$$= -\underline{1{\cdot}10\,\text{ft}}$$

i.e. Z is 1·10 ft below Y

By Eq. (5.37),

$$\text{Total error } (c - r) + e = \tfrac{1}{2}[(a_2 - b_2) - (a_1 - b_1)]$$
$$= \tfrac{1}{2}[-1{\cdot}46 + 0{\cdot}74]$$
$$= -0{\cdot}36\,\text{ft}$$

By Eq. (5.28), $(c - r) \simeq 0{\cdot}57\,d^2$

$$= 0{\cdot}57\left(\frac{2\,400}{5\,280}\right)^2$$
$$= 0{\cdot}118\,\text{ft}$$
$$\therefore\quad e = -0{\cdot}478\,\text{ft per } 2\,400\,\text{ft}$$

i.e. $-\underline{0{\cdot}02\,\text{ft per } 100\,\text{ft}}$

(collimation depressed)

Check

Difference in level $= 4{\cdot}80 - 5{\cdot}54 - 0{\cdot}48 + 0{\cdot}12 = -1{\cdot}10\,\text{ft}$

also $3{\cdot}25 - 4{\cdot}71 + 0{\cdot}48 - 0{\cdot}12 = -1{\cdot}10\,\text{ft}$

5.71 The use of two instruments

To improve the observations by removing the likelihood of climatic change two instruments should be used, as in the following example.

Example 5.18

Instrument		Staff at A	Mean	Staff at B	Mean	Apparent Difference in level	Remarks
I	L	6·934		9·424			Inst. I on same
	M	6·784	6·784	8·072	8·073	−1·289	side as A
	U	6·634		6·722			
II	L	6·335		6·426			Inst. II on same
	M	4·985	4·984	6·276	6·276	−1·292	side as B
	U	3·633		6·126			
I	L	6·452		6·514			Inst. I on same
	M	5·098	5·099	6·364	6·364	−1·265	side as B
	U	3·747		6·214			
II	L	6·782		9·249			Inst. II on same
	M	6·632	6·632	7·893	6·895	−1·263	side as A
	U	6·482		6·543			

4) −5·109

True difference in level −1·277

Thus B is 1·277 ft below A.

Exercises 5(f) (Reciprocal levelling)

23. The results of reciprocal levelling between stations A and B 1 500 ft apart on opposite sides of a wide river were as follows:

Level at	Height of Eyepiece (ft)	Staff Readings
A	4·59	8·26 on B
B	4·37	1·72 on A

Find (a) the true difference in level between the stations

(b) the error due to imperfect adjustment of the instrument assuming the mean radius of the earth 3 956 miles.

(L.U./E Ans. (a) −3·16 ft; (b) +0·031 ft / 100 ft)

24. In levelling across a wide river the following readings were taken:

Instrument at	Staff Reading at A	Staff Reading at B
A	5·98 ft (1·823 m)	8·14 ft (2·481 m)
B	8·20 ft (2·499 m)	10·44 ft (3·182 m)

If the reduced level at A is 102·63 ft (31·282 m) above datum what is the reduced level of B?

(Ans. 100·43 ft (30·612 m))

5.8 Levelling for Construction

5.81 Grading of constructions

The gradient of the proposed construction will be expressed as 1 in x, i.e. 1 vertical to x horizontal.

The reduced formation level is then computed from the reduced level of a point on the formation, e.g. the starting point, and the proposed gradient.

By comparing the existing reduced levels with the proposed reduced levels the amount of cut and fill is obtained.

If formation > existing, fill is required.

If formation < existing, cut is required.

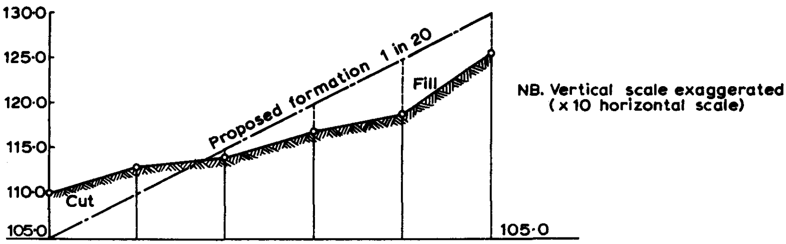
Example 5.19 The following notes of a sectional levelling were taken along a line of a proposed road on the surface.

B.S.	I.S.	F.S.	Height of Collimation	Reduced Level	Horizontal Distance	Remarks
10·24				104·52		B.M.
	4·63				0	Station 1
	1·47				100	Station 2
8·52		0·41			200	Station 3
	5·23				300	Station 4
12·64		3·37			400	Station 5
		5·87			500	Station 6

Calculate the reduced level of each station and apply the conventional arithmetical checks. Thereafter calculate the depth of cutting and filling necessary at each station to form an even gradient rising at 1 in 20 and starting at a level of 105 ft above datum at station 1.

(M.Q.B./M)

B.S.	I.S.	F.S.	Height of Collimation	Reduced Level	Formation Level	Cut	Fill	Station
10·24			114·76	104·52				
	4·63			110·13	105·00	5·13		1
	1·47			113·29	110·00	3·29		2
8·52		0·41	122·87	114·35	115·00		0·65	3
	5·23			117·64	120·00		2·36	4
12·64		3·37	132·14	119·50	125·00		5·50	5
		5·87		126·27	130·00		3·73	6
31·40	11·33	9·65		701·18				
9·65								
21·75								



Chainage	0	1	2	3	4	5
Existing level	113·13	113·29	114·35	117·64	119·50	126·27
Formation level	105·0	110·0	115·0	120·0	125·0	130·0
Cut	5·13	3·29				
Fill			0·65	2·36	5·50	3·73

Fig. 5.20

Check $701{\cdot}18 + 11{\cdot}33 + 9{\cdot}65 = 722{\cdot}16$

$$(114{\cdot}76 \times 3) + (122{\cdot}87 \times 2) + (132{\cdot}14 \times 2) = 722{\cdot}16$$

5.82 The use of sight rails and boning (or travelling) rods

Sight rails and boning rods are used for excavation purposes associated with the grading of drains and sewers.

The sight rails are established at fixed points along the excavation line, at a height above the formation level equal to the length of the boning rod. The formation level compared with the surface level gives the depth of excavation, Fig. 5.21.

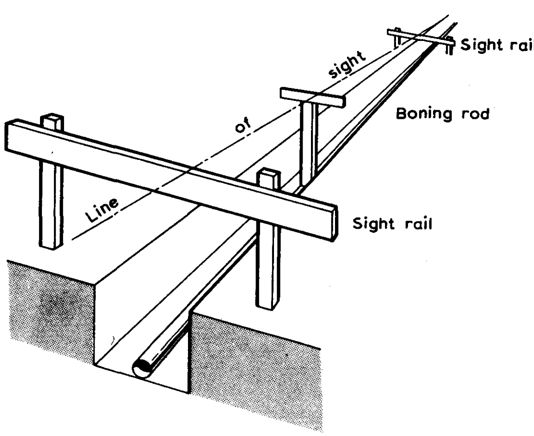


Fig. 5.21

When the boning rod is in line with the sight rails the excavation is at the correct depth, Fig. 5.22.

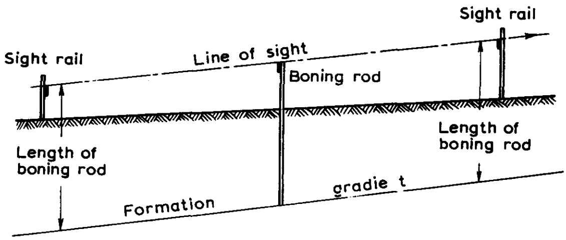


Fig. 5.22

Example 5.20 In preparing the fixing of sight rails, the following consecutive staff readings were taken from one setting of the level:

Bench mark (165·65 ft A.O.D.)	2·73
Ground level at A	5·92
Invert of sewer at A	10·63
Ground level at B	4·27
Ground level at C	3·54

If the sewer is to rise at 1 in 300 and the distance AB 105 ft and BC 153 ft, what will be the height of the sight rails for use with 10 ft boning rods?

What is the reduced level of the invert at A, B and C?

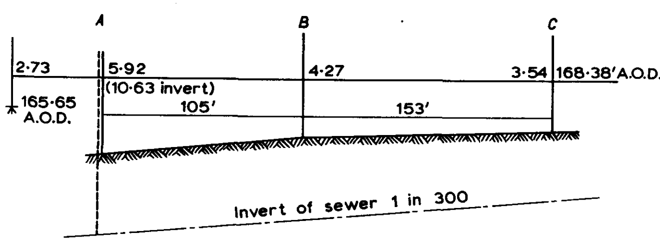


Fig. 5.23

In Fig. 5.23,

Height of collimation	=	165·65 + 2·73	=	168·38 ft A.O.D.
Invert of sewer at A	=	168·38 − 10·63	=	157·75 ft
Sight rail at A	=	157·75 + 10·00	=	167·75 ft
Ground level at A	=	168·38 − 5·92	=	162·46 ft
Height of sight rail above ground at A			=	5·29 ft

Gradient of sewer 1 in 300				
Invert of sewer at B	=	Invert at A + rise due to gradient		
	=	157·75 + 105/300	=	158·10 ft
Sight rail at B			=	168·10 ft
Ground level at B	=	168·38 − 4·27	=	164·11 ft
Height of sight rail above ground at B			=	3·99 ft
Invert of sewer at C	=	Invert at B + rise due to gradient		
	=	158·10 + 153/300	=	158·61 ft
Sight rail at C	=	158·61 + 10·00	=	168·61 ft
Ground level at C	=	168·38 − 3·54	=	164·84 ft
				3·77 ft

5.83 The setting of slope stakes

A slope stake is set in the ground at the intersection of the ground and the formation slope of the cutting or embankment.

The position of the slope stake relative to the centre line of the formation may be obtained:

(a) by scaling from the development plan, or
(b) by calculation involving the cross-slope of the ground and the formation slope, using the rate of approach method suggested in §8.3.

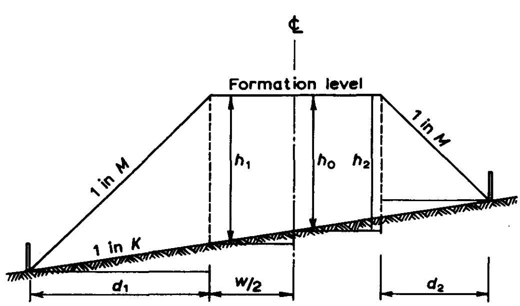


Fig. 5.24

By the rate of approach method, in Fig. 5.24,

$$h_1 = h_0 + \frac{w}{2K} \tag{5.38}$$

$$h_2 = h_0 - \frac{w}{2K} \tag{5.39}$$

By Eq. (8.14), $$d_1 = \frac{h_1}{\frac{1}{M} - \frac{1}{K}} \tag{5.40}$$

Similarly, $$d_2 = \frac{h_2}{\frac{1}{M} + \frac{1}{K}} \tag{5.41}$$

Example 5.21 To determine the position of slope stakes, staff readings were taken at ground level as follows:

Point A	Centre line of proposed road (Reduced level 103·72 ft A.O.D.)	5·63 ft
Point B	50 ft from centre line and at right angles to it	6·13 ft

If the reduced level of the formation at the centre line is to be 123·96 A.O.D., the formation width 20 ft, and the batter is to be 1 in 2, what will be the staff reading, from the same instrument height at the slope stake and how far will the peg be from the centre line point A?

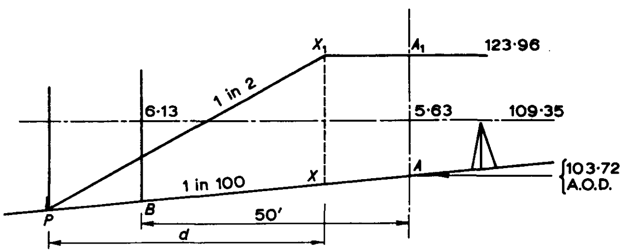


Fig. 5.25

Gradient of AB = (6·13 – 5·63) in 50 ft i.e. 0·5 in 50 ft
$\underline{\text{1 in 100}}$

In Fig. 5.25, $AA_1 = 123{\cdot}96 - 103{\cdot}72 = 20{\cdot}24$ ft

By Eq. (5.38), $XX_1 = h = 20{\cdot}24 + \dfrac{20}{2 \times 100} = 20{\cdot}34$ ft

By Eq. (5.40), the horizontal distance d,i.e. XP,

$$= \frac{h}{\frac{1}{M} - \frac{1}{K}} = \frac{20{\cdot}34}{\frac{1}{2} - \frac{1}{100}}$$

$$= \frac{100 \times 20{\cdot}34}{50 - 1}$$

$$= \underline{41{\cdot}51 \text{ ft}}$$

$\therefore$ The distance from the centre line point A = 51·51 ft

the inclined length $XP = \dfrac{41\cdot51 \times \sqrt{(100^2 + 1)}}{100}$

$= \underline{41\cdot72\text{ ft}}$

Level of A = 103·72

Difference in level $AP = \dfrac{41\cdot51 + 10}{100}$ = 0·52 ft

Level of P = 103·20

Height of collimation = 103·72 + 5·63 = 109·35 ft

$\therefore$ Staff reading at P = $\underline{6\cdot15\text{ ft}}$

Exercises 5(g) (Construction levelling)

25. Sight rails are to be fixed at A and B 350 ft apart for the setting out of a sewer at an inclination of 1 in 200 rising towards B.

If the levels of the surface are A 106·23 and B 104·46 and the invert level at A is 100·74, at what height above ground should the sight rails be set for use with boning rods 10 ft long?

(Ans. 4·51 at A; 8·03 at B)

26. A sewer is to be laid at a uniform gradient of 1 in 200 between two points X and Y, 800 ft apart. The reduced level of the invert at the outfall X is 494·82.

In order to fix sight rails at X and Y, readings are taken with a level in the following order:

	Reading	Staff Station
B.S.	2·66	T.B.M. (near X) Reduced level 504·64
I.S.	a	Top of sight rail at X
I.S.	3·52	Peg at X
F.S.	1·80	T.P. between X and Y
B.S.	7·04	T.P. between X and Y
I.S.	b	Top of sight rail at Y
F.S.	6·15	Peg at Y

(i) Draw up a level book and find the reduced levels of the pegs.

(ii) If a boning rod of length 9′-6″ is to be used, find the readings a and b.

(iii) Find the height of the sight rails above the pegs at X and Y.

(L.U. Ans. (ii) 2·98, 4·22; (iii) 0·54, 1·93)

27. The levelling shown on the field sheet given below was undertaken during the laying out of a sewer line. Determine the height of

the ground at each observed point along the sewer line and calculate the depth of the trench at points X and Y if the sewer is to have a gradient of 1 in 200 downwards from A to B and is to be 4·20 ft below the surface at A.

B.S.	I.S.	F.S.	Distance (ft)	Remarks
11·21				B.M. 321·53
4·56		5·82	0	
	3·78		100	
11·65		3·66	200	Point X
2·40		3·57	300	
7·82		10·81	400	
	5·91		500	
	6·56		600	Point Y
6·32		8·65	700	Point B
		3·81		B.M. 329·15

(R.I.C.S./M/L Ans. 6·10, 9·03)

Exercises 5(h) (General)

28. The following staff readings in fact were taken successively with a level, the instrument having been moved forward after the second, fourth and eighth reading. 1·54, 7·24, 4·03, 1·15, 8·62, 8·52, 6·41, 1·13, 7·31, 2·75 and 5·41.

The last reading was taken with the staff on a bench mark having an elevation of 103·74 ft.

Enter the readings in level book form, complete the reduced levels and apply the usual checks.

29. The following readings were taken using a dumpy level on a slightly undulating underground roadway.

B.S.	I.S.	F.S.	Reduced Level	Distance	Remarks
5·32			+ 8752·20	0	Point A + 8752·20
	6·43			100	
	5·23			200	
3·06		4·12		300	
	4·30			400	
	2·23			500	
3·02		1·09		600	
	4·01			700	
	5·12			800	
		6·67		900	Point B

Work out the reduced levels relative to the assumed datum of mean

sea level + 10 000 ft (as used by the National Coal Board to avoid negative reduced levels).

State the amount of excavation necessary at point B to form an even gradient dipping 1 in 300 from A to B, the reduced level of A to remain at 8 752·20 ft.

(Ans. 2·52 ft)

30. Reduce the page of a level book and plot the result to a scale of 1″ = 100′ horizontal and 1″ = 10′ vertical.

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level	Distance	Remarks
9·92					25·23	0	B.M.
	8·22					0	Start of Section
	5·98					100	
7·35		3·15				200	
	5·59					300	
8·13		2·12				400	
	6·05					460	
	5·63					500	
	5·00					560	
5·63		3·65				600	
	4·19					630	
	5·91					700	
4·71		8·04				800	
	5·35					830	
	4·01					900	
6·24		2·73				1 000	
	5·82					1 100	
	4·36					1 200	End of Section
		3·72					B.M.

(R.I.C.S./Q)

31. A levelling party ran a line of levels from point A at elevation 135·43 to point B for which the reduced level was found to be 87·15. A series of flying levels (as below) was taken back to the starting point A.

B.S.	F.S.	Remarks
9·67		*B*
11·54	1·38	
8·22	4·81	
7·94	3·35	
10·56	2·07	
9·92	5·33	
8·88	1·04	
	0·42	*A*

Find the misclosure on the starting point.

(L.U./E Ans. 0·05 ft)

32. (a) Explain the difference between 'rise and fall' and 'height of collimation' method of reducing levels, stating the advantages and disadvantages of each.

(b) The following is an extract from a level book. Reduce the levels by whichever method you think appropriate, making all the necessary checks and insert the staff readings in the correct blank spaces for setting in the levels pegs *A*, *B* and *C* so that they have the reduced levels given in the book.

B.S.	I.S.	F.S.	Reduced Level	Distance	Remarks
3·24			58·63		B.M. 1
5·03		9·65			C.P.
	6·42			0	Beginning of Section
	9·69			50	
	–		49·69	50	Peg *A*
4·19		10·87		100	C.P.
	–		48·55	100	Peg *B*
	5·54			150	
	–		47·41	150	Peg *C*
	4·30			200	End of Section
11·73		2·32			C.P.
		8·61	51·35		B.M. 2

(R.I.C.S./M/L Ans. Staff Readings, *A* 7·56, *B* 2·02, *C* 3·16; Error in levelling, 0·02)

33. The level book refers to a grid of levels taken at 100 ft intervals on 4 parallel lines 100 ft apart.

(A) Reduce and check the level book.

(B) Draw a grid to a scale of 50 ft to 1 in. and plot the contours for a 2 ft vertical interval.

B.S.	I.S.	F.S.	Reduced Level	Distance on Line	Remarks
1·23			89·14	0	Line *A* T.B.M.
	2·75			100	
	3·51			200	
	4·26			300	
	12·35			300	Line *B*
	9·06			200	
	6·78			100	
	4·18			0	
4·15		6·97		0	Line *C*
	5·51			100	

B.S.	I.S.	F.S.	Reduced Level	Distance on Line	Remarks
	7·88			200	
	10·45			300	
	8·93			300	Line *D*
	7·18			200	
	5·34			100	
	4·59			0	
8·37		2·62			C.P.
		4·14			T.B.M.

(R.I.C.S./L/M)

34. Discuss the various ways in which 'errors' can occur in levelling and measures that can be adopted to keep each source of error to a minimum.

In levelling from a bench mark 347·79 ft above O.D. and closing on to another 330·61 ft above O.D., staff readings were taken in the following order:

3·72, 8·21; 4·91, 8·33, 7·28; 0·89, 4·27; 2·28, 3·91, 3·72, 9·23.

The position of the instrument was moved immediately after taking the 2nd, 5th, and 7th readings indicated by semi-colons in the above series of readings.

Show how these readings would be booked and the levels reduced using either the 'collimation' or the 'rise and fall' method. Carry out the usual arithmetical checks and quote the closing error.

Explain briefly why it is particularly important not to make a mistake in reading an intermediate sight.

(I.C.E.)

35. The record of a levelling made some years ago has become of current importance. Some of the data are undecipherable but sufficient remain to enable all the missing values to be calculated. Reproduce the following levelling notes and calculate and insert the missing values.

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level	Remarks
2·36					121·36	B.M. at No. 1 Shaft
				1·94		
4·05		7·29				
				4·46		
	4·31					
	6·93					
					113·32	
		7·79		0·63		B.M. on School
	3·22		1·58			
					112·01	
	6·53					

B.S.	I.S.	F.S.	Rise	Fall	Reduced Level	Remarks
					113·53	
		5·86				
				3·10		B.M. on Church
14·96				21·18		

36. The following is an extract from a level book

B.S.	I.S.	F.S.	Reduced Level	Remarks
4·20				Point *A*
	2·70			
2·64		11·40	119·30	C.P. B.M. 119·30
	3·42			
	9·51			
	11·74			
2·56		13·75		C.P. *B*
	3·10			
	6·91			
3·61		11·23		C.P. *C*
	5·60			
12·98		3·61		C.P. *C*
13·62		3·31		C.P. *B*
12·03		2·51		C.P. B.M.
		4·83		Point *A*

(a) Reduce the above levels.

(b) If you consider a mistake has been made suggest how it occurred.

(c) Give reasons for choice of 'Rise and Fall' or 'Height of Collimation' for reducing the levels. The B.S. and F.S. lengths were approximately equal.

(L.U. Ans. probably 11·98 instead of 12·98)

37. The following are the levels along a line *ABC*.

Distance	Reduced Level	Remarks
0	0·00	At *A*
10	1·21	
20	2·46	
30	3·39	
40	4·54	At *B*
50	6·03	
60	7·65	
70	9·03	
80	10·32	At *C*

Plot the reduced levels to a scale of 10 ft to 1 in. for the horizontal scale and 1 ft to 1 in. for the vertical scale.

A roadway is to be constructed from A to C at a uniform gradient. From the section state the height of filling required at each plotting point. (R.I.C.S./M)

38. In order to check the underground levellings of a colliery it was decided to remeasure the depth of the shaft and connect the levelling to a recently established Ordnance Survey Bench Mark A, 272·45 ft above O.D.

The following levels were taken with a dumpy level starting at A to the mouth of the shaft at D.

B.S.	F.S.	Reduced Level	Remarks
2·17		272·45 ft	B.M. at A
3·36	11·32		Mark B
5·79	7·93		Mark C
	0·00		Mark D on rails

The vertical depth of the shaft was then measured from D to E at the pit bottom and found to be 1 745 ft $8\frac{1}{2}$ in.

A backsight underground to E was found to be 3·98 and a foresight to the colliery Bench Mark F on a wall near the pit bottom was 2·73.

Tabulate the above readings and find the value of the underground B.M. at F expressing this as a depth below Ordnance Datum.

(Ans. 1 479·94 ft)

39. Levels were taken at 100 ft intervals down a road with a fairly uniform gradient and the following staff readings booked:

B.S.	I.S.	F.S.	Distance	
7·00			0	T.B.M. 98·50
	9·50		100	
11·75		6·00	200	
	8·55		300	
	10·81		400	
13·05		5·60	500	
	7·90		600	
	10·20		700	
12·70		6·00	800	
	8·48		900	
	10·98		1 000	
	13·20		1 100	

Errors were made in booking, correct these and reduce the levels.
(L.U. Ans. B.S. and F.S. transposed)

40. Spot levels are given below at 200 ft intervals on a grid *ABCD*. Draw the plan to a scale of 200 ft to 1 in. and show on it where you would place the level in order to take readings.

Draw up a level book by the 'height of collimation' method showing your readings. Take the level at *A* as a T.B.M.

A	100·70	102·00	103·50	105·20	106·80	108·20	109·50	*B*
	101·30	103·40	104·10	106·30	108·20	109·30	110·70	
	105·00	106·20	107·30	109·10	110·40	111·50	112·30	
D	108·00	107·10	108·60	110·40	111·30	112·20	113·80	*C*

(L.U.)

41. In levelling up a hillside, the sight lengths were observed with stadia lines, the average length of the ten backsights and foresights being 70 ft and 35 ft respectively.

Since the observed difference of the reduced level of 78·40 ft was disputed, the level was set up midway between two pegs *A* and *B* 300 ft apart, and the reading on *A* was 4·60 and on *B* 5·11; and when set up in line *AB*, 30 ft behind *B*, the reading on *A* was 5·17 and on *B* 5·64.

Calculate the true difference of reduced level.
(L.U. Ans. 78·35 ft; 0·013 ft per 100 ft)

42. *A, B, C, D, E* and *F* are the sites of manholes, 300 ft apart on a straight sewer. The natural ground can be considered as a plane surface rising uniformly from *A* to *F* at a gradient of 1 vertically in 500 horizontally, the ground level at *A* being 103·00. The level of the sewer invert is to be 95·00 at *A*, the invert then rising uniformly at 1 in 200 to *F*. Site rails are to be set up at *A, B, C, D, E* and *F* so that a 10 ft boning rod or traveller can be used. The backsights and foresights were made approximately equal and a peg at ground level at *A* was used as datum.

Draw up a level book showing the readings. (L.U.)

43. The following staff readings were obtained when running a line of levels between two bench marks *A* and *B*:

3·56 (*A*) 6·68, 7·32, 9·89 change point, 2·01, 6·57, 7·66, C.P.
5·32, 4·21, 1·78, C.P. 4·68, 5·89, 2·99 (*B*)

Enter and reduce the readings in an accepted form of field book. The reduced levels of the bench marks at *A* and *B* were known to be 143·21 ft and 136·72 ft respectively.

It is found after the readings have been taken with the staff

supposedly vertical, as indicated by a level on the staff, that the level is 5° in error in the plane of the staff and instrument.

Is the collimation error of the instrument elevated or depressed and what is its value in seconds if the backsights and foresights averaged 100 ft and 200 ft respectively.

(L.U. True difference in level 6·72; collimation elevated 119 sec)

44. Undernoted are levels taken on the floor of an undulating underground roadway AB, 10 ft in width and 6 ft in height, which is to be regraded and heightened.

Distance (ft)	B.S.	I.S.	F.S.	Rise	Fall	Reduced Level	Remarks
0	6·95					30·00	Floor level at A
50		3·40					
100		0·65					
150	2·50		0·00				
200		3·50					
250		6·00					
300		6·50					
350	7·75		5·00				
400		4·50					
450		1·25					
500		2·20					Floor level at B
	1·00		6·65				
			6·55				Floor level at A

Plot the section of the roadway to a scale of 1 in. = 50 ft for horizontals and 1 in. = 10 ft for verticals. Thereafter calculate and mark on the section the amount of ripping and filling at the respective 50 ft intervals to give a uniform gradient from A to B and a minimum height of 8 ft.

Calculate the volume, in cubic yards, of the material to be ripped from the roof in giving effect to the above improvements.

(M.Q.B./M Ans. 320 yd^3)

45. The centre-line of a section of a proposed road in cutting is indicated by pegs at equal intervals and the corresponding longitudinal section gives the existing ground level and the proposed formation level at each peg, but no cross-sections have been taken, or sidelong slopes observed.

Given the proposed formation width (d) and the batter of the sides (S horizontal to 1 vertical) how would you set out the batter pegs marking the tops of the slopes at each centre line peg, without taking and plotting the usual cross-sections?

An alternative method would be acceptable. (I.C.E.)

46. (a) Determine from first principles the approximate distance at which correction for curvature and refraction in levelling amounts to 0·01 ft, assuming that the effect of refraction is one seventh that of the earth's curvature and that the earth is a sphere of 7 920 miles diameter.

(b) Two survey stations *A* and *B* on opposite sides of a river are 2 510 ft apart, and reciprocal levels have been taken between them with the following results:

Instrument at	Height of instrument	Staff at	Staff reading (ft)
A	4·83	*B*	6·02
B	4·91	*A*	3·98

Compute the ratio of refraction correction to curvature correction, and the difference in level between *A* and *B*.

(I.C.E. Ans. (a) $\simeq$ 700 ft;
(b) *A* is 1·06 ft above *B*.
Ratio $\simeq$ 0·14 to 1)

Bibliography

HOLLAND, J.L., WARDELL, K. and WEBSTER, A.G., *Surveying*, Coal Mining Series (Virtue).
SANDOVER, J.A. *Plane Surveying* (Edward Arnold).
TRUTMAN, O., *Levelling* (Wild (Heerbrugg)).
PARRY, R. and JENKINS, W.R., *Land Surveying* (Estates Gazette).
BANNISTER, A. and RAYMOND, S., *Surveying* (Pitman).
CURTEN, W. and LANE, R.F., *Concise Practical Surveying* (English Universities Press).

6 TRAVERSE SURVEYS

The purpose of traverse surveys is to control subsequent detail, i.e. the fixing of specific points to which detail can be related. *The accuracy of the control survey must be superior to that of the subsidiary survey.*

A traverse consists of a series of related points or stations, which when connected by angular and linear values form a framework.

6.1 Types of Traverse

6.11 Open

Traverse *ABCDE*, Fig. 6.1. The start and finish are not fixed points.

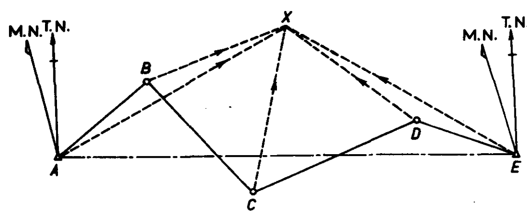


Fig. 6.1 Open traverse

A check on the angles may be made by (a) taking meridian observations at the start and finish or (b) taking observations to a common fixed point X.

6.12 Closed

(a) *On to fixed points.* If the start and finish are fixed points, e.g. A and E, then the length and bearing between them is known. From the traverse the distance may be computed.

(b) *Closed polygon,* Fig. 6.2.

Checks (i) The sum of the deflection angles should equal 360°, i.e.

$$\Sigma\alpha = \alpha_1 + \alpha_2 + \alpha_3 \dots \alpha_n = 360^\circ = 4 \times 90^\circ$$

or (ii) The sum of the internal angles should equal

$$(2n - 4) \times 90^\circ$$

where n = no. of angles or sides, i.e.

$$\Sigma\beta = \beta_1 + \beta_2 + \beta_3 \dots \beta_n = (2n - 4)90^\circ \qquad (6.1)$$

or (iii) The sum of the external angles should equal

$$(2n + 4)\,90^\circ$$

i.e.
$$\Sigma\theta = \theta_1 + \theta_2 + \theta_3 \dots \theta_n = (2n + 4)\,90^\circ \qquad (6.2)$$

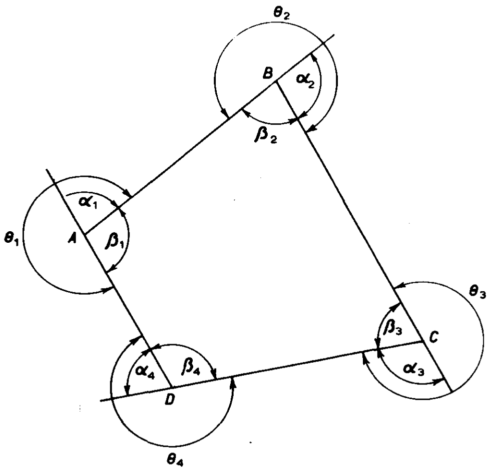


Fig. 6.2 Closed traverse

N.B.
$$\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \alpha_n + \beta_n = 2 \times 90^\circ$$
$$\Sigma(\alpha + \beta) = (4 \times 90) + (2n - 4)\,90 = (2 \times 90) \times n.$$

The sum will seldom add up exactly to the theoretical value and the 'closing error' must be distributed before plotting or computing.

6.2 Methods of Traversing

The method is dependent upon the accuracy required and the equipment available. The following are alternative methods.

(1) *Compass traversing* using one of the following:
   (a) a prismatic compass
   (b) a miners' dial
   (c) a tubular or trough compass fitted to a theodolite
   (d) a special compass theodolite.

(2) *Continuous azimuth* (fixed needle traversing) using either (a) a miners' dial or (b) a theodolite.

(3) *Direction method* using any angular measuring equipment.

(4) *Separate angular measurement* (double foresight method) using any angular measuring equipment.

6.21 Compass traversing (loose needle traversing), Fig. 6.3

Application. Reconnaissance or exploratory surveys.

Advantages. (1) Rapid surveys.
(2) Each line is independent – errors tend to compensate.
(3) The bearing of a line can be observed at any point along the line.
(4) Only every second station needs to be occupied (this is not recommended because of the possibility of local attraction)

Disadvantages. (1) Lack of accuracy. (2) Local attraction.

Accuracy of survey. Due to magnetic variations, instrument and observation errors, the maximum accuracy is probably limited to ± 10 min, i.e. linear equivalent 1 in 300.

Detection of effects of local attraction. Forward and back bearings should differ by 180° assuming no instrumental or personal errors exist.

Elimination of the effect of local attraction. The effect of local attraction is that all bearings from a given station will be in error by a constant value, the angle between adjacent bearings being correct.

Where forward and back bearings of a line agree this indicates that the terminal stations are both free of local attraction.

Thus, starting from bearings which are unaffected, a comparison of forward and back bearings will show the correction factors to be applied.

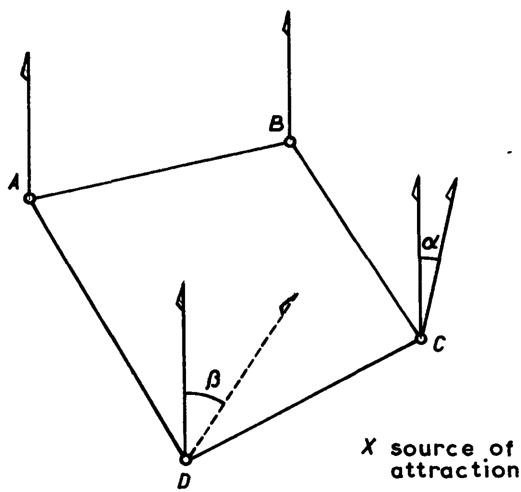


Fig. 6.3 Compass traversing

In Fig. 6.3 the bearings at A and B are correct. The back bearing of CB will be in error by α compared with the forward bearing BC.

The forward bearing CD can thus be corrected by α. Comparison of the corrected forward bearing CD with the observed back bearing DC will show the error β by which the forward bearing DA must be corrected. This should finally check with back bearing AD.

Example 6.1

Line	Forward Bearing	Back Bearing
AB	120° 10′	300° 10′
BC	124° 08′	306° 15′
CD	137° 10′	310° 08′
DE	159° 08′	349° 08′
EF	138° 15′	313° 10′

Station	Back Bearing	Forward Bearing	Correction	Corrected Forward Bearing	± 180°
A	–	120° 10′	–		
B	300° 10′	124° 08′	–	124° 08′	304° 08′
C	306° 15′	137° 10′	−2° 07′	135° 03′	315° 03′
D	310° 08′	159° 08′	+4° 55′	164° 03′	344° 03′
E	349° 08′	138° 15′	−5° 05′	133° 10′	313° 10′
F	313° 10′				

Thus stations A, B, and F are free from local attraction.

N.B. (1) Line AB. Forward and Back bearings agree; therefore stations A and B are free from attraction

(2) Corrected forward bearing at B + 180° compared with back bearing at C shows an error of 2° 07′, i.e. 304° 08′ − 306° 15′ = −2° 07′.

(3) Corrected forward bearing CD 137° 10′ − 2° 07′ = 135° 03′.

(4) Comparison of corrected forward bearing EF + 180° agrees with back bearing FE. Therefore station F is also free from local attraction

6.22 Continuous azimuth method (Fig. 6.4)

This method was ideally suited to the old type of miners' dial with open-vane sights which could be used in either direction.

The instrument is orientated at each station by observing the backsight, with the reader clamped, from the reverse end of the 'dial' sights.

The recorded value of each foresight is thus the bearing of each line relative to the original orientation. For mining purposes this was the magnetic meridian and hence the method was known as 'the fixed needle method'.

The method may be modified for use with a theodolite by changing face between backsight and foresight observations.

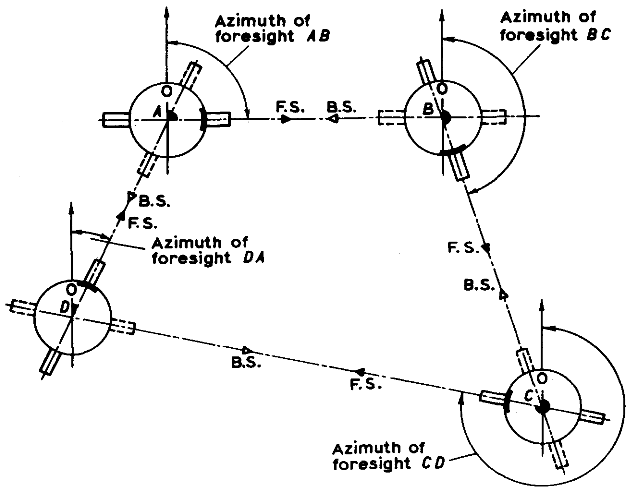


Fig. 6.4 Continuous azimuth method of traversing

6.23 Direction method

The continuous azimuth method of traversing is restrictive for use with the theodolite where the accuracy must be improved, as the observations cannot be repeated.

To overcome this difficulty, and still retain the benefit of carrying the bearing, the Direction method may be employed.

This requires only approximate orientation, corrections being made either on a 'Direction' bearing sheet if one arc on each face is taken, or, alternatively, on the field booking sheet.

N.B. No angles are extracted, the theodolite showing approximate bearings of the traverse lines as the work proceeds.

Direction bearing sheet

Set at	Obs. to	Mean observed directions	Corr- ection	Back bearing	Forward bearing	Final correction	Final bearing
A	*E*	0 08·00		0 08·00			0 08·00
	B	283 09·05			283 09·05	−0·18	283 08·87
B	*A*	103 00·60	+8·45	103 09·05			
	C	345 37·05	+8·45		345 45·50	−0·36	345 45·14
C	*B*	165 36·40	+9·10	165 45·50			
	D	039 40·05	+9·10		039 49·15	−0·54	039 48·61
D	*C*	219 55·50	−6·35	219 49·15			
	E	101 31·35	−6·35		101 25·00	−0·72	101 24·28
E	*D*	281 31·20	−6·20	281 25·00			
	A	180 15·10	−6·20		180 08·90	−0·90	180 08·00
			Initial bearing		180 08·00		
				Error	0·90		

N.B. (1) Here the instrument has been approximately orientated at each station, i.e. the reciprocal of the previous forward bearing is set as a back bearing. Any variation from the previous mean forward bearing thus requires an orientation correction.

(2) In the closed traverse the closing error is seen immediately by comparing the first back bearing with the final forward bearing.

(3) As a simple adjustment the closing error is distributed equally amongst the lines.

Method of booking by the direction method

Station set at A *Back bearing AE* $0°\,08'$

Arc	Obs. to	F.L.	F.R.	Mean	Correction	Back bearing	Forward bearing
		° ′	° ′	° ′	° ′	° ′	° ′
1	*E*	0 08·0	180 08·0			0 08·00	
	B	283 09·0	103 09·1				283 09·05
2	*E*	090 12·2	270 12·2	090 12·20	−90 04·20	0 08·00	
	B	013 13·2	193 13·4	013 13·30	−90 04·20		283 09·10
3	*E*	27·6	27·4	27·50	− 19·50	0 08·00	
	B	28·5	28·5	28·50	− 19·50		283 09·00
					Mean Forward Bearing		283 09·05

Station set at B *Back bearing BA* $103°\,09{\cdot}05'$

Arc	Obs. to	F.L.	F.R.	Mean	Correction	Back bearing	Forward bearing
		° ′	° ′	° ′	° ′	° ′	° ′
1	*A*	103 00·6	283 00·5	103 00·55	+ 08·50	103 09·05	
	C	345 37·1	165 37·2	345 37·15	+ 08·50		345 45·65
2	*A*	199 04·9	019 05·0	199 04·95	−95 55·90	103 09·05	
	C	081 41·3	261 41·5	081 41·40	−95 55·90		345 45·50
3	*A*	15·6	15·7	15·65	− 06·60	103 09·05	
	C	52·0	52·2	52·10	− 06·60		345 45·50
					Mean Forward Bearing		345 45·55

N.B. (1) No angles are extracted.

(2) The back bearing is approximately set on the instrument for the backsight, e.g. *AE* is set exactly here $0°\,08{\cdot}0'$

$\therefore$ *AB* is the correctly oriented bearing $283°\,09{\cdot}0'$.

(3) On arc 2 the instrument zero is changed. After taking out the mean of the faces, a correction is applied to give the back bearing, i.e.

$$\begin{array}{lr} & 090°\,12{\cdot}20' \\ & -\quad 0°\,08{\cdot}00' \\ \hline \text{Correction} & 090°\,04{\cdot}20' \end{array}$$

This correction is now applied to give the second forward bearing.

(4) On the 3rd and subsequent arcs, if required, only the minutes are booked, a new zero being obtained each time.

(5) The mean forward bearing is now extracted and carried forward to the next station.

6.24 Separate angular measurement

Angles are measured by finding the difference between adjacent recorded pointings.

After extracting the mean values these are converted into bearings for co-ordinate purposes.

In the case of a closed polygon, the angles may be summated to conform with the geometrical properties, Eq. (6.1) or (6.2).

Exercises 6(a)

1. A colliery plan has been laid down on the national grid of the Ordnance Survey and the co-ordinates of the two stations A and B have been converted into feet and reduced to A as a local origin.

	Departure (ft)	Latitude (ft)
Station A	0	0
Station B	East 109·2	South 991·7.

Calculate the Grid bearing of the line AB. The mean magnetic bearing of the line AB is S 3° 54′ W and the mean magnetic bearing of an underground line CD is N 17° 55′ W.

State the Grid bearing of the line CD.

(M.Q.B./M Ans. 331° 54′)

2. The following angles were measured in a clockwise direction, from the National Grid North lines on a colliery plan:

(a) 156° 15′ (b) 181° 30′ (c) 354° 00′ (d) 17° 45′

At the present time in this locality, the magnetic north is found to be 10° 30′ W of the Grid North.

Express the above directions as quadrant bearings to be set off using the magnetic needle.

(M.Q.B./UM Ans. (a) S 13° 15′ E; (b) S 12° 00′ W; (c) N 4° 30′ E; (d) N 28° 15′ E)

3. The following underground survey was made with a miners' dial in the presence of iron rails. Assuming that station A was free from local attraction calculate the correct magnetic bearing of each line.

Station	BS	FS
A		352° 00′
B	358° 30′	12° 20′
C	14° 35′	282° 15′

Station	BS	FS
D	280° 00′	164° 24′
E	168° 42′	200° 22′

(Ans. 352° 00′; 05° 50′; 273° 30′; 157° 54′; 189° 34′)

(N.B. A miners' dial has vane sights, i.e. B.S. = F.S., not reciprocal bearings).

4. The geographical azimuth of a church spire is observed from a triangulation station as 346° 20′. At a certain time of the day a magnetic bearing was taken of this same line as 003° 23′. On the following day at the same time an underground survey line was magnetically observed as 195° 20′ with the same instrument.

Calculate (a) the magnetic declination,
(b) the true bearing of the underground line.

(Ans. 17° 03′W; 178° 17′)

5. Describe and sketch a prismatic compass. What precautions are taken when using the compass for field observations?

The following readings were obtained in a short traverse $ABCA$. Adjust the readings and calculate the co-ordinates of B and C if the co-ordinates of A are 250 ft E, 75 ft N.

Line	Compass bearing	Length (ft)
AC	00° 00′	195·5
AB	44° 59′	169·5
BA	225° 01′	169·5
BC	302° 10′	141·7
CB	122° 10′	141·7
CA	180° 00′	195·5

(R.I.C.S. Ans. B 370 E, 195 N C 250 E, 270 N)

6. The following notes were obtained during a compass survey made to determine the approximate area covered by an old dirt-tip.

Correct the compass readings for local attraction. Plot the survey to a scale of 1 in 2400 and adjust graphically by Bowditch's rule.

Thereafter find the area enclosed by equalising to a triangle.

Line	Forward bearing	Back bearing	Length (ft)
AB	N 57° 10′E	S 58° 20′ W	750
BC	N 81° 40′E	S 78° 00′ W	828
CD	S 15° 30′E	N 15° 30′ W	764
DE	S 10° 20′W	N 12° 00′ E	405
EF	S 78° 50′W	N 76° 00′ E	540
FG	N 69° 30′E	S 68° 30′ W	950
GA	N 22° 10′W	S 19° 30′ E	383.

(N.R.C.T. Ans. AB 54° 40′; BC 78° 00′; CD 164° 30′; DE 190° 20′; EF 257° 10′; FG 291° 40′; GA 338° 00′; area 32·64 acres)

7. A and B are two reference stations in an underground roadway, and it is required to extend the survey through a drift from station B to a third station C. The observations at B were as follows:

Horizontal angle ABC 271° 05′ 20″.
Vertical angle to staff at C + 10° 15′ 00″.
Staff reading at C 1·50 ft.
Instrument height at B 5 ft 7 in.
Measured distance BC 284·86 ft.

The bearing of AB was 349° 56′ 10″ and the co-ordinates of B E 4689·22 ft, N 5873·50 ft.

Calculate the true slope of BC to the nearest 10 seconds, the horizontal length of BC, its bearing, and the co-ordinates of C.

(N.R.C.T. Ans. 11° 06′20″; 279·55 ft; 081° 01′ 30″; E 4965·35, N 5917·11)

6.3 Office Tests for Locating Mistakes in Traversing

6.31 A mistake in the linear value of one line

If the figure is closed the co-ordinates can be computed and the closing error found.

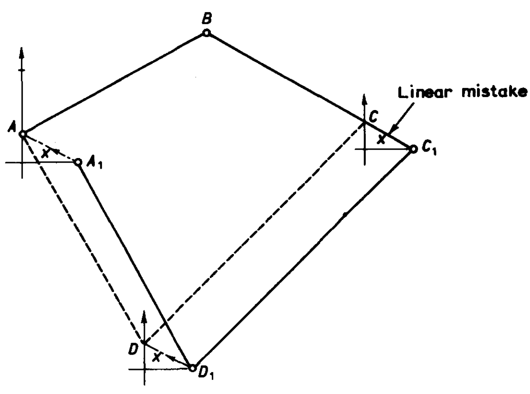


Fig. 6.5 Location of a linear mistake

Let the computed co-ordinates give values for $ABC_1D_1A_1$, Fig. 6.5.

The length and bearing of AA_1 suggests that the mistake lies in this direction, and if it is parallel to any given line of the traverse this is where the mistake has been made.

The amount AA_1 is therefore the linear mistake, and a correction to the line BC gives the new station values of C_1D and thus closes on A.

If the closing error is parallel to a number of lines then a repetition of their measurements is suggested.

6.32 A mistake in the angular value at one station

Let the traverse be plotted as $ABCD_1A_1$, Fig. 6.6

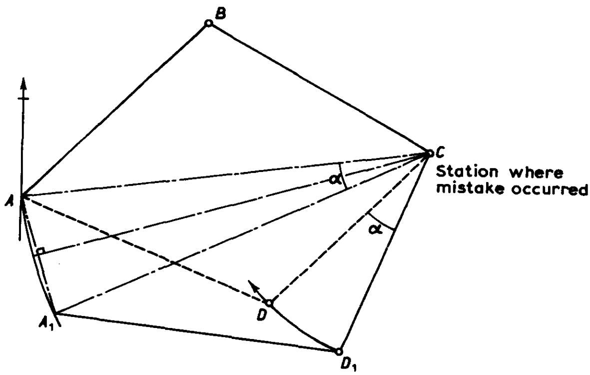


Fig. 6.6 Location of an angular mistake

The closing error AA_1 is not parallel to any line but the perpendicular bisector of AA_1 when produced passes through station C. Here an angular mistake exists.

Proof. AA_1 represents a chord of a circle of radius AC, the perpendicular bisector of the chord passing through the centre of the circle of centre C.

The line CD_1 must be turned through the angle $\alpha = ACA_1$.

6.33 When the traverse is closed on to fixed points and a mistake in the bearing is known to exist

The survey should be plotted or computed from each end in turn, i.e. $ABCDE - E_1D_1C_1BA$, Fig. 6.7.

The station which is common to both systems will suggest where the mistake has been made.

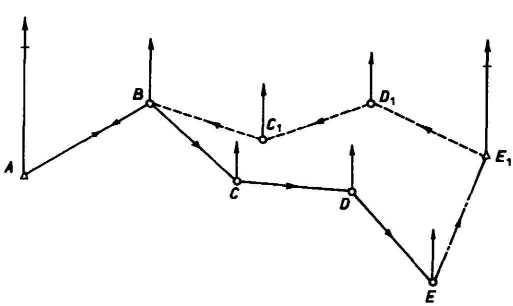


Fig. 6.7

If there are two or more mistakes, either in length or bearing, then it is impossible to locate their positions but they may be localised.

6.4 Omitted Measurements in Closed Traverses

Where it is impossible to measure all the values (either linear, angular or a combination of both) in a closed traverse, the missing quantities can be calculated provided they do not exceed two.

As the traverse is closed the algebraic sum of the partial co-ordinates must each sum to zero, i.e.

$$l_1 \sin\theta_1 + l_2 \sin\theta_2 + l_3 \sin\theta_3 + \dots l_n \sin\theta_n = 0$$
$$l_1 \cos\theta_1 + l_2 \cos\theta_2 + l_3 \cos\theta_3 + \dots l_n \cos\theta_n = 0$$

where the lengths of the lines are $l_1 l_2 l_3$ etc., and the bearings $\theta_1 \theta_2 \theta_3$ etc.

As only 2 independent equations are involved only 2 unknowns are possible.

Failure to close the traverse in any way transfers all the traverse errors to the unknown quantities. Therefore use of the process is to be deprecated unless there is no other solution.

Six cases may arise:

(1) Bearing of one line.
(2) Length of one line.
(3) Length and bearing of one line.
(4) Bearing of two lines.
(5) Length of two lines.
(6) Bearing of one line and length of another line.

Cases 1, 2 and 3 are merely part of the calculation of a join between two co-ordinates.

6.41 Where the bearing of one line is missing

$l_n \sin\theta_n = P$ (1) where P = the sum of the other partial departures

$l_n \cos\theta_n = Q$ (2) where Q = the sum of the other partial latitudes

Dividing (1) by (2),

$$\tan\theta_n = \frac{P}{Q} \quad \text{i.e.} \quad \frac{\Delta E}{\Delta N} = \text{the difference in the total co-ordinates of the stations forming the ends of the missing line} \qquad (6.3)$$

6.42 Where the length of one line is missing

$$l_n \sin\theta_n = P \qquad (1)$$
$$l_n \cos\theta_n = Q \qquad (2)$$

By squaring each and adding

$$l_n^2 \sin^2\theta_n = P^2$$
$$l_n^2 \cos^2\theta_n = Q^2$$

$$\therefore \quad l_n^2(\sin^2\theta_n + \cos^2\theta_n) = P^2 + Q^2$$

i.e.

$$l_n = \sqrt{(P^2 + Q^2)}$$
$$= \sqrt{(\Delta E^2 + \Delta N^2)} \qquad (6.4)$$
$$= \frac{\Delta E}{\sin\theta_n} \qquad (6.5)$$
$$= \frac{\Delta N}{\cos\theta_n} \qquad (6.6)$$

6.43 Where the length and bearing of a line are missing

The two previous cases provide the required values.

6.44 Where the bearings of two lines are missing

(1) *If the bearings are equal*

$$l_p \sin\theta_p + l_q \sin\theta_q = P$$
$$l_p \cos\theta_p + l_q \cos\theta_q = Q$$

if $\theta_p = \theta_q = \theta$

Then

$$l_p \sin\theta + l_q \sin\theta = P$$
$$\therefore \quad (l_p + l_q)\sin\theta = P$$
$$\sin\theta = \frac{P}{l_p + l_q}$$

or

$$\cos\theta = \frac{Q}{l_p + l_q}$$

or

$$\tan\theta = \frac{P}{Q} \qquad (6.7)$$

(2) *If the bearings are adjacent*

$$l_1 \sin\theta_1 + l_2 \sin\theta_2 = P$$
$$l_1 \cos\theta_1 + l_2 \cos\theta_2 = Q$$

In Fig. 6.8, l_1, l_2, l_3, l_4, θ_3, and θ_4 are known.

$AC = l_5$ can be found with the bearing AC.

In triangle ABC,

$$\tan\frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

where $$s = \frac{a+b+c}{2}$$

$$\sin B = \frac{b \sin\alpha}{a}$$

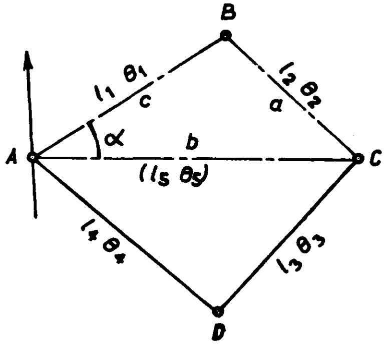


Fig. 6.8

From the value of the angles the required bearings can be found.

Bearing AB = bearing AC − angle α

Bearing BC = bearing BA − angle B

(3) *If the bearings are not adjacent*

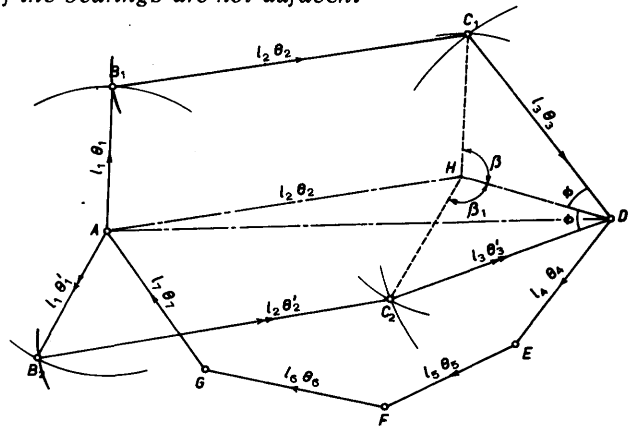


Fig. 6.9

Assume θ_1 and θ_3 are missing.

Graphical solution (Fig. 6.9)

Plot the part of the survey in which the lengths and bearings are known, giving the relative positions of A and D.

At A draw circle of radius $AB = l_1$; this gives the locus of station B.

At D draw circle of radius $DC = l_3$; this gives the locus of station C.

From A plot length and bearing $l_2\theta_2$ to give line AH.

At H draw arcs HC_1 and HC_2, radius l_1, cutting the locus of station C at C_1 and C_2. At C_1 and C_2 draw arcs of radius $BC = l_2$, cutting the other locus at B_1 and B_2.

Mathematical solution

Using the graphical solution:

Find the length and bearing AD. Solve triangle AHD to give HD. Solve triangle HC_1D to give ϕ and β and thence obtain the bearings of HC_1 = bearing AB_1, HC_2 = bearing AB_2, C_1D and C_2D.

N.B. There are two possible solutions in all cases (1), (2) and (3), and some knowledge of the shape or direction of the lines is required to give the required values.

Alternative solution

Let
$$l_1 \sin\theta_1 + l_3 \sin\theta_3 = P \qquad (1)$$
$$l_1 \cos\theta_1 + l_3 \cos\theta_3 = Q \qquad (2)$$
i.e.
$$l_1 \sin\theta_1 = P - l_3 \sin\theta_3 \qquad (3)$$
$$l_1 \cos\theta_1 = Q - l_3 \cos\theta_3 \qquad (4)$$

Squaring (3) and (4) and adding,
$$l_1^2 = P^2 + Q^2 + l_3^2 - 2l_3(P \sin\theta_3 + Q \cos\theta_3)$$

$$\frac{P}{\sqrt{(P^2+Q^2)}} \sin\theta_3 + \frac{Q}{\sqrt{(P^2+Q^2)}} \cos\theta_3 = \frac{P^2+Q^2+l_3^2-l_1^2}{2l_3\sqrt{(P^2+Q^2)}}$$

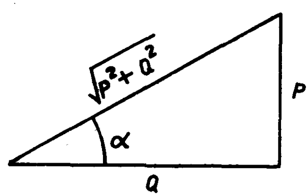


Fig. 6.10

Referring to Fig. 6.10

$$\sin\alpha \sin\theta_3 + \cos\alpha \cos\theta_3 = \frac{P^2+Q^2+l_3^2-l_1^2}{2l_3\sqrt{(P^2+Q^2)}} = k \qquad (6.8)$$

i.e.
$$\cos(\theta_3 - \alpha) = k$$
$$\theta_3 - \alpha = \cos^{-1} k$$
$$\alpha = \tan^{-1}\frac{P}{Q}$$
$$\therefore \quad \theta_3 = \cos^{-1} k + \tan^{-1}\frac{P}{Q} \qquad (6.9)$$

from (3),
$$\sin\theta_1 = \frac{P - l_3 \sin\theta_3}{l_1} \qquad (6.10)$$

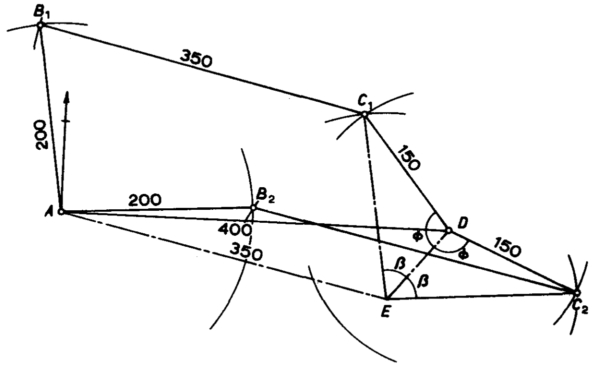
Example 6.2 The following data relate to a closed traverse $ABCD$ in which the bearings of the lines AB and CD are missing.

	Length	Bearing
AB	200	–
BC	350	102° 36′
CD	150	–
DA	400	270° 00′

Calculate the missing data.

Method (1) Fig. 6.11

Fig. 6.11



In triangle ADE, $AE = BC = 350$

$AD = 400$

By co-ordinates relative to A,

st. (D) 400 E 0 N

st. (E) $350 \sin 102° 36' = +350 \sin 77° 24' = +341{\cdot}57$

$350 \cos 102° 36' = -350 \cos 77° 24' = -\ 76{\cdot}34$

$$\tan \text{bearing } ED = \frac{400 - 341{\cdot}57}{0 + 76{\cdot}34} = \frac{58{\cdot}43}{76{\cdot}34}$$

$$\text{bearing } ED = \text{N}\,37° 26'\text{E} = 037° 26'$$

$$\text{length } ED = 76{\cdot}34 \sec 37° 26' = 96{\cdot}14$$

$$58{\cdot}43 \operatorname{cosec} 37° 26' = 96{\cdot}13 \quad (\textit{check})$$

In triangle EDC,

$$\tan \phi/2 = \sqrt{\left\{\frac{(s - ED)(s - DC)}{s(s - EC)}\right\}} \qquad \text{where } s = \frac{ED + DC + EC}{2}$$

$$= \sqrt{\left\{\frac{(126{\cdot}93)(73{\cdot}07)}{(223{\cdot}07)(23{\cdot}07)}\right\}}$$

check 126·93
73·07
23·07
s 223·07

$$\phi/2 = 53° 19'$$

$$\phi = 106° 38'$$

$$\sin\beta = \frac{DC \sin\phi}{EC}$$

$$= \frac{150 \sin 106^\circ 38'}{200}$$

$$\beta = 45^\circ 56'$$

$$\text{Bearing } AB_2 = 037^\circ 26' + 45^\circ 56' = \underline{083^\circ 22'}$$

$$\text{or } AB_1 = 037^\circ 26' - 45^\circ 56' = \underline{351^\circ 30'}$$

$$\text{Bearing } DC_1 = 217^\circ 26' + 106^\circ 38' = 324^\circ 04'$$

$$\text{or } DC_2 = 217^\circ 26' - 106^\circ 38' = 110^\circ 48'$$

$$\therefore \text{ Bearing } CD = \underline{144^\circ 04'} \text{ or } \underline{290^\circ 48'}$$

Method (2)

$$200 \sin\theta_1 + 350 \sin 102^\circ 36' + 150 \sin\theta_2 + 400 \sin 270^\circ = 0 \quad (1)$$

$$200 \cos\theta_1 + 350 \cos 102^\circ 36' + 150 \cos\theta_2 + 400 \cos 270^\circ = 0 \quad (2)$$

i.e.

$$200 \sin\theta_1 + 150 \sin\theta_2 = -341{\cdot}57 + 400 = 58{\cdot}43 \quad (3)$$

$$200 \cos\theta_1 + 150 \cos\theta_2 = 76{\cdot}34 + 0 = 76{\cdot}34 \quad (4)$$

$$\therefore \quad 200 \sin\theta_1 = 58{\cdot}43 - 150 \sin\theta_2 \quad (5)$$

$$200 \cos\theta_1 = 76{\cdot}34 - 150 \cos\theta_2 \quad (6)$$

Squaring and adding,

$$200^2 = 58{\cdot}43^2 + 76{\cdot}34^2 + 150^2 - 2 \times 150\,(58{\cdot}43 \sin\theta_2 + 76{\cdot}34 \cos\theta_2)$$

$$\therefore\ 58{\cdot}43 \sin\theta_2 + 76{\cdot}34 \cos\theta_2 = \frac{58{\cdot}43^2 + 76{\cdot}34^2 + 150^2 - 200^2}{300}$$

$$\cos(\theta_2 - \alpha) = \frac{58{\cdot}43^2 + 76{\cdot}34^2 + 150^2 - 200^2}{300\sqrt{(58{\cdot}43^2 + 76{\cdot}34^2)}}$$

$$= \frac{3414{\cdot}07 + 5827{\cdot}79 + 22\,500 - 40\,000}{300\sqrt{(3414{\cdot}07 + 5827{\cdot}79)}}$$

$$\theta_2 - \alpha = -73^\circ 22'$$

$$= 106^\circ 38' \text{ or } 253^\circ 22'$$

$$\text{but} \quad \tan\alpha = \frac{58{\cdot}43}{76{\cdot}34}$$

$$\alpha = 37^\circ 26'$$

$$\therefore \quad \underline{\theta_2 = 144^\circ 04' \text{ or } 290^\circ 48'}$$

from (5) $$\sin\theta_1 = \frac{58{\cdot}43 - 150\sin 144^\circ\, 04'}{200}$$

$$\underline{\theta_1 = 351^\circ\, 30'}$$

or $$\sin\theta_1 = \frac{58{\cdot}43 - 150\sin 290^\circ\, 48'}{200}$$

$$\underline{\theta_1 = 83^\circ\, 21'}$$

6.45 Where two lengths are missing

Let
$$l_1\sin\theta_1 + l_2\sin\theta_2 = P$$
$$l_1\cos\theta_1 + l_2\cos\theta_2 = Q$$

(a) The simultaneous equations may be solved to give values for l_1 and l_2 regardless of their position.

(b) If they are adjacent lines the solution of a triangle ADE will give the required values (Fig. 6.12), as length AD together with angles α and β are obtainable*

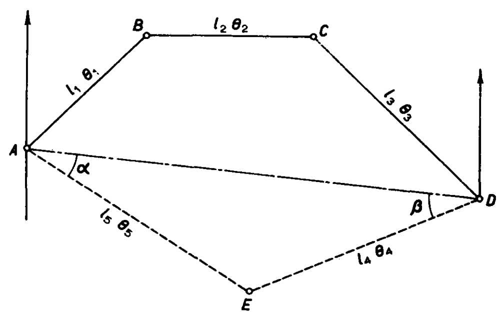


Fig. 6.12

(c) If $\theta_1 = \theta_2 = \theta$ (i.e. the lines are parallel)

$$(l_1 + l_2)\sin\theta = P$$
$$(l_1 + l_2)\cos\theta = Q$$

Squaring and adding,
$$(l_1 + l_2)^2 = P^2 + Q^2$$

Therefore this is not determinate.

(d) If $l_1 = l_2 = l$ and $\theta_1 = \theta_2 = \theta$

$$2l\sin\theta = P$$

$$l = \frac{P}{2\sin\theta} \tag{6.11}$$

*The lines can be adjusted so that the missing values are adjacent. Solution (b) can then be applied.

6.46 Where the length of one line and the bearing of another line are missing

Let
$$l_1 \sin\theta_1 + l_2 \sin\theta_2 = P$$
$$l_1 \cos\theta_1 + l_2 \cos\theta_2 = Q$$

where θ_1 and l_2 are missing.

Then, as before,
$$l_1 \sin\theta_1 = P - l_2 \sin\theta_2 \quad (1)$$
$$l_1 \cos\theta_1 = Q - l_2 \cos\theta_2 \quad (2)$$

Square and add,
$$l_1^2 = P^2 + Q^2 + l_2^2 - 2l_2(P \sin\theta_2 + Q \cos\theta_2)$$

this resolves into a quadratic equation in l_2.
$$l_2^2 - 2l_2(P \sin\theta_2 + Q \cos\theta_2) + P^2 + Q^2 - l_1^2 = 0 \qquad (6.12)$$

Then from the value of l_2, θ_1 may be obtained from equation (1).

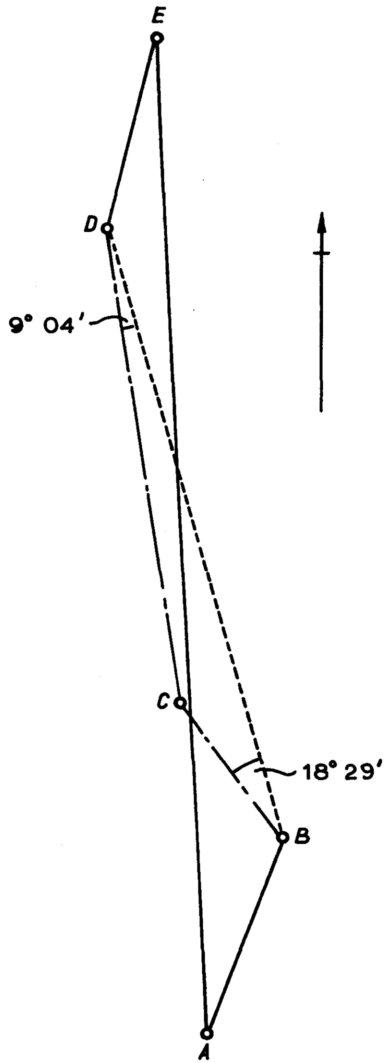
Example 6.3 Using the data of a closed traverse given below, calculate the lengths of the lines BC and CD.

Line	Length (ft)	W.C.B.	Reduced bearing	Latitude	Departure
AB	344	014° 31′	N 14° 31′ E	+333·0	+ 86·2
BC		319° 42′	N 40° 18′ W		
CD		347° 15′	N 12° 45′ W		
DE	300	005° 16′	N 05° 16′ E	+298·8	+ 27·6
EA	1958	168° 12′	S 11° 48′ E	−1916·4	+400·4

(I.C.E.)

Assuming the co-ordinates of A to be + 1000 E, + 1000 N, from the given co-ordinates:

E_A	+ 1000·0	N_A	+ 1000·0
ΔE_{AE}	− 400·4	ΔN_{AE}	+ 1916·4
E_E	+ 599·6	N_E	2916·4
ΔE_{ED}	− 27·6	ΔN_{ED}	− 298·8
E_D	+ 572·0	N_D	2617·6
E_A	+ 1000·0	N_A	+ 1000·0
ΔE_{AB}	+ 86·2	ΔN_{AB}	+ 333·0
E_B	+ 1086·2	N_B	+ 1333·0
ΔE_{BD}	− 514·2	ΔN_{BD}	+ 1284·6.



Bearing $BD = \tan^{-1} - 514{\cdot}2/+1284{\cdot}6$

$= \underline{\text{N } 21° 49' \text{ W}} = \underline{338° 11'}$

$$\begin{aligned} BC &= 319° 42' \\ \text{Angle } CBD &= 18° 29' \quad (\hat{B}) \\ DB &= 158° 11' \\ DC &= 167° 15' \\ \text{Angle } BDC &= 9° 04' \quad (\hat{D}) \\ (B+D) &= 27° 33' \end{aligned}$$

Length $BD = 1284{\cdot}6/\cos 21° 49'$

$= \underline{1383{\cdot}7}$

In triangle BCD,

$$DC = \frac{DB \sin B}{\sin(B+D)} = \frac{1383{\cdot}7 \sin 18° 29'}{\sin 27° 33'}$$

$$= \underline{948{\cdot}4 \text{ ft}}$$

$$BC = \frac{DB \sin D}{\sin(B+D)} = \frac{1383{\cdot}7 \sin 9° 04'}{\sin 27° 33'}$$

$$= \underline{471{\cdot}4 \text{ ft}}$$

Fig. 6.13

Exercises 6(b) (Omitted values)

8. A clockwise traverse $ABCDEA$ was surveyed with the following results:

AB 331·4 ft	$E\hat{A}B$ 128° 10′ 20″	$B\hat{C}D$ 84° 18′ 10″
BC 460·1 ft		
CD 325·7 ft	$A\hat{B}C$ 102° 04′ 30″	$C\hat{D}E$ 121° 30′ 30″

The angle DEA and the sides DE and EA could not be measured direct. Assuming no error in the survey, find the missing lengths and their bearings if AB is due north.

(L.U. Ans. $EA = 223{\cdot}1$ ft, $DE = 293{\cdot}7$ ft, $308^\circ 10' 20''$, $232^\circ 06' 50''$)

9. An open traverse was run from A to E in order to obtain the length and bearing of the line AE which could not be measured direct, with the following results:

Line	AB	BC	CD	DE
Length	1025	1087	925	1250
W.C.B.	$261^\circ 41'$	$9^\circ 06'$	$282^\circ 22'$	$71^\circ 30'$

Find by calculation the required information.

(L.U. Ans. 1901; $342^\circ 51'$)

10. The following measurements were obtained when surveying a closed traverse $ABCDEA$:

Line	EA	AB	BC	
Length (ft)	$793{\cdot}7$	$1512{\cdot}1$	$863{\cdot}7$	
	DEA	EAB	ABC	BCD
Included angles	$93^\circ 14'$	$122^\circ 36'$	$131^\circ 42'$	$95^\circ 43'$

It is not possible to occupy D, but it could be observed from both C and E.

Calculate the angle CDE, and the lengths CD and DE, taking DE as the datum, and assuming all observations to be correct.

(L.U. Ans. $96^\circ 45'$; $1848{\cdot}0$ ft, $1501{\cdot}6$ ft)

11. In a traverse $ABCDEFG$, the line BA is taken as the reference meridian. The latitudes and departures of the sides AB, BC, CD, DE and EF are:

Line	AB	BC	CD	DE	EF
Latitude	$-1190{\cdot}0$	$-565{\cdot}3$	$+590{\cdot}5$	$+606{\cdot}9$	$+1097{\cdot}2$
Departure	0	$+736{\cdot}4$	$+796{\cdot}8$	$-468{\cdot}0$	$+370{\cdot}4$

If the bearing of FG is N $75^\circ 47'$ W and its length is $896{\cdot}0$ ft, find the length and bearing of GA.

(L.U. Ans. $947{\cdot}6$ ft; S $36^\circ 45'$ W)

6.5 The Adjustment of Closed Traverses

(a) Traverses connecting two known points.
(b) Traverses which return to their starting point.

6.51 Where the start and finish of a traverse are fixed

The length and bearing of the line joining these points are known and must be in agreement with the length and bearing of the closing line of the traverse.

Where the traverse is not orientated to the fixed line an angle of swing (α) has to be applied.

Where there is discrepancy between the closing lengths, a scale factor k must be applied to all the traverse lengths.

$$k = \frac{\text{length between the fixed points}}{\text{closing length of traverse}}$$

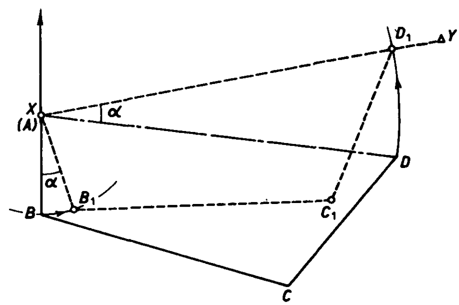


Fig. 6.14

In Fig. 6.14, the traverse is turned through angle α so that traverse $ABCD$ becomes AB_1C_1D and AD_1 is orientated on to line XY. The scale factor $k = \dfrac{XY}{AD}$ must be applied to the traverse lines.

Traverses are often orientated originally on their first line. Co-ordinates are then computed, and from these the length and bearing of the closing line (AD). The latter is then compared with the length and bearing XY.

Co-ordinates of the traverse can now be adjusted by either

(1) recomputing the traverse by adjusting the bearings by the angle α and the length by multiplying by k, or

(2) transposing the co-ordinates by changing the grid (Eqs. 3.33/3.34) and also applying the scale factor k, or

(3) applying one of the following adjustment methods, e.g. Bowditch.

N.B. The factor k can be a compounded value involving:

(a) traverse error,

(b) local scale factor – (ground distance to national grid),

(c) change of units, e.g. feet to metres.

Example 6.4. A traverse XaY is made between two survey stations X (E 1000 N 1000) and Y (E 1424·5 N 754·9).

Based upon an assumed meridian, the following partial co-ordinates are computed:

	ΔE	ΔN
Xa	69·5	− 393·9
aY	199·3	− 17·4

Adjust the traverse so that the co-ordinates conform to the fixed stations X and Y.

Ans.

	E	N
X	1000·0	1000·0
Y	1424·5	754·9
	ΔE + 424·5	ΔN − 245·1

$$\text{Bearing of control line } XY = \tan^{-1} 424{\cdot}5/-245{\cdot}1$$
$$= \underline{S\ 60^\circ\ 00'E} \quad \text{i.e. } 120^\circ\ 00'$$
$$\text{Length of control line } XY = 424{\cdot}5/\sin 60^\circ$$
$$= \underline{490{\cdot}2.}$$

From the partial co-ordinates,

$$\Delta E_{XY} = 69{\cdot}5 + 199{\cdot}3 = +268{\cdot}8$$
$$\Delta N_{XY} = -393{\cdot}9 - 17{\cdot}4 = -411{\cdot}3$$

$$\text{Bearing of traverse line } XY = \tan^{-1} +268{\cdot}8/-411{\cdot}3$$
$$= \underline{S\ 33^\circ\ 10'E} \quad \text{i.e. } 146^\circ\ 50'$$
$$\text{Length of traverse line } XY = 411{\cdot}3/\cos 33^\circ\ 10'$$
$$= \underline{491{\cdot}4.}$$

$$\text{Angle of swing } \alpha = \text{traverse bearing } XY - \text{fixed bearing } XY$$
$$= 146^\circ\ 50' - 120^\circ\ 00'$$
$$= \underline{+26^\circ\ 50'}$$

$$\text{Scale factor } k = \text{fixed length/traverse length}$$
$$= 490{\cdot}2/491{\cdot}4$$
$$= \underline{0{\cdot}997\,56}$$

Using Eqs. (3.33) and (3.34),

$$\Delta E' = +m\Delta E - n\Delta N$$
$$\Delta N' = m\Delta N + n\Delta E$$
$$m = k\cos\alpha = 0{\cdot}997\,56 \cos 26^\circ\ 50' = 0{\cdot}890\,14$$
$$n = k\sin\alpha = 0{\cdot}997\,56 \sin 26^\circ\ 50' = 0{\cdot}450\,30$$

			−			+		
Line	ΔE	ΔN	$m\Delta$E	$n\Delta$N	ΔE′	$m\Delta$N	$n\Delta$E	ΔN′
Xa	+ 69·5	−393·9	+ 61·86	−177·37	+239·23	−350·63	+31·30	−319·33
aY	+199·3	− 17·4	+177·40	− 7·84	+185·24	− 15·48	+89·74	+ 74·26
					+424·47			−245·07

Total co-ordinates	E	N
X	1000·0	1000·0
a	1239·2	680·7
Y	1424·5	754·9

Example 6.5 If in the previous example the co-ordinates of Y are E 1266·9 N 589·1,

$$\begin{aligned} \text{then the bearing of } XY &= \tan^{-1} + 266{\cdot}9/-410{\cdot}9 \\ &= \text{S } 33^\circ\ 00'\text{E} \quad \text{i.e. } 147^\circ\ 00' \\ \text{length } XY &= 410{\cdot}9/\cos 33^\circ \\ &= 490{\cdot}0 \\ \text{angle of swing } \alpha &= 146^\circ\ 50' - 147^\circ\ 00' \\ &= -\ 0^\circ\ 10' \\ \alpha_{\text{rad}} &= -\ 0{\cdot}002\,91 \\ \text{Scale factor } k &= 490{\cdot}0/491{\cdot}4 \\ &= 0{\cdot}997\,16 \end{aligned}$$

Using Eqs. (3.35) and (3.36),

$$\Delta E' = k[\Delta E - \Delta N\alpha]$$
$$\Delta N' = k[\Delta N + \Delta E\alpha]$$

Line	ΔE	ΔN	$\Delta N\alpha$	$\Delta E\alpha$	$\Delta E - \Delta N\alpha$	$\Delta N + \Delta E\alpha$	ΔE′	ΔN′
Xa	+ 69·5	−393·9	+1·15	−0·20	+ 68·35	−394·10	+ 68·16	−392·98
aY	+199·3	− 17·4	+0·05	−0·58	+199·25	− 17·98	+198·68	− 17·93
						Σ	+266·84	−410·91

Example 6.6 An underground traverse between two wires in shafts A and D based on an assumed meridian gives the following partial co-ordinates:

	ΔE (ft)	ΔN (ft)
AB	0	− 263·516
BC	+ 523·684	+ 21·743
CD	+ 36·862	+ 421·827

If the grid co-ordinates of the wires are:

	E (metres)	N (metres)
A	552 361·63	441 372·48
D	552 532·50	441 428·18

Transform the underground partials into grid co-ordinates.

Ans.

Bearing of traverse line AD = $\tan^{-1} + 560{\cdot}546/+180{\cdot}054$ = N 72°11′33″E

Length of traverse line AD = $560{\cdot}546/\sin 72°\,11'33''$ = 588·755 ft

Bearing of grid line AD = $\tan^{-1} 170{\cdot}87/55{\cdot}70$ = N 71° 56′ 45″E

Length of grid line AD = $170{\cdot}87/\sin 71°\,56'45''$ = 179·713 m

Angle of swing α = 72° 11′33″ − 71° 56′45″ = + 0° 14′48″

α_{rad} = 0·004 31.

Scale factor k = 179·713/588·755 = 0·305 23

Using Eqs. (3.35) and (3.36),

$$\Delta E' = k[\Delta E - \Delta N\alpha]$$
$$\Delta N' = k[\Delta N + \Delta E\alpha]$$

Line	ΔE	ΔN	ΔNα	ΔEα	ΔE − ΔNα	ΔN + ΔEα	ΔE′	ΔN′
AB	0·0	− 263·516	− 1·136	0	+ 1·136	− 263·516	+ 0·35	− 80·43
BC	+ 523·684	+ 21·743	+ 0·094	+ 2·257	+ 523·590	+ 24·000	+ 159·82	+ 7·33
CD	+ 36·862	+ 421·827	+ 1·818	+ 0·159	+ 35·044	+ 421·986	+ 10·70	+ 128·80
						Σ	+ 170·87	+ 55·70

	E	N
A	552 361·63 m	441 372·48 m
B	552 361·98 m	441 292·05 m
C	552 521·80 m	441 299·38 m
D	552 532·50 m	441 428·18 m

Example 6.7 Fig. 6.15 shows a short 'dial' traverse connecting two theodolite lines in a mine survey. The co-ordinates of T.M.64 are E 45603·1 ft N 35 709·9 ft and of T.M.86 E 46 163·6 ft N 35 411·8 ft. Calculate the co-ordinates of the traverse and adjust to close on T.M.86.

(N.R.C.T.)

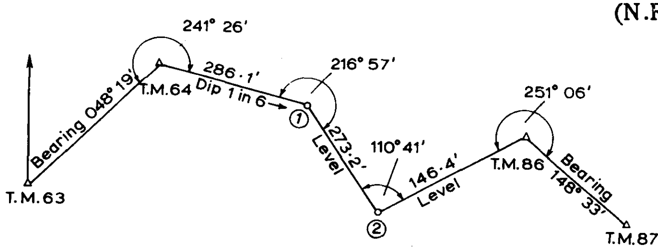


Fig. 6.15

This is a subsidiary survey carried out with a 1 minute instrument (miners' dial) and the method of adjustment should be as simple as possible.

Bearing T.M. 63 – T.M. 64	048° 19′				
+ angle	241° 26′				
	289° 45′		Adjusted Bearings		
	– 180°				
Bearing T.M. 64 – (1)	109° 45′	+ 01′	109° 46′	S 70° 14′ E	
+ angle	216° 57′				
	326° 42′				
	– 180°				
Bearing 1 – 2	146° 42′	+ 02′	146° 44′	S 33° 16′ E	
+ angle	110° 41′				
	257° 23′				
	– 180°				
Bearing 2 – T.M. 86	077° 23′	+ 03′	077° 26′	N 77° 26′ E	
+ angle	251° 06′				
	328° 29′				
	– 180°				
Bearing T.M. 86 – T.M. 87	148° 29′	+ 04′	148° 33′	S 31° 27′ E	
Fixed bearing T.M.86–T.M.87	148° 33′				
∴ Error	04′				

Horizontal length T.M. 64 – 1 (1 in 6 = 9° 28′) = 286·1 cos 9° 28′ = 282·2 ft

Co-ordinates

Line	Length	Bearing	ΔE	δE	ΔE′	ΔN	δN	ΔN′
64 – 1	282·2	S 70° 14′ E	+ 265·6	– 0·5	+ 265·1	– 95·4	– 0·5	– 95·9
1 – 2	273·2	S 33° 16′ E	+ 153·2	– 0·5	+ 152·7	– 233·5	– 0·4	– 233·9
2 – 86	146·4	N 77° 26′ E	+ 142·9	– 0·2	+ 142·7	+ 31·9	– 0·2	+ 31·7
			+ 561·7	– 1·2		+ 31·9	– 1·1	
						– 328·9		
						– 297·0		

From the theodolite station co-ordinates,

	E	N
T.M. 64	45 603·1	35 709·9
T.M. 86	46 163·6	35 411·8
	ΔE + 560·5	ΔN – 298·1

∴ Error in traverse = E + 1·2, N + 1·1

Applying Bowditch's method (see p. 330),

$$\delta E = \frac{1{\cdot}2 \times \text{length}}{\Sigma \text{ length}} = \frac{1{\cdot}2 \times l}{701{\cdot}8} = 1{\cdot}71 \times 10^{-3} \times l.$$

$$\delta N = \frac{1{\cdot}1 \times l}{\Sigma l} = \frac{1{\cdot}1 \times l}{701{\cdot}8} = 1{\cdot}57 \times 10^{-3} \times l.$$

Total co-ordinates (adjusted)

	E	N
T.M.64	45603·1	35709·9
1	45868·2	35614·0
2	46020·9	35380·1
86	46163·6	35411·8

6.52 Traverses which return to their starting point

The closing error may be expressed as either (a) the length and bearing of the closing line or (b) the errors in latitude and departure.

To make the traverse consistent, the error must be distributed and this can be done by adjusting either

(a) the lengths only, without altering the bearings or

(b) the length and bearing of each line by adjusting the co-ordinates.

6.53 Adjusting the lengths without altering the bearings

Where all the angles in a closed traverse have been measured, the closing angular error may be distributed either (a) equally or (b) by weight inversely proportional to the square of the probable error.

It may therefore be assumed that the most probable values for the bearings have been obtained and that any subsequent error relates to the lengths, i.e. a similar figure should be obtained.

Three methods are proposed:

(1) Scale factor axis method.

(2) *xy* method (Ormsby method).

(3) Crandal's method.

In the following description of these methods, to simplify the solution, the normal co-ordinate notation will be altered as follows: partial departure ΔE becomes d with error in departure δd. The sum of the errors in departure $\Sigma \delta d$ becomes Δd. Similarly, partial latitude ΔN becomes l with error in latitude δl. The sum of the errors in latitude $\Sigma \delta l$ becomes Δl.

(1) *Scale factor axis method* (after R.E. Middleton and O. Chadwick). This follows the principles proposed for traverses closed on fixed points.

Graphical solution (Fig. 6.16)

The traverse is plotted and the closing error obtained.

It is intended that this error produced should divide the figure into two approximately equal parts. To decide on the position of this line, the closing error bearing is drawn through each station until the above condition is obtained.

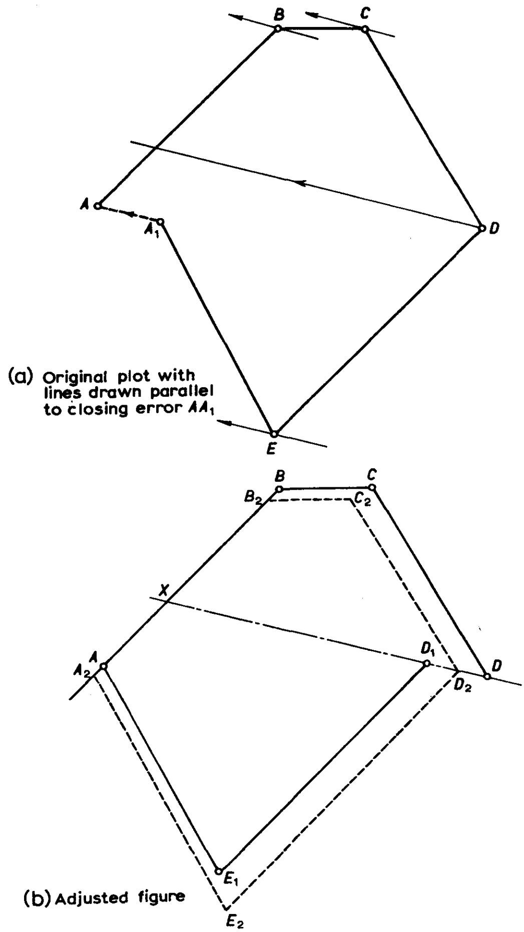


Fig. 6.16 Scale factor axis method

The lines above XD need to be reduced by a scale factor so as to finish at D_2 midway between D and D_1. The lines below XD need to be enlarged by a scale factor so as to finish at D_2.

Example 6.8 (Fig. 6.17)

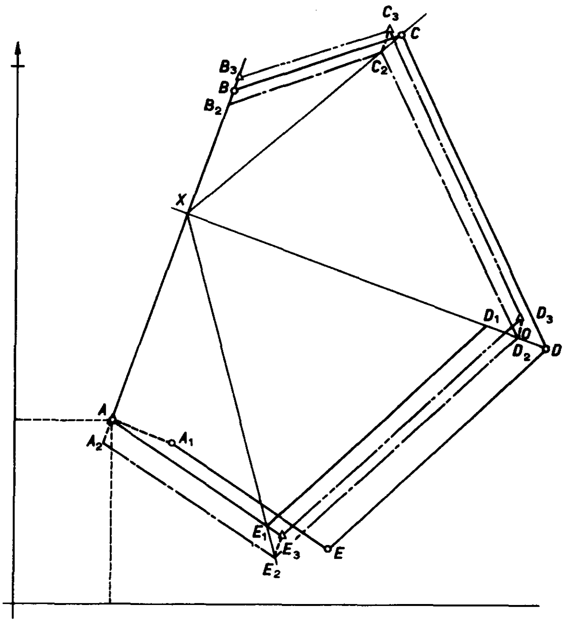


Fig. 6.17 Graphical adjustment

(1) The traverse is plotted as $ABCDEA_1$ – closing error AA_1.

(2) The closing error is transferred to station D, so that the traverse now plots as D_1E_1ABCD. D_1D_2 is produced to cut the traverse into two parts at X on line AB.

(3) From X, rays are drawn through B, C, D, E_1 and A.

(4) From O, midway between D_1 and D, lines are drawn parallel to D_1E_1, giving OE_2, and parallel to DC, giving OC_2. Lines parallel to BC and EA_1 through C_2 and E_2 give B_2 and A_2.

The figure $A_2B_2C_2OE_2A_2$ is the adjusted shape with the bearings of the lines unaltered, i.e.

Lines XB, BC and CD are reduced in length in the ratio $\dfrac{OX}{OD}$

Lines XA, AE and ED_1 are enlarged in length in the ratio $\dfrac{OX}{OD_1}$

(5) If the traverse is to be plotted relative to the original station A, then all the new stations will require adjustment in length and bearing A_2A, giving B_3, C_3, D_3, E_3.

The final figure is $AB_3C_3D_3E_3A$.

The graphical solution can now be expressed mathematically as follows (after R.E. Middleton and O. Chadwick).

In the figure $XBCD$, Fig. 6.18, the lengths need to be reduced by a scale factor

$$k_1 = \frac{XD_2}{XD}.$$

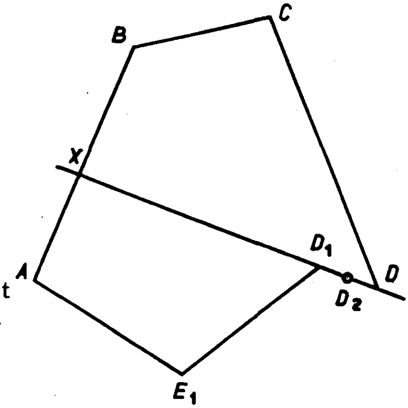


Fig. 6.18

As the co-ordinates are dependent on their respective lengths, the partial co-ordinates can be multiplied by the factor k_1.

Let the partial departure be d and the partial latitude l, the error in d be δd and the error in l be δl. Expressing this as a correction,

$$\delta d = d - dk_1 = d(1 - k_1)$$

$$= d\frac{XD - XD_2}{XD}$$

$$= d\frac{D_2 D}{XD}$$

$$= \frac{\text{partial departure} \times \tfrac{1}{2}\text{error}}{\text{length of axis of error}} \quad (6.13)$$

Thus for all lines above the line XD,

$$\delta d = \lambda d \quad (6.14)$$

where $$\lambda = \frac{\text{traverse error } (D_1 D)}{2 \times \text{axis of error } (XD)}$$

Also $$\delta l = \lambda l \quad (6.15)$$

Similarly in figure XD_1E_1A the lengths need to be enlarged by a scale factor

$$k_2 = \frac{XD_2}{XD_1}$$

$$\therefore \quad \delta d = d(k_2 - 1)$$

$$= d\left(\frac{XD_2 - XD_1}{XD_1}\right)$$

$$= d.\frac{D_2 D_1}{XD_1}$$

Thus similarly for all lines below line XD.

$$\delta d = \mu d \tag{6.16}$$

where $$\mu = \frac{\text{traverse error } (DD_1)}{2 \times \text{axis of error } (XD_1)}.$$

Also $$\delta l = \mu l \tag{6.17}$$

By comparison with the traverse lines DD_1 is small and thus

$$XD_1 \simeq XD \simeq XD_2$$

$$\therefore \quad \lambda \simeq \mu$$

Summary

$$\delta d = \lambda d$$
$$\delta l = \lambda l$$

The sign of the correction depends on the position of the line, i.e. whether the line needs to be reduced or enlarged.

N.B. A special case needs to be dealt with, viz. the line AB intersected by line DD_1 produced to X.

XB must be reduced

AX must be enlarged.

$$\delta d_{XB} = -\lambda d_{AB} \times \frac{XB}{AB}$$

$$\delta d_{AX} = +\mu d_{AB} \times \frac{AX}{AB}$$

$$\therefore \quad \delta d_{AB} = \mu d_{AB}\left[\frac{AX - XB}{AB}\right] \tag{6.18}$$

Also $$\delta l_{AB} = \mu l_{AB}\left[\frac{AX - XB}{AB}\right] \tag{6.19}$$

(2) *xy (Ormsby) method* (Fig. 6.19)

If any line is varied in length by a fractional value then the partial co-ordinates will be varied in the same proportion without altering the bearing, i.e.

$$\frac{\delta s}{s} = \frac{\delta d}{d} = \frac{\delta l}{l}$$

Let (a) the lines in the NE/SW quadrants be altered by a factor x and the lines in the NW/SE quadrants by a factor y,

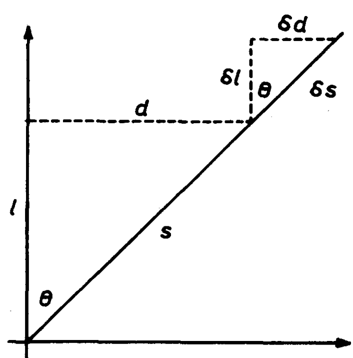


Fig. 6.19

(b) the sign of all the terms in summation of the partial co-ordinates in one of the equations (say Eq. 6.20) be the same as the sign of the greater closing error,

(c) the sign in the other equation (say Eq. 6.21) be made consistent with the figure, to bring the corrections back to the same bearing (Fig. 6.20).

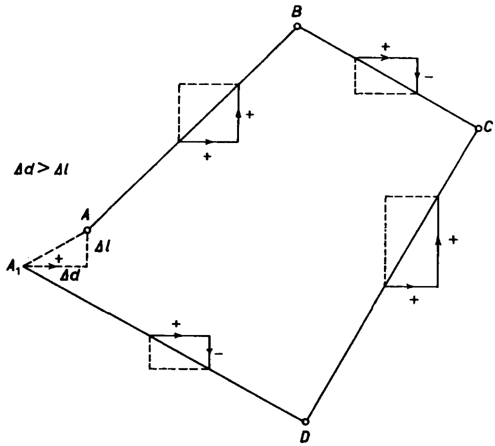


Fig. 6.20

$$+\Delta d = xd_1 + yd_2 + xd_3 + yd_4.$$

i.e. $$+\Delta d = x(d_1 + d_3) + y(d_2 + d_4) \qquad (6.20)$$

$$+\Delta l = xl_1 - yl_2 + xl_3 - yl_4.$$

i.e. $$+\Delta l = x(l_1 + l_3) - y(l_2 + l_4) \qquad (6.21)$$

where the partial co-ordinates are $d_1\ l_1$; $d_2\ l_2$, etc.

The solution of these simultaneous equations gives values for x and y which are then applied to each value in turn to give the corrected values of the partial co-ordinates.

(3) *Crandal's method* by applying the principle of least squares, Fig. 6.21.

Let the length of each side be varied by a fraction x of its lengths, then if l and d be the partial latitude and departure of the line AB of bearing θ, they also will be varied by xl and xd respectively.

If the probable error in length is assumed to be proportional to $\sqrt{s}$, then the weight to be applied to each line will be $1/s$, i.e.

$$\text{wt} \propto \frac{1}{(\text{probable error})^2}$$

$$\text{P.E.} \propto \sqrt{s}$$

$$\therefore \quad \text{wt} \propto \frac{1}{s}$$

(The value of x is thus dependent on the length of the line.)

By the theory of 'Least Squares', the sum of the weighted residual errors should be a minimum.

i.e. $\Sigma \dfrac{x^2 s^2}{s} = \Sigma x^2 s = \text{minimum.}$

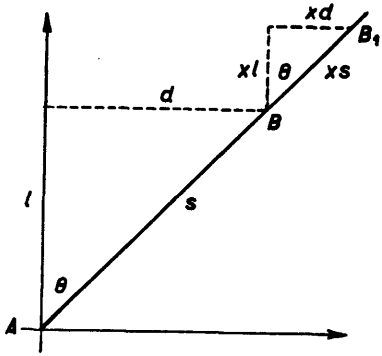


Fig. 6.21

Let $\Sigma xl = \Sigma \delta l = \Delta l$, total error in latitude.

$\Sigma xd = \Sigma \delta d = \Delta d$, total error in departure.

Then differentiating these equations and equating them to zero will give the minima values, i.e.

$$xs\,\delta x = 0 \qquad (1)$$
$$l\,\delta x = 0 \qquad (2)$$
$$d\,\delta x = 0 \qquad (3)$$

The differentiated equations (2) and (3), being conditional equations, should now be multiplied by factors (correlatives) $-k_1$ and $-k_2$ (Reference: Rainsford *Survey Adjustments and Least Squares*).

Adding all three equations together and equating the coefficients of each δx to zero, we have

$$\delta x(x_1 s_1 - k_1 l_1 - k_2 d_1) = 0$$
$$\delta x(x_2 s_2 - k_1 l_2 - k_2 d_2) = 0 \quad \text{etc.}$$

Thus

$$x_1 = \frac{k_1 l_1 + k_2 d_1}{s_1}$$

$$x_2 = \frac{k_1 l_2 + k_2 d_2}{s_2} \qquad \text{etc.}$$

Substituting the values of x into the original equations (2) and (3), we have

$$l_1\left(\frac{k_1 l_1 + k_2 d_1}{s_1}\right) + l_2\left(\frac{k_1 l_2 + k_2 d_2}{s_2}\right) + \ldots = \Delta l$$

i.e. $$k_1 \Sigma \frac{l^2}{s} + k_2 \Sigma \frac{ld}{s} = \Delta l \tag{6.22}$$

Also $$d_1 \left(\frac{k_1 l_1 + k_2 d_1}{s_1}\right) + d_2 \left(\frac{k_1 l_2 + k_2 d_2}{s_2}\right) + \dots = \Delta d$$

i.e. $$k_1 \Sigma \frac{ld}{s} + k_2 \Sigma \frac{d^2}{s} = \Delta d \tag{6.23}$$

Solving Eqs. (6.22) and (6.23), we obtain values for k_1 and k_2. The corrections to the partial co-ordinates then become

$$\delta l_1 = k_1 \frac{l_1^2}{s_1} + k_2 \frac{l_1 d_1}{s_1} \quad \text{etc.} \tag{6.24}$$

$$\delta d_1 = k_1 \frac{l_1 d_1}{s} + k_2 \frac{d_1^2}{s} \quad \text{etc} \tag{6.25}$$

It should be noted that a check on the equations is given by

$$\frac{k_1 \dfrac{l_1^2}{s_1} + k_2 \dfrac{l_1 d_1}{s_1}}{k_1 \dfrac{l_1 d_1}{s} + k_2 \dfrac{d_1^2}{s}} = \frac{l_1 [k_1 l_1 + k_2 d_1]}{d_1 [k_1 l_1 + k_2 d_1]} = \frac{l_1}{d_1}$$

i.e. no change in bearing.

The assumption that the probable error in length is proportional to $\sqrt{s}$ applies to compensating errors. It has been shown that, where the accuracy of the linear measurement decreases, the probable error in length becomes proportional to the length itself, i.e.

$$\text{P.E.} \propto s$$

i.e. $$\text{wt} \propto \frac{1}{s^2}$$

The effect of this on the foregoing equations is to remove the factor s from them, i.e.

$$\delta l_1 = k_1 l_1^2 + k_2 l_1 d_1 \quad \text{etc} \tag{6.26}$$

$$\delta d_1 = k_1 l_1 d_1 + k_2 d_1^2 \quad \text{etc} \tag{6.27}$$

6.54 Adjustment to the length and bearing

Three methods are compared:

(1) Bowditch, (2) Transit (Wilson's method), (3) Smirnoff.

(1) *The Bowditch method* (Fig. 6.22). This method is more widely used than any other because of its simplicity. It was originally devised for the adjustment of compass traverses.

Bowditch assumed that (a) the linear errors were compensating and thus the probable error (P.E.) was proportional to the square root of the distance s, i.e.

$$\text{P.E.} \propto \sqrt{s}$$

and (b) the angular error $\delta\theta$ in the θ would produce an equal displacement B_1B_2 at right angles to the line AB.

A resultant AB_2 is thus developed with the total probable error.

$$BB_2 = \sqrt{B_1B^2 + B_1B_2^2}$$
$$= \sqrt{2}B_1B$$
$$(B_1B = B_1B_2)$$

also
$$= \sqrt{\delta l^2 + \delta d^2}$$

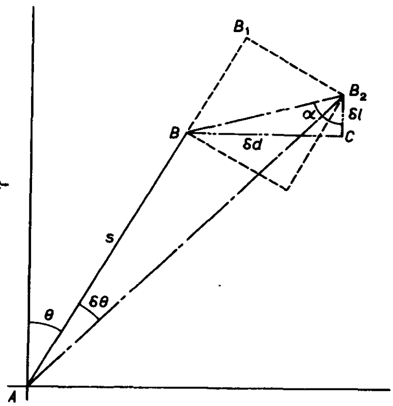


Fig. 6.22

where δl and δd are the corrections to the partial co-ordinates.

The weight, as before, becomes $1/s$.

By the theory of least squares

$$\Sigma\left(\frac{\delta l^2 + \delta d^2}{s}\right) = \text{a minimum}$$

The conditional equations are:

$$\Sigma\,\delta l = \Delta l \qquad (1)$$
$$\Sigma\,\delta d = \Delta d \qquad (2)$$

As in the previous method, using correlatives, differentiation of each equation gives:

$$\Sigma\frac{\delta l\;\delta(\delta l)}{s} + \frac{\delta d\;\delta(\delta d)}{s} = 0$$

$$\Sigma\,\delta(\delta l) = 0$$
$$\Sigma\,\delta(\delta d) = 0$$

Multiplying the last two equations by the correlatives $-k_1$ and $-k_2$ respectively, adding the equations and equating the coefficients of $\delta(\delta l)$ and $\delta(\delta d)$ to zero we have:

$$\delta(\delta l)\left[\frac{\delta l}{s} - k_1\right] = 0 \quad \text{i.e.} \quad \delta l_1 = s_1k_1 \quad dl_2 = s_2k_1 \quad \text{etc.}$$

$$\delta(\delta d)\left[\frac{\delta d}{s} - k_2\right] = 0 \quad \text{i.e.} \quad \delta d_1 = s_1k_2 \quad \delta d_2 = s_2k_2 \quad \text{etc.}$$

Substituting the values into equations (1) and (2),

$$k_1 \Sigma s = \Delta l \quad \text{i.e.} \quad k_1 = \frac{\Delta l}{\Sigma s}$$

$$k_2 \Sigma s = \Delta d \quad \text{i.e.} \quad k_2 = \frac{\Delta d}{\Sigma s}$$

$\therefore$ The corrections to the latitudes $\delta l_1 = s_1 \dfrac{\Delta l}{\Sigma s}$

$$sl_2 = s_2 \frac{\Delta d}{\Sigma s} \quad \text{etc.}$$

to the departures $\delta d_1 = s_1 \dfrac{\Delta d}{\Sigma s}$

$$\delta d_2 = s_2 \frac{\Delta d}{\Sigma s} \quad \text{etc.}$$

i.e. Correction to the partial co-ordinate = total correction

$$\times \frac{\text{length of corresponding side}}{\text{total length of traverse}} \qquad (6.28)$$

The effect at a station is that the resultant BB_2 will be equal to the closing error $\times\ s/\Sigma s$ and parallel to the bearing of the closing error,

i.e. through an angle $\alpha = \tan^{-1} \dfrac{\Delta d}{\Delta l}$

$$= \tan^{-1} \frac{\delta d}{\delta l}$$

The total movement of each station is therefore parallel to the closing error and equal to

$$\frac{\Sigma\,(\text{lengths up to that point})}{\text{total length of traverse}} \times \text{closing error.}$$

The correction can thus be applied either graphically in the manner originally intended or mathematically to the co-ordinates.

Jameson points out that the bearings of all the lines are altered unless they lie in the direction of the closing error and that the maximum alteration in the bearing occurs when the line is at right angles to the closing bearing, when it becomes

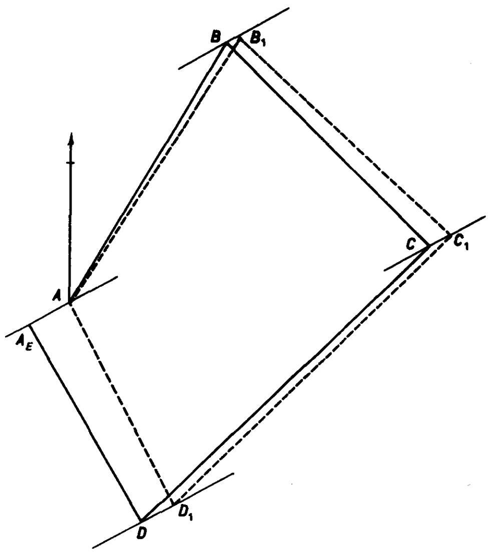
$$\delta\theta_{rad} = \frac{\dfrac{S_n}{\Sigma s} \times \text{closing error}}{S_n} = \frac{\text{closing error}}{\Sigma s}$$

The closing error expressed as a fraction of the length of the traverse may vary from 1/1000 to 1/10 000, so taking the maximum error as 1/1000

$$\delta\theta'' = \frac{206\,265}{1000} = 206''$$
$$= 03'26''$$

a value far in excess of any theodolite station error. A change of bearing of 20″ represents 1/10 000 and this would be excessive even using a 20″ theodolite.

Graphical Solution by the Bowditch Method (Fig. 6.23)



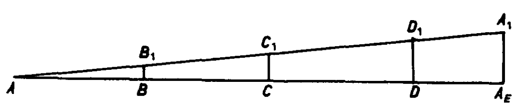


Fig. 6.23

(1) Plot the survey and obtain the closing error AA_E.

(2) Draw a line representing the length of each line of the traverse to any convenient scale.

(3) At A_E draw a perpendicular A_EA_1, equal to the closing error and to the same scale as the plan.

(4) Join AA_1, forming a triangle AA_1A_E, and then through B, C and D similarly draw perpendiculars to cut the line AA_1 at B_1, C_1 and D_1.

(5) Draw a line through each station parallel to the closing error

and plot lines equal to BB_1, CC_1 and DD_1, giving the new figure $AB_1C_1D_1A$.

(2) *The Transit or Wilson method.* This is an empirical method which can only be justified on the basis that (a) it is simple to operate, (b) it has generally less effect on the bearings than the Bowditch method.

It can be stated as:
the correction to the partial co-ordinate

$$= \text{the partial co-ordinate} \times \frac{\text{closing error in the co-ordinate}}{\Sigma\,\text{partial co-ordinates}}$$
(ignoring the signs) (6.29)

$$\text{i.e.} \quad \delta l_1 = l_1 \frac{\Delta l}{\Sigma l} \tag{6.30}$$

$$\delta d_1 = d_1 \frac{\Delta d}{\Sigma d} \tag{6.31}$$

(3) *The Smirnoff method.* The partial latitude l of a line length s and bearing θ is given by

$$l = s\cos\theta.$$

If the two variables s and $\cos\theta$ are subjected to errors of δs and $\delta(\cos\theta)$ respectively, then

$$l + \delta l = (s + \delta s)[\cos\theta + \delta(\cos\theta)]$$

Subtracting the value of l from each side and neglecting the small value $\delta s\,\delta(\cos\theta)$, gives

$$\delta l = \delta s\cos\theta + s\,\delta(\cos\theta)$$

Dividing both sides by l,

$$\frac{\delta l}{l} = \frac{\delta s\cos\theta}{s\cos\theta} + \frac{s\,\delta(\cos\theta)}{s\cos\theta}$$

$$\text{i.e.} \quad \underline{\frac{\delta l}{l} = \frac{\delta s}{s} + \frac{\delta(\cos\theta)}{\cos\theta}} \tag{6.32}$$

i.e. the relative accuracy in latitude

= the relative accuracy in distance +
the relative accuracy in the cosine of the bearing

Thus, in a traverse of n lines,

$$\delta l_1 = l_1\frac{\delta s_1}{s_1} + l_1\frac{\delta(\cos\theta_1)}{\cos\theta_1}$$

$$\delta l_2 = l_2\frac{\delta s_2}{s_2} + l_2\frac{\delta(\cos\theta_2)}{\cos\theta_2} \qquad \text{etc.}$$

$$\Sigma\delta l = \Delta l \quad \text{(total error in latitude)}$$

$$= \Sigma l \frac{\delta s}{s} + l_1 \frac{\delta(\cos\theta_1)}{\cos\theta_1} + l_2 \frac{\delta(\cos\theta_2)}{\cos\theta_2} + \dots\ l_n \frac{\delta(\cos\theta_n)}{\cos\theta_n}$$

Similarly, as $d = s \sin\theta$,

$$\frac{\delta d}{d} = \frac{\delta s}{s} + \frac{\delta(\sin\theta)}{\sin\theta} \tag{6.33}$$

$$\Sigma\delta d = \Delta d = \Sigma d\frac{\delta s}{s} + d_1 \frac{\delta(\sin\theta_1)}{\sin\theta_1} + d_2 \frac{\delta(\sin\theta_2)}{\sin\theta_2} + \dots\ d_n \frac{\delta(\sin\theta_n)}{\sin\theta_n}$$

where the linear relative accuracy $\delta s/s$ is considered constant for all lines.

From the above equations,

$$\frac{\delta s}{s} = \frac{1}{\Sigma l}\left[\Delta l - \Sigma\, l\, \frac{\delta(\cos\theta)}{\cos\theta}\right] \tag{6.34}$$

$$\frac{\delta s}{s} = \frac{1}{\Sigma d}\left[\Delta d - \Sigma\, d\, \frac{\delta(\sin\theta)}{\sin\theta}\right] \tag{6.35}$$

The value of $\delta s/s$ should closely approximate to the actual accuracy in linear measurement attained if the traverse consists of a large number of lines, but in short traverses there may be quite a large discrepency. In such cases the ratio shows the accuracy attained as it affects the closing error.

The ratio is first worked out separately for latitude and departure from Eqs. 6.34/6.35 and these allow subsequent corrections to be applied as in Eqs. 6.32/6.33.

N.B. The precision ratios for cosine and sine of the bearings are obtained by extraction from trigonometrical tables. Special attention is necessary when values of 0° or 90° are involved as the trigonometrical values of ∞ will be obtained. The traverse containing such bearings may be rotated before adjustment and then re-orientated to the original bearings.

To calculate the precision ratio for $\cos\theta$,

$$\text{Let } \theta = 60° \pm 6''$$

$$\cos\theta = 0{\cdot}5$$

$$\delta(\cos\theta) \text{ difference}/6'' = 0{\cdot}000\,025$$

$$\therefore \quad \frac{\delta(\cos\theta)}{\cos\theta} = \frac{0{\cdot}000\,025}{0{\cdot}5} = \frac{1}{20\,000}$$

The ratio is therefore proportional to the angular accuracy.

Where a negative ratio $\delta s/s$ is obtained it implies that the angular precision has been over-estimated.

As the precision of the angular values increases relative to the linear values, the precision ratios of the former reduce and become negligible.

Thus
$$\frac{\delta s}{s} = \frac{\Delta l}{\Sigma l}$$

and, substituting this into the partial latitude equation,
$$\delta l_1 = l_1 \frac{\Delta l}{\Sigma l}$$

Similarly,
$$\delta d_1 = d_1 \frac{\Delta d}{\Sigma d}$$

These equations thus reduce to method of adjustment (2), the Transit Rule.

6.55 Comparison of methods of adjustment

Example 6.8

Line	Bearing θ	Length s	d +	d −	l +	l −
AB	045° 00′	514·63	363·898		363·898	
BC	090° 00′	341·36	341·360		0·0	
CD	180° 00′	324·15	0·0			324·150
DE	210° 00′	462·37		231·185		400·420
EF	300° 00′	386·44		334·667	193·220	
FA	320° 16′	217·42		138·978	167·202	
	Σs	2246·37	705·258	704·830	724·320	724·570
			704·830			724·320
		Δd	+0·428		Δl	−0·250

Assuming the co-ordinates of A (0,0)

Total Co-ordinates

	D	L
A	0·0	0·0
B	+363·898	+363·898
C	+705·258	+363·898
D	+705·258	+ 39·748
E	+474·073	−360·672
F	+139·406	−167·452
A_1	+ 0·428	− 0·250

$$\text{Bearing of closing line } AA_1 = \tan^{-1} +0{\cdot}428/-0{\cdot}250$$
$$= \underline{\text{S } 59^\circ 43' \text{ F}}$$

$$\text{Length of closing line } AA_1 = 0{\cdot}428 \operatorname{cosec} 59^\circ 43'$$
$$= \underline{0{\cdot}496}$$

(1) *'Axis' scale factor method* (Figs. 6.24 and 6.25)

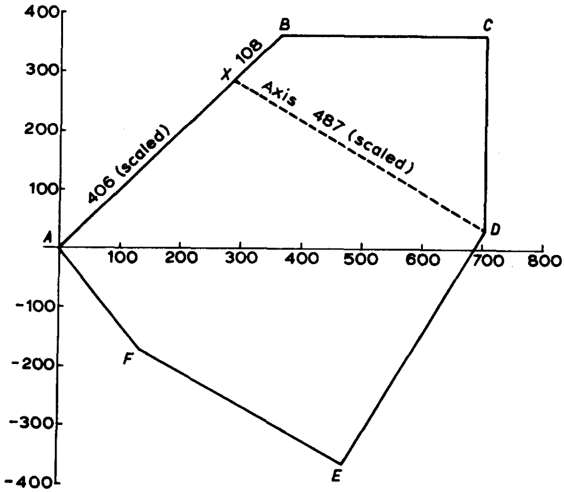


Fig. 6.24

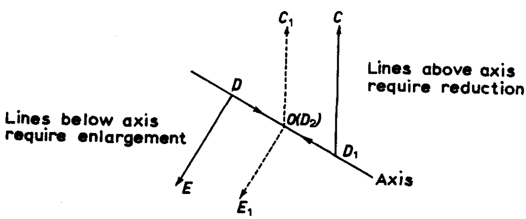


Fig. 6.25

Scaled values from plotting (station D is chosen to contain the closing error as the axis approximately bisects the figure):

$$DX = 487$$
$$AX = 406$$
$$XB = 108$$

From Eqs. 6.14/6.15, $\delta d = \lambda d$

$$\delta l = \lambda l$$

where $\lambda = \dfrac{AA_1}{2 \times \text{Axis}(DX)} = \dfrac{0{\cdot}496}{2 \times 487} = \underline{5{\cdot}1 \times 10^{-4}}$

$$\delta d_{AB} = \frac{AX - XB}{AB} \times \lambda d_{AB}$$

$$= \frac{406 - 108}{514} \times +5{\cdot}1 \times 10^{-4} \times 363{\cdot}9$$

$$= +0{\cdot}105$$

(AX requires enlarging, $AX > XB$)

$$\delta d_{BC} = -5\cdot1 \times 10^{-4} \times +341\cdot4 = -0\cdot174$$
$$\delta d_{CD} = 0\cdot0$$
$$\delta d_{DE} = +5\cdot1 \times 10^{-4} \times -231\cdot2 = -0\cdot118$$
$$\delta d_{EF} = +5\cdot1 \times 10^{-4} \times -334\cdot7 = -0\cdot171$$
$$\delta d_{FA} = +5\cdot1 \times 10^{-4} \times -139\cdot0 = -0\cdot070$$

$$\delta l_{AB} = +5\cdot1 \times 10^{-4} \times \frac{406 - 108}{514} \times +363\cdot9 = +0\cdot105$$
$$\delta l_{BC} = 0\cdot0$$
$$\delta l_{CD} = -5\cdot1 \times 10^{-4} \times -324\cdot2 = +0\cdot165$$
$$\delta l_{DE} = +5\cdot1 \times 10^{-4} \times -400\cdot4 = -0\cdot204$$
$$\delta l_{EF} = +5\cdot1 \times 10^{-4} \times +193\cdot2 = +0\cdot098$$
$$\delta l_{FA} = +5\cdot1 \times 10^{-4} \times +167\cdot2 = +0\cdot086$$

Co-ordinates

	d_o	δd	d_n	l_o	δl	
AB	+363·898	+0·105	+364·063	+363·898	+0·105	+364·063
BC	+341·360	−0·174	+341·186	0·0	0·0	0·0
CD	0·0	0·0	0·0	−324·150	+0·165	−323·985
DE	−231·185	−0·118	−231·303	−400·420	−0·204	−400·624
EF	−334·667	−0·171	−334·838	+193·220	+0·098	+193·318
FA	−138·978	−0·070	−139·048	+167·202	+0·086	+167·288
		−0·533			+0·454	
		+0·105			−0·204	
	Δd	−0·428		Δl	+0·250	

(2) *Ormsby's xy method* (Fig. 6.26)

	Bearing	Term	d	δd	l	δl
AB	045° 00″	x	+363·898	−0·063	+363·898	−0·063
BC	090° 00′	x	+341·360	−0·060	0·0	0·0
CD	180° 00′	y	0·0	0·0	−324·150	+0·181
DE	210° 00′	x	−231·185	−0·040	−400·420	−0·069
EF	300° 00′	y	−334·667	−0·187	+193·220	+0·108
FA	320° 16′	y	−138·978	−0·078	+167·202	+0·093
			Δd	−0·428		+0·382
						−0·132
					Δl	+0·250

N.B. Term x is assumed to be $0° \rightarrow 90°$ inclusive
y is assumed to be $90° \rightarrow 180°$

As the error in departure is greater, the equation takes the sign of the correction, i.e.

$$\underset{(AB)}{-364\,x}\ \underset{(BC)}{-341\,x}\ \underset{(CD)}{-\,Oy}\ \underset{(DE)}{-231\,x}\ \underset{(EF)}{-335\,y}\ \underset{(FA)}{-139y} = -0{\cdot}428 \qquad (1)$$

Adjusting the latitude values and signs to make them consistent,

$$-364\,x + Ox + 324y - 400\,x + 193\,y + 167\,y = +0{\cdot}250 \qquad (2)$$

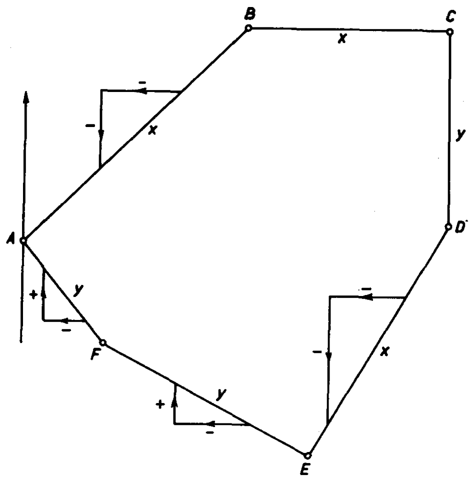


Fig. 6.26

Simplifying the equations,

$$-936\,x - 474\,y = -0{\cdot}428 \qquad (1)$$
$$-764\,x + 684\,y = +0{\cdot}250 \qquad (2)$$

Solving the equations simultaneously,

$$x = +1{\cdot}74 \times 10^{-4}$$
$$y = +5{\cdot}59 \times 10^{-4}$$

The values of x and y are now applied to each term to give corrections as above.

(3) *Crandal's method (least squares)*

(a) Probable error $\propto$ length s

Using equations,

$$k_1\,\Sigma ld + k_2\,\Sigma d^2 = \Delta d$$
$$k_1\,\Sigma l^2 + k_2\,\Sigma ld = \Delta l$$

	d	l	d^2	ld	l^2
AB	+363·898	+363·898	+132 421·8	+132 421·8	+132 421·8
BC	+341·360	0·0	116 526·6	0·0	0·0
CD	0·0	−324·150	0·0	0·0	105 073·2
DE	−231·185	−400·420	53 446·5	+ 92 571·1	160 336·2
EF	−334·667	+193·220	112 002·0	− 64 664·4	37 334·0
FA	−138·978	+167·202	19 314·9	− 23 237·4	27 956·5
	Δd −0·428	Δl +0·250	433 711·8		463 121·7
			(Σd^2)	+224 992·9	(Σl^2)
				− 87 901·8	
				+137 091·1	
				(Σld)	

$$\therefore \qquad 137\,091{\cdot}1\,k_1 + 433\,711{\cdot}8\,k_2 = -0{\cdot}428 \qquad (1)$$
$$463\,121{\cdot}7\,k_1 + 137\,091{\cdot}1\,k_2 = +0{\cdot}250 \qquad (2)$$

Solving simultaneously,

$$k_1 = +9{\cdot}1768 \times 10^{-7}$$
$$k_2 = -1{\cdot}2769 \times 10^{-6}$$

Substituting in the equations

$$k_1 l_1 d_1 + k_2 d_1^2 = \delta d_1$$
$$k_1 l_1^2 + k_2 l_1 d_1 = \delta l_1$$

gives the correction for each partial co-ordinate:

	$k_1 ld$	$k_2 d^2$	δd	$k_1 l^2$	$k_2 ld$	δl
AB	+0·122	−0·169	−0·047	+0·122	−0·169	−0·047
BC	0·0	−0·149	−0·149	0·0	0·0	0·0
CD	0·0	0·0	0·0	+0·096	0·0	+0·096
DE	+0·085	−0·068	+0·017	+0·147	− 0·118	+0·029
EF	−0·059	−0·143	−0·202	+0·034	+ 0·083	+0·117
FA	−0·021	−0·026	−0·047	+0·026	+ 0·029	+0·055
	+0·207	−0·555	−0·445	+0·425	− 0·287	+0·297
	−0·080	+0·127	+0·017	−0·175	+ 0·112	−0·047
	+0·127	−0·428	−0·428	+0·250	− 0·175	+0·250

(b) Probable error $\propto \sqrt{s}$

	s	$1/s$	d^2/s	ld/s	l^2/s
AB	514·63	0·001 945	257·295	+257·295	257·295
BC	341·36	0·002 929	341·306	0·0	0·0
CD	324·15	0·003 085	0·0	0·0	324·151
DE	462·37	0·002 163	115·605	+200·231	346·807
EF	386·44	0·002 588	289·861	−167·351	96·620
FA	217·42	0·004 599	88·292	−106·869	128·572
			$\Sigma d^2/s$ 1092·359	+457·526	$\Sigma l^2/s$ 1153·445
				−274·220	
				$\Sigma ld/s$ +183·306	

Using equations

$$k_1\,\Sigma\frac{ld}{s} + k_2\Sigma\frac{d^2}{s} = \Delta d$$

$$k_1\,\Sigma\frac{l^2}{s} + k_2\,\Sigma\frac{ld}{s} = \Delta l$$

and substituting values gives

$$+\ 183{\cdot}306\,k_1 + 1092{\cdot}359\,k_2 = -0{\cdot}428 \qquad (1)$$
$$+1153{\cdot}445\,k_1 + \ 183{\cdot}306\,k_2 = +0{\cdot}250 \qquad (2)$$

Solving simultaneously,

$$k_1 = +2{\cdot}867 \times 10^{-4}$$
$$k_2 = -4{\cdot}398 \times 10^{-5}$$

Substituting values into equations

$$k_1\,\frac{l_1 d_1}{s_1} + k_2\,\frac{d_1^2}{s_1} = \delta d_1 \qquad k_1\,\frac{l_1^2}{s_1} + k_2\,\frac{l_1 d_1}{s_1} = \delta l_1$$

	$k_1\,ld/s$	$k_2\,d^2/s$	δd	$k_1 l^2/s$	$k_2 ld/s$	δl
AB	+0·074	−0·113	−0·039	+0·074	−0·113	−0·039
BC	0·0	−0·150	−0·150	0·0	0·0	0·0
CD	0·0	0·0	0·0	+0·093	0·0	+0·093
DE	+0·057	−0·051	+0·006	+0·099	−0·088	+0·011
EF	−0·048	−0·127	−0·175	+0·027	+0·074	+0·101
FA	−0·031	−0·039	−0·070	+0·037	+0·047	+0·084
	+0·131	−0·480	−0·434	+0·330	−0·201	+0·289
	−0·079	+0·052	+0·006	−0·080	+0·121	−0·039
	+0·052	−0·428	−0·428	+0·250	−0·080	+0·250

(4) *Bowditch's method*

	s	d	l	δd	δl
AB	514·63	+363·898	+363·898	−0·098	+0·057
BC	341·36	+341·360	0·0	−0·065	+0·038
CD	324·15	0·0	−324·150	−0·062	+0·036
DE	462·37	−231·185	−400·420	−0·088	+0·052
EF	386·44	−334·667	+193·220	−0·074	+0·043
FA	217·42	−138·978	+167·202	−0·041	+0·024
	Σs 2246·37			−0·428	+0·250
		Δd −0·428	Δl +0·250		
		Σd 1410·088	Σl 1448·890		

Using the formulae,

$$\delta d = \frac{\Delta d}{\Sigma s} \times s = \frac{-0{\cdot}428\, s}{2246{\cdot}37} = -1{\cdot}906 \times 10^{-4} s$$

$$\delta l = \frac{\Delta l}{\Sigma s} \times s = \frac{+0{\cdot}250\, s}{2246{\cdot}37} = +1{\cdot}112 \times 10^{-4} s$$

Ex.

$$\delta d_1 = -1{\cdot}906 \times 10^{-4} \times 514{\cdot}63 = -0{\cdot}098$$

$$\delta l_1 = +1{\cdot}112 \times 10^{-4} \times 514{\cdot}63 = +0{\cdot}057$$

(5) *Transit or Wilson's method*

Using the formulae,

$$\delta d = \frac{\Delta d}{\Sigma d} \times d = \frac{-0{\cdot}428\, d}{1410{\cdot}088} = -3{\cdot}035 \times 10^{-4} d$$

$$\delta l = \frac{\Delta l}{\Sigma l} \times l = \frac{+0{\cdot}250\, l}{1448{\cdot}890} = +1{\cdot}725 \times 10^{-4} l$$

Ex.

$$\delta d_1 = -3{\cdot}035 \times 10^{-4} \times +363{\cdot}898 = -0{\cdot}111$$

$$\delta l_1 = +1{\cdot}725 \times 10^{-4} \times +363{\cdot}898 = +0{\cdot}063$$

	δd	δl
AB	−0·111	+0·063
BC	−0·103	0·0
CD	−0·0	+0·056
DE	−0·070	+0·069
EF	−0·102	+0·033
FA	−0·042	+0·029
	Δd −0·428	Δl +0·250

(6) *Smirnoff's method*

N.B. As bearings of BC and CD are 90° and 180° respectively, the values of $\dfrac{\delta(\cos 90)}{\cos 90}$ and $\dfrac{\delta(\sin 180)}{\sin 180}$ will be infinity.

Thus the whole survey is turned clockwise through 20° giving new bearings (with an accuracy of $\pm 10''$)

	θ	s
AB	065·00	514·630
BC	110·00	341·360
CD	200·00	324·150
DE	230·00	462·370
EF	320·00	386·440
FA	340·16	217·420

	$\sin\theta$	$\delta(\sin\theta) \times 10^{-6}$	d	$\dfrac{d\,\delta(\sin\theta)}{\sin\theta}$	$d\dfrac{\delta s}{s}$	δd	d (Adj)
AB	0·906 308	20	+466·413	+0·010	0·074	−0·084	+466·329
BC	0·939 693	17	+320·774	+0·006	0·051	−0·057	+320·717
CD	0·342 020	45	−110·866	−0·013	0·017	−0·030	−110·896
DE	0·766 044	31	−354·196	−0·014	0·056	−0·070	−354·266
EF	0·642 788	37	−248·399	−0·014	0·039	−0·053	−248·452
FA	0·337 643	46	− 73·410	−0·010	0·012	−0·022	− 73·432
			+787·187				
			−786·871				
			Δd + 0·316	0·067	0·249	− 0·316	
			Σd 1574·058				

	$\cos\theta$	$\delta(\cos\theta) \times 10^{-6}$	l	$\dfrac{l\,\delta(\cos\theta)}{\cos\theta}$	$l\dfrac{\delta s}{s}$	δl	l (Adj)
AB	0·422 618	44	+217·492	0·023	0·046	+0·069	+217·561
BC	0·342 020	45	−116·752	0·015	0·025	+0·040	−116·712
CD	0·939 693	17	−304·601	0·006	0·065	+0·071	−304·530
DE	0·642 788	37	−297·206	0·017	0·064	+0·081	−297·125
EF	0·766 044	31	+296·030	0·012	0·064	+0·076	+296·106
FA	0·941 274	17	+204·652	0·004	0·044	+0·048	+204·700
			+718·174				
			−718·559				
			Δl − 0·385	0·077		+0·385	
			Σl 1436·733				

Departure

$$\frac{\delta s}{s} = \frac{1}{\Sigma d}\left[\Delta d - \frac{d\,\delta(\sin\theta)}{\sin\theta}\right]$$

$$= \frac{1}{1574 \cdot 1}[+0\cdot316 - 0\cdot067] = \frac{0\cdot249}{1574\cdot1} = 1\cdot581 \times 10^{-4}$$

Latitude

$$\frac{\delta s}{s} = \frac{1}{\Sigma l}\left[\Delta l - \frac{l\,\delta(\cos\theta)}{\cos\theta}\right]$$

$$= \frac{1}{1436\cdot7}[0\cdot385 - 0\cdot077] = \frac{0\cdot308}{1436\cdot7} = 2\cdot14 \times 10^{-4}$$

The co-ordinates must now be transposed to their original orientation.

Using Eqs. (3.29/30)

$$x_2 = x_1 \cos 20 - y_1 \sin 20°$$
$$y_2 = y_1 \cos 20 + x_1 \sin 20°$$
$$\cos 20° = 0\cdot939\,693$$
$$\sin 20° = 0\cdot342\,020$$

Line AB

x_1 +466·329	+438·206 − 74·410	= +363·796	δd	=	−0·102
y_1 +217·561	+204·441 + 159·494	= +363·935	δl	=	+0·037

Line BC

x_1 +320·717	+301·376 + 39·918	= +341·294	δd	=	−0·074
y_1 +116·712	−109·673 + 109·692	= + 0·019	δl	=	+0·019

Line CD

x_1 −110·869	−104·208 + 104·155	= − 0·053	δd	=	−0·053
y_1 =304·530	−286·165 − 37·929	= −324·094	δl	=	+0·056

Line DE

x_1 −354·266	−332·901 + 101·623	= −231·278	δd	=	−0·093
y_1 −297·125	−297·206 − 121·166	= −400·372	δl	=	−0·048

Line EF

x_1 −248·452	−233·469 − 101·274	= −334·743	δd	=	−0·076
y_1 +296·106	+278·249 − 84·976	= +193·273	δl	=	+0·053

Line FA

x_1 − 73·432	− 69·004 − 70·011	= −139·015	δd	=	−0·037
y_1 +204·700	+192·355 − 25·115	= +167·240	δl	=	+0·038

Analysis of corrections (Figs. 6.27 – 6.31)

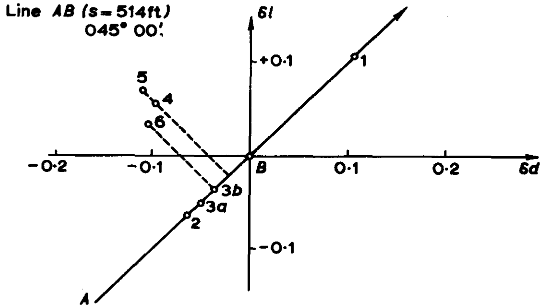


Fig. 6.27

Method	$\delta\theta$	δs	δd	δl
1	0	+0·15	+0·105	+0·105
2	0	−0·09	−0·063	−0·063
3a	0	−0·07	−0·047	−0·047
3b	0	−0·05	−0·039	−0·039
4	−44″	−0·03	−0·098	+0·057
5	−48″	−0·03	−0·111	+0·063
6	−36″	−0·05	−0·102	+0·037

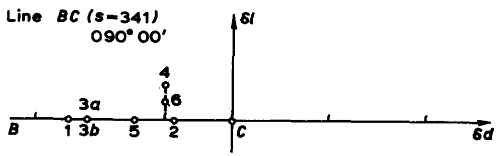


Fig. 6.28

Method	$\delta\theta$	δs	δd	δl
1	0	−0·17	−0·174	0·0
2	0	−0·06	−0·060	0·0
3a	0	−0·15	−0·149	0·0
3b	0	−0·15	−0·150	0·0
4	+22″	−0·07	−0·065	+0·038
5	0	−0·10	−0·103	0·0
6	+11″	−0·07	−0·074	+0·019

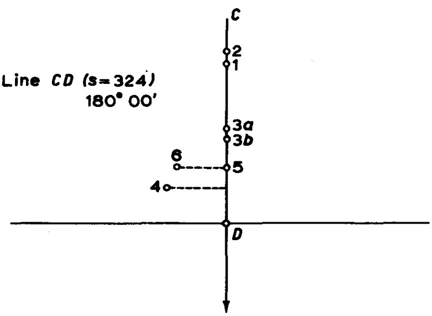


Fig. 6.29

Method	$\delta\theta$	δs	δd	δl
1	0	+0·17	0·0	+0·165
2	0	+0·18	0·0	+0·181
3a	0	+0·10	0·0	+0·096
3b	0	+0·09	0·0	+0·093
4	+44″	0·07	−0·062	+0·036
5	0	+0·06	0·0	+0·056
6	+38″	0·06	−0·053	+0·056

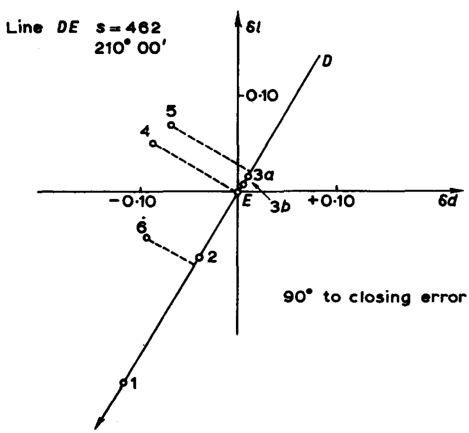


Fig. 6.30

Method	$\delta\theta$	δs	δd	δl
1	0	0·23	−0·118	−0·204
2	0	0·08	−0·040	−0·069
3a	0	0·04	+0·017	+0·029
3b	0	0·02	+0·006	+0·011

Method	$\delta\theta$	δs	δd	δl
4	+45″	0·0	−0·088	+0·052
5	+44″	0·03	−0·070	+0·069
6	+26″	0·09	−0·093	−0·048

(N.B. 90° to closing error)

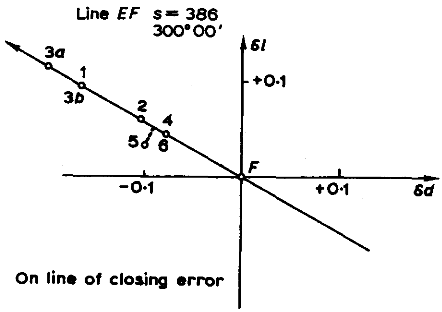


Fig. 6.31

Method	$\delta\theta$	δs	δd	δl
1	0	0·19	− 0·171	+0·098
2	0	0·12	− 0·187	+0·108
3a	0	0·23	− 0·202	+0·117
3b	0	0·19	− 0·175	+0·101
4	0	0·09	− 0·074	+0·043
5	−5″	0·10	−0·102	+0·033
6	0	0·09	−0·076	+0·053

(N.B. On line of closing error)

The following may be conjectured:

(1) The first four methods do not change the bearings.

(2) Method (1) has a greater effect on the linear values than any other.

(3) There is no change in bearing in Wilson's method when the line coincides with the axes.

(4) There is little or no change in bearing on any line parallel to the closing error in any of the methods analysed – maximum linear correction.

(5) Wilson's method has less effect on the bearings than Bowditch's, but more than Smirnoff's.

(6) The maximum change in bearing occurs at 90° to the closing error – maximum linear correction.

Exercises 6 (c) (Traverse Adjustment)

12. The mean observed internal angles and measured sides of a closed traverse *ABCDA* (in anticlockwise order) are as follows:

Angle	Observed Value	Side	Measured Length (ft)
DAB	97° 41′	*AB*	221·1
ABC	99° 53′	*BC*	583·4
BCD	72° 23′	*CD*	399·7
CDA	89° 59′	*DA*	521·0

Adjust the angles, compute the latitudes and departures assuming that *D* is due N of *A*, adjust the traverse by the Bowditch method; and give the co-ordinates of *B*, *C* and *D* relative to *A*.

Assess the accuracy of these observations and justify your assessment. (I.C.E. Ans. *B* −30·3 N, +219·7 E,
C +523·9 N, +397·5 E,
D +522·6 N, − 1·2 E)

13. The measured lengths and reduced bearings of a closed theodolite traverse *ABCD* are as follows:

Line	Length (ft)	Bearing
AB	454·9	Due N
BC	527·3	Due W
CD	681·0	S 25° 18′ W
DA	831·2	N 78° 54′ E

(a) Adjust the traverse by the Bowditch method and taking *A* as the origin, find the co-ordinates of *B*, *C* and *D*.

(b) Assess the accuracy of the unadjusted traverse.

(c) Suggest, and outline briefly, an alternative method of adjusting the traverse so that the bearing of *AB* is unaltered by the adjustment.
(I.C.E. Ans. *B* 455·0 N, 0·5 E,
C 455·1 N, 526·2 W,
D 160·3 S, 816·5 W)

14. The following lengths, latitudes and departures refer to a closed traverse *ABCDEA* :

	Length	Latitude	Departure
AB	3425·9	0	3425·9
BC	938·2	812·6	469·1
CD	4573·4	2287·1	−3961·0
DE	2651·3	−2295·7	−1325·9
EA	1606·4	− 803·0	1391·1

Adjust the traverse by the Bowditch method, finding the corrected latitudes and departures to the nearest 0·1 ft.

Discuss the merits and demerits of this method, with particular reference to its effect on lines CD and DE.

(L.U. Ans. AB	$-$ 0·3,	+3426·1
BC	+ 812·5,	+ 469·2
CD	+2286·8,	−3960·7
DE	−2295·9,	−1325·8
EA	− 803·1,	+1391·2)

15. In a closed traverse $ABCDEFA$ the lengths, latitudes and departures of lines (in ft) are as follows:

Line	AB	BC	CD	DE	EF	FA
Length	1342·0	860·4	916·3	1004·1	1100·0	977·3
Latitude	−1342·00	−135·58	+910·35	+ 529·11	+ 525·99	−483·23
Departure	0·0	+849·65	+104·26	+ 853·38	− 966·08	−849·42

Adjust the traverse by the Bowditch method and give the corrected co-ordinates of A as (0, 0)

(L.U. Ans. A	0·0,	0·0	D	−569·56,	+958·04
B	−1343·00,	+1·77	E	− 41·20,	+1812·74
C	−1479·22,	+852·56	F	+483·96,	+ 848·12)

16. A traverse $ACDB$ is surveyed by theodolite and chain. The lengths and quadrantal bearings of the lines, AC, CD and DB are given below.

If the co-ordinates of A are $x = 0$, $y = 0$ and those of B are $x = 0$, $y = +897{\cdot}05$, adjust the traverse and determine the co-ordinates of C and D.

The co-ordinates of A and B must not be altered.

Line	AC	CD	DB
Length	480·6	292·0	448·1
Bearing	N 25° 19′ E	N 37° 53′ E	N 59° 00′ W

(L.U. Ans. C	+205·2,	435·0
D	+384·4,	665·8)

17. The lengths, latitudes and departures of the lines of a closed traverse are given below.

In one of the lines it appears that a chainage has been misread by 40 ft. Select the line in which the error is most likely to have occurred, correct it and adjust the latitudes and departures by the Bowditch method to the nearest 0·1 ft.

Line	Length (ft)	Latitude	Departure
AB	310·5	+301·2	+ 75·4
BC	695·8	+267·1	−642·5
CD	492·8	−299·8	−391·1

Line	Length (ft)	Latitude	Departure
DE	431·7	−359·1	+239·6
EF	343·1	+173·5	+296·0
FA	401·9	− 49·0	+398·9

(L.U. Ans. Line *DE* 40 ft too long

AB	+301·1	+ 75·6	*DE*	−392·5	+262·0
BC	+267·0	−642·1	*EF*	+173·4	+296·2
CD	−299·9	−390·8	*FA*	− 49·1	+399·1)

18. (a) Why is the accuracy of angular measurement so important in a traverse for which a theodolite and steel tape are used?

(b) *A* and *D* are the terminals of traverse *ABCD*. Their plane rectangular co-ordinates on the survey grid are:

	Eastings	Northings
A	+5861·14 ft	+3677·90 ft
D	+6444·46 ft	+3327·27 ft

The bearings adjusted for angular misclosure and the lengths of the legs are:

AB	111° 53′ 50″	306·57 ft
BC	170° 56′ 30″	256·60 ft
CD	86° 43′ 10″	303·67 ft

Calculate the adjusted co-ordinates of *B* and *C*

(N.U. Ans. *B* E 6100·70 N 3563·47
C E 6141·19 N 3309·99)

19. From an underground traverse between two shaft-wires, *A* and *D*, the following partial co-ordinates in feet were obtained:

AB	E 150·632 ft	S 327·958 ft
BC	E 528·314 ft	N 82·115 ft
CD	E 26·075 ft	N 428·862 ft

Transform the above partials to give the total Grid co-ordinates of *station* *B* given that the Grid co-ordinates of *A* and *D* were:

A	E 520 163·462 metres	N 432 182·684 metres
D	E 520 378·827 metres	N 432 238·359 metres

(aide memoire : $X = x_1 + k(x - y\theta)$
$Y = y_1 + k(y + x\theta)$)

(N.R.C.T. Ans. *B* E 520 209·364 N 432 082·480)

20. (a) A traverse to control the survey of a long straight street forms an approximate rectangle of which the long sides, on the pavements, are formed by several legs, each about 300 ft long and the short sides are about 40 ft long; heavy traffic prevents the measurement of lines obliquely across the road. A theodolite reading to 20″ and a tape

graduated to 0·01 ft are used and the co-ordinates of the stations are required as accurately as possible.

Explain how the short legs in the traverse can reduce the accuracy of results and suggest a procedure in measurement and calculation which will minimize this reduction.

(b) A traverse *TABP* was run between the fixed stations *T* and *P* of which the co-ordinates are:

	E	N
T	+6155·04	+9091·73
P	+6349·48	+9385·14

The co-ordinate differences for the traverse legs and the data from which they were calculated are:

	Length	Adjusted Bearing	ΔE	ΔN
TA	354·40	210°41′40″	−180·91	−304·75
AB	275·82	50°28′30″	+212·75	+175·54
BP	453·03	20°59′50″	+162·33	+422·95

Applying the Bowditch rule, calculate the co-ordinates of *A* and *B*.

(L.U. Ans. *A* E5974·22 N8786·87
B E6187·04 N8962·33)

21. The co-ordinates in feet of survey control stations *A* and *B* in a mine are as follows:

Station *A*	E8432·50	N6981·23
Station *B*	E9357·56	N4145·53

Undernoted are azimuths and distances of a traverse survey between *A* and *B*.

Line	Azimuth	Horizontal Distance
A − 1	151°54′20″	564·31
1 − 2	158°30′25″	394·82
2 − 3	161°02′10″	953·65
3 − 4	168°15′00″	540·03
4 − *B*	170°03′50″	548·90

Adjust the traverse on the assumption that the co-ordinates of stations *A* and *B* are correct and state the corrected co-ordinates of the traverse station

(M.Q.B./S Ans.

	E	N
A	8432·50	6981·23
1	8698·24	6483·55
2	8842·91	6116·28
3	9152·83	5214·64
4	9262·82	4686·07
B	9357·56	4145·53)

22. Define the terms 'error of closure' and 'fractional linear closing error' as applied to closed traverse surveys. What error of closure would be acceptable for a main road traverse survey underground?

Starting with the equations

$$l = s \cos\alpha$$
$$d = s \sin\alpha$$

derive the Smirnoff equations

$$\frac{ds}{s} = \frac{1}{\Sigma l}\left\{(dL) - \Sigma l \; \frac{d(\cos\alpha)}{\cos\alpha}\right\}$$

and

$$\frac{ds}{s} = \frac{1}{\Sigma d}\left\{(dD) - \Sigma d \; \frac{d(\sin\alpha)}{\sin\alpha}\right\}$$

where

α = bearing angle

s = length of traverse draft

ds/s = linear precision ratios

$\dfrac{d(\cos\alpha)}{\cos\alpha}$ and $\dfrac{d(\sin\alpha)}{\sin\alpha}$ = angular precision ratios

Σl = sum of latitudes

Σd = sum of departures

dL = total closing error in latitudes

dD = total closing error in departures.

(N.U.)

Exercises 6 (d) (General)

23. The following are the notes of a theodolite traverse between the faces of two advancing roadways BA and FG, which are to be driven until they meet.

Calculate the distance still to be driven in each roadway.

Line	Azimuth	Distance (ft)
AB	267° 55′	150
BC	355° 01′	350
CD	001° 41′	315
DE	000° 53′	503
EF	086° 01′	1060
FG	203° 55′	420

(Ans. BA produced 352·6 ft
FG produced 916·5 ft)

24. The following measurements were made in a closed traverse, $ABCD$

$$\hat{A} = 70° 45' \; ; \quad \hat{D} = 39° 15'$$

$AB = 400$ ft ; $CD = 700$ ft ; $AD = 1019$ ft

Calculate the missing measurements.

(L.U./E Ans. $\hat{B} = 119° 58'$, $\hat{C} = 130° 02'$, $BC = 351{\cdot}1$ ft)

25. Particulars of a traverse survey are as follows:

Line	Length (ft)	Deflection Angle
AB	330	*B* 76° 23′ right
BC	515	*C* 118° 29′ right
CD	500	*D* 79° 02′ right
DA	375	*A* 86° 06′ right

Bearing of line *AB* 97° 15′

Prepare a traverse sheet and so calculate the length and bearing of the closing error.

(L.U./E Ans. 6·4 ft, N 347° 04′ W)

26. The interior angles of a closed (clockwise) traverse *ABCDEA* have been measured with a vernier theodolite reading to 20″, with results as follows:

Angle at *A*	88° 03′ 20″
B	117° 41′ 40″
C	126° 13′ 00″
D	119° 28′ 40″
E	88° 35′ 00″

Adjust the measurements to closure and find the reduced bearings of the other lines if that for line *AB* is S 42° 57′ 20″ E.

(L.U./E. Ans. *BC* S 48° 59′ 40″ W *DE* N 14° 54′ 20″ W
CD N 68° 41′ 40″ W *EA* N 45° 37′ 20″ E)

27. An approximate compass traverse carried out over marshy ground yielded the following results:

Line	Length (ft)	Bearing
AB	386	139°
BC	436	50°
CD	495	335°
DE	271	249°
EA	355	200°

Plot the traverse to a scale of 100 ft to the inch and adjust it graphically to closure.

28. A plot of land is up for sale and there is some doubt about its area. As a quick check, a compass traverse is run along the boundaries.

Determine the area enclosed by the traverse from the following data:

Line	Bearing	Feet
AB	195°	528
BC	275°	548
CD	182½°	813
DE	261½°	1293
EF	343°	788
FG	5°	653
GH	80½°	1421
HA	102½°	778

(I.C.E. Ans. 56 acres)

29. The traverse table below refers to a closed traverse run from station D, through O, G and H and closing on D. The whole-circle bearing of O from D is $06° 26'$ and G and H lie to the west of the line OD.

Compute the latitudes and departures of O, G and H with reference to D as origin, making any adjustments necessary.

Observed	Internal Angles	Length in feet
HDO	79° 47′	*DO* 547·7
DOG	102° 10′	*OG* 939·8
OGH	41° 11′	*GH* 840·2
GHD	136° 56′	*HD* 426·5

(I.C.E. Ans. O +545·1, + 61·0
G +846·1, −830·7
H +121·6, −408·2)

30. The field results for a closed traverse are:

Line	Whole Circle Bearing	Length (ft)
AB	0° 00′	166
BC	63° 49′	246
CD	89° 13′	220
DE	160° 55′	202
EF	264° 02′	135
FA	258° 18′	399

The observed values of the included angles check satisfactorily, but there is a mistake in the length of a line.

Which length is wrong and by how much?

As the lengths were measured by an accurate 100 ft chain, suggest how the mistake was made.

(I.C.E. Ans. Line BC 20 ft short)

31. The following traverse was run from station I to station V between which there occur certain obstacles.

Line	Length (ft)	Bearing
I – II	351·3	N 82° 28′ E
II – III	149·3	N 30° 41′ E
III – IV	447·3	S 81° 43′ E
IV – V	213·3	S 86° 21′ E

It is required to peg the mid-point of I – V.

Calculate the length and bearing of a line from station III to the required point. (I.C.E. Ans. 171·1 ft S 42° 28′E)

32. Two shafts, A and B, have been accurately connected to the National Grid of the Ordnance Survey and the co-ordinates of the shaft centres, reduced to a local origin, are as follows:

Shaft A	E 10 055·02 metres	N 9768·32 metres
Shaft B	E 11 801·90 metres	N 8549·68 metres

From shaft A, a connection to an underground survey was made by wires and the grid bearing of a base line was established from which the underground survey was calculated. Recently, owing to a holing through between the collieries, an opportunity arose to make an underground traverse survey between the shafts A and B. This survey was based on the grid bearing as established from A by wires, and the co-ordinates of B in relation to A as origin were computed as

E 5720·8 ft S 4007·0 ft

Assuming that the underground survey between A and B is correct, state the adjustment required on the underground base line as established from shaft A to conform to the Nation Grid bearing of that line. (M.Q.B./S Ans. 00° 06′ 30″)

33. It is proposed to sink a vertical staple shaft to connect X on a roadway CD on the top horizon at a colliery with a roadway GH on the lower horizon which passes under CD.

From the surveys on the two horizons, the undernoted data are available:

Upper Horizon

Station	Horizontal Angle	Inclination	Inclined Length (ft)	Remarks
A				
		+ 1 in 200	854·37	co-ordinates of A E 6549·10 ft N 1356·24 ft
B	276° 15′ 45″			
		+ 1 in 400	943·21	Bearing AB N 30° 14′ 00″ E
C	88° 19′ 10″			
		level	736·21	
D				

Lower Horizon

Station	Horizontal Angle	Inclination	Inclined Length (ft)	Remarks
E				co-ordinates of *E*
		+1 in 50	326·17	E 7704·08 ft
F	193° 46′ 45″			N 1210·88 ft
		+1 in 20	278·66	Bearing *EF*
G	83° 03′ 10″			N 54° 59′ 10″ E
		level	626·10	
H				

Calculate the co-ordinates of *X*

(M.Q.B./S Ans. *X* = E 8005·54 ft
N 1918·79 ft)

34. Calculate the co-ordinate values of the stations *B*, *C*, *D* and *E* of the traverse *ABCDEA*, the details of which are given below.

Data: Co-ordinates of *A* 1000·0 ft E 1000·0 ft N
Bearing of line *AB* 0° 00′
Length of line *AB* 342·0 ft

Interior	Angle	Length (ft)	
BAE	27° 18′ 00″	*AB*	342
CBA	194° 18′ 40″	*BC*	412
DCB	146° 16′ 00″	*CD*	412
EDC	47° 27′ 20″	*DE*	592
AED	124° 40′ 00″	*EA*	683

(R.I.C.S. Ans. *B* 1000·0 E, 1342·0 N *C* 898·2 E, 1741·1 N
D 1035·2 E, 2129·6 N *E* 1313·4 E, 1607·0 N)

35. The table below gives the forward and back quadrantal bearings of a closed compass traverse.

Tabulate the whole-circle bearings corrected for local attraction, indicating clearly your reasons for any corrections.

Line	Length (ft)	Forward Bearing	Back Bearing
AB	650	N 55° E	S 54° W
BC	328	S 67½° E	N 66° W
CD	325	S 25° W	N 25° E
DE	280	S 77° W	N 75½° E
EA	440	N 64½° W	S 63½° E

A gross mistake of 100 ft has been made in the measurement or booking of one of the lines. State which line is in error. Using this corrected length, adjust the departure and latitude of each line of the traverse to close, using Bowditch's method of adjustment.

(L.U. Ans. Local attraction at *B* and *E*, *CD* 100 ft too small)

36. It is proposed to extend a straight road AB in the direction AB produced. The centre line of the extension passes through a small farm and in order to obtain the centre line of the road beyond the farm a traverse is run from B to a point C, where A, B and C lie in the same straight line.

The following angles and distance were recorded, the angles being measured clockwise from the back to the forward station:

ABD	$= 87°42'$	BD	$= 95{\cdot}2$ ft
BDE	$= 282°36'$	DE	$= 253{\cdot}1$ ft
DEC	$= 291°06'$		

Calculate (a) the length of the line EC

(b) the angle to be measured at C so that the centre line of the road can be extended beyond C.

(c) the chainage of C taking the chainage of A as zero and $AB = 362$ ft.

(L.U. Ans. (a) 58·3 ft; (b) 58° 36′; (c) 576·8 ft)

37. The following are the notes of a traverse made to ascertain the position if the point F was in line with BA produced.

Line	Azimuth	Distance
AB	355° 30′	600 ft level
BC	125° 00′	310 ft rising 1 in 2
CD	210° 18′	378 ft level
DE	130° 36′	412 ft level
EF	214° 00′	465 ft level

Calculate the difference in the azimuths of AF and BA and the extent to which the point F is out of alignment with BA produced.

(N.R.C.T. Ans. 0° 01′; 0·3 ft)

38. The following notes were made when running a traverse from a station A to a station E:

Side	W.C. Bearing	Length (ft)
AB	119° 32′	264·8
BC	171° 28′	162·4
CD	223° 36′	188·3
DE	118° 34′	316·5

A series of levels were also taken along the same route as follows;

BS	I.S.	F.S.	R.L.	Remarks
6·84			246·20	B.M. near A
	3·86			Sta. A
	11·02			Sta. B
1·32		13·66		C.P. 1

BS	I.S.	F.S.	R.L.	Remarks
	9·66			Sta. *C*
	12·96			Sta. *D*
0·82		13·44		C.P. 2
		12·88		Sta. *E*

Calculate the plan length, bearing and average gradient of the line *AE*. (L.U. Ans. 705·1 ft; 145° 11′; 1 in 22·75)

39. The following are the notes of an underground theodolite traverse.

Line	Azimuth	Distance (ft)	Vertical Angle
AB	180° 00′	–	
BC	119° 01′	181·6	+15° 25′
CD	160° 35′	312·0	+12° 45′
DE	207° 38′	320·0	–19° 30′
EF	333° 26′	200·0	–14° 12′

It is proposed to drive a cross-measures drift dipping from station *B* at a gradient of 1 in 10 on the line of *AB* produced to intersect at a point *X*, a level cross-measures drift to be driven from station *F*.

Calculate the azimuth and length of the proposed drift *FX*.

(Ans. 340° 34′; 83·1 ft)

40. The following are the notes of an underground theodolite traverse:

Line	Azimuth	Distance (ft)	Vertical Angle
AB	089° 54′	350	–
BC	150° 12′	190	–
CD	180° 00′	600	–
DE	140° 18′	155	+28°
EF	228° 36′	800	–12°

It is proposed to drive a cross-measures drift to connect stations *A* and *F*. Calculate the gradient and length of the cross-measures drift, and the azimuth of the line *FA*.

(M.Q.B./M Ans. 1 in 14·8 (3° 52′); 1391·3 ft (incl); 182° 17′)

Bibliography

MIDDLETON, R.E. and CHADWICK, O., *A Treatise on Surveying* (Spon)
JAMESON, A,H., *Advanced Surveying* (Pitman)
CLARK, D., *Plane and Geodetic Surveying* (Constable)
RICHARDSON, P., *Project Surveying* (North Holland Publishing Co.)
HOLLAND, J.L. WARDELL, K and WEBSTER, A.G., *Surveying*, Coal Mining Series (Virtue)
SMIRNOFF, M.V., *Measurements for Engineering and Other Surveys* (Prentice-Hall)
RAINSFORD, HUME F., *Survey Adjustments and Least Squares* (Constable)

7 TACHEOMETRY

The word tacheometry is derived from the Greek ταχυς swift, μετρον a measure. This form of surveying is usually confined to the optical measurement of distance.

In all forms of tacheometry there are two alternatives:

(a) A fixed angle with a variable length observed.

(b) A variable angle with a fixed length observed.

In each case the standard instrument is the theodolite, modified to suit the conditions.

The alternatives are classified as:

(1) Fixed angle: (a) Stadia systems, (b) Optical wedge systems.

(2) Variable angle: (a) Tangential system – vertical staff, (b) Subtense system – horizontal staff.

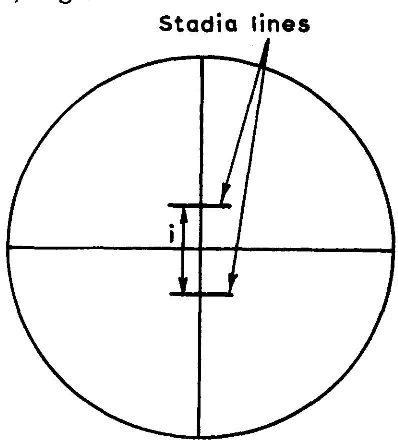
There are two forms of stadia:

(1) Fixed stadia, found in all theodolites and levels.

(2) Variable stadia, used in special tacheometers.

7.1 Stadia systems – Fixed stadia

The stadia lines are fine lines cut on glass diaphragms placed close to the eyepiece of the telescope, Fig. 7.1.



From Chapter 4, the basic formulae are:

$$D = ms + K \qquad \text{(Eq. 4.29)}$$

$$= \frac{f}{i}\, s + (f + d) \qquad \text{(Eq. 4.28)}$$

Fig. 7.1. Diaphragm

where $m = \dfrac{f}{i}$ = the multiplying constant,

f = the focal length of the object lens,

i = the spacing of the stadia lines on the diaphragm,

d = the distance from the object lens to the vertical axis.

7.2 Determination of the Tacheometric Constants m and K

Two methods are available:

(a) by physical measurement of the instrument itself,

(b) by reference to linear base lines.

7.21 By physical measurement of the instrument

From the general equation,

$$D = ms + K$$

where $m = \frac{f}{i}$ and $K = f + d$.

In the equation

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \qquad \text{(Eq. 4.19)}$$

where u = the distance from the objective to the staff is very large compared with f and v and thus $1/u$ is negligible compared with $1/v$ and $1/f$,

$$\frac{1}{f} \simeq \frac{1}{v}$$

i.e.

$$f \simeq v$$

i.e. $f \simeq$ the length from the objective to the diaphragm with the focus at ∞.

With the external focussing telescope, this distance can be changed to correspond to the value of u in one of two ways:

(1) by moving the objective forward,

(2) by moving the eyepiece backwards.

In the former case the value of K varies with u, whilst the latter gives a constant value.

The physical value i cannot easily be measured, so that a linear value D is required for the substitution of the value of f to give the factors i and K, i.e.

$$D = s\frac{f}{i} + (f + d)$$

Thus a vertical staff is observed at a distance D, the readings on the staff giving the value of s.

Example 7.1. A vertical staff is observed with a horizontal external focussing telescope at a distance of 366 ft 3 in.

Measurements of the telescope are recorded as:

Objective to diaphragm	9 in.
Objective to vertical axis	6 in.

If the readings taken to the staff were 3·52, 5·35 and 7·17 ft, calculate

(a) the distance apart of the stadia lines (i),

(b) the multiplying constant (m),

(c) the additive constant (K).

From Eqs. (4.28) and (4.29),

$$D = ms + K$$

$$= \frac{f}{i} \cdot s + (f + d)$$

$$\therefore \quad i = \frac{fs}{D - (f+d)}$$

$$= \frac{9{\cdot}0 \times (7{\cdot}17 - 3{\cdot}52)}{366{\cdot}25 - (0{\cdot}75 + 0{\cdot}50)} \text{ in.}$$

$$= \frac{9{\cdot}0 \times 3{\cdot}65}{366{\cdot}25 - 1{\cdot}25}$$

$$= \underline{0{\cdot}09 \text{ in.}}$$

$$\therefore \quad m = \frac{f}{i} = \frac{9}{0{\cdot}09} = \underline{100}$$

$$K = f + d = 9 \text{ in.} + 6 \text{ in.} = \underline{1{\cdot}25 \text{ ft}}$$

7.22 By field measurement

The more usual approach is to set out on a level site a base line of say 400 ft with pegs at 100 ft intervals.

The instrument is then set up at one end of the line and stadia readings are taken successively on to a staff held vertically at the pegs.

By substitution into the formula for selected pairs of observations, the solution of simultaneous equations will give the factors m and K.

$$\text{i.e.} \quad \left.\begin{aligned} D_1 &= m s_1 + K \\ D_2 &= m s_2 + K \end{aligned}\right\}$$

Example 7.2 The following readings were taken with a vernier theodolite on to a vertical staff:

Stadia Readings	Vertical Angle	Horizontal Distance
2·613 3·359 4·106	0°	150 ft
6·146 7·150 8·154	5°00′	200 ft

Calculate the tacheometric constants.

$$D = m(4{\cdot}106 - 2{\cdot}613) + K = 150$$
$$= 1{\cdot}493\,m + K = 150$$
$$D_2 = m(8{\cdot}154 - 6{\cdot}146)\cos^2 5^\circ + K\cos 5^\circ = 200$$
$$= 2{\cdot}008\,m \times 0{\cdot}99620^2 + 0{\cdot}996\,20\,K = 200$$
$$= 1{\cdot}99276\,m + 0{\cdot}99620\,K = 200$$

Solving these two equations simultaneously gives

$$\underline{m = 100{\cdot}05} \quad \text{(say 100)}$$
$$\underline{K = 0{\cdot}7\text{ ft}}$$

N.B. The three readings at each staff station should produce a check, i.e. Middle–Upper = Lower–Middle

$3{\cdot}359 - 2{\cdot}613 = 0{\cdot}746$ $\quad$ $4{\cdot}106 - 3{\cdot}359 = 0{\cdot}747$

$7{\cdot}150 - 6{\cdot}146 = 1{\cdot}004$ $\quad$ $8{\cdot}154 - 7{\cdot}150 = 1{\cdot}004$

7.3 Inclined Sights

The staff may be held (a) normal to the line of sight or (b) vertical.

7.31 Staff normal to the line of sight (Fig. 7.2)

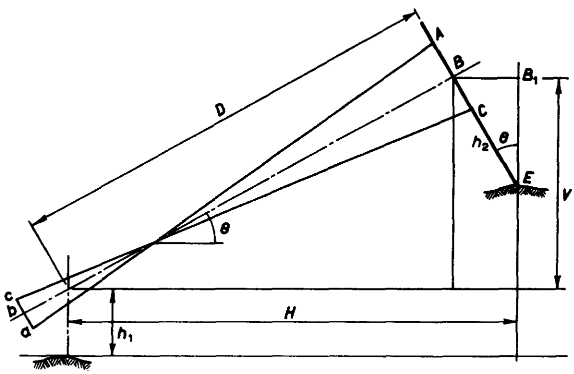


Fig. 7.2 Inclined sights with staff normal

As before, $\quad D = ms + K$

but $\quad H = D\cos\theta + BB_1$

$$\underline{= D\cos\theta + BE\sin\theta} \tag{7.1}$$

i.e. $\quad H = (ms + K)\cos\theta + BE\sin\theta \quad (7.2)$

N.B. $BE = h_2$ = staff reading of middle line of diaphragm. BB_1 is $-ve$ when θ is a depression.

Vertical difference $\quad \underline{V = D\sin\theta} \quad (7.3)$

i.e. $$V = (ms + K)\sin\theta \tag{7.4}$$

As the factor K may be neglected generally,

$$\underline{H = ms\cos\theta + BE\sin\theta} \tag{7.5}$$

$$\underline{V = ms\sin\theta} \tag{7.6}$$

If the height of the instrument to the trunnion axis is h_1 and the middle staff reading h_2, then the *difference in elevation*

$$= \underline{h_1 \pm V - h_2\cos\theta} \tag{7.7}$$

Setting the staff normal to the line of sight is not easy in practice and it is more common to use the vertical staff.

7.32 **Staff vertical** (Fig. 7.3)

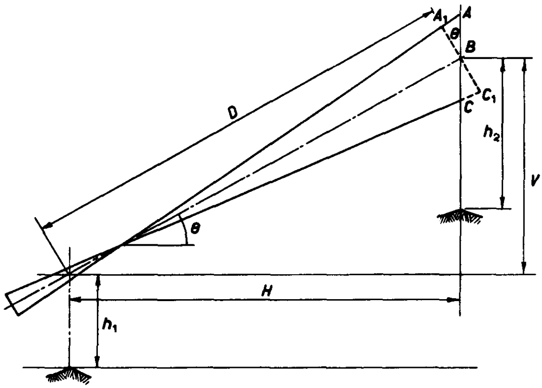


Fig. 7.3 Inclined sights with staff vertical

As before $D = ms_1 + K$

i.e. $= m(A_1C_1) + K$

but AC are the staff readings

thus $D = m(AC\cos\theta) + K$ (assuming $B\hat{A_1}A = B\hat{C_1}C = 90$)

$$= ms\cos\theta + K \tag{7.8}$$

$\therefore$ $H = D\cos\theta$

$$\underline{= ms\cos^2\theta + K\cos\theta} \tag{7.9}$$

Also $V = D\sin\theta$

$$\underline{= ms\sin\theta\cos\theta + K\sin\theta} \tag{7.10}$$

Also $$V = H\tan\theta \tag{7.11}$$

It can be readily seen that the constant $K = 0$ simplifies the equations.

Therefore the equations are generally modified to

$$\underline{H = ms \cos^2 \theta} \tag{7.12}$$

$$\underline{V = \tfrac{1}{2} ms \sin 2\theta} \tag{7.13}$$

If it is felt that the additive factor is required, then the following approximations are justified:

$$H = (ms + K) \cos^2 \theta \tag{7.14}$$

$$V = \tfrac{1}{2}(ms + K) \sin 2\theta \tag{7.15}$$

The difference in elevation now becomes

$$= h_1 \pm V - h_2 \tag{7.16}$$

Example 7.3 A line of third order levelling is run by theodolite, using tacheometric methods with a staff held vertically. The usual three staff readings, of centre and both stadia hairs, are recorded together with the vertical angle (V.A.) A second value of height difference is found by altering the telescope elevation and recording the new readings by the vertical circle and centre hair only.

The two values of the height differences are then meaned. Compute the difference in height between the points A and B from the following data:

The stadia constants are multiplying constant = 100.
additive constant = 0.

Backsights V.A.	Backsights Staff	Foresights V.A.	Foresights Staff	Remarks (all measurements in ft)
$+0°02'$	6·20			Point A
	4·65			
	3·10			
$+0°20'$	6·26			
		$-0°18'$	10·20	Point B
			6·60	
			3·00	
		$0°00'$	10·37	

(*Aide memoire*: Height difference between the two ends of the theodolite ray $= 100 s \sin\theta \cos\theta$, where s = stadia intercept and θ = V.A.) (R.I.C.S.)

$$V = 100 s \sin\theta \cos\theta$$
$$= \underline{50 s \sin 2\theta}$$

To A, $V = 50(6{\cdot}20 - 3{\cdot}10) \sin 0°04'$
$= +0{\cdot}18$ ft

Difference in level from instrument axis	$+0{\cdot}18$
	$-4{\cdot}65$
	$-4{\cdot}47$

Check reading

$$V = 50(3{\cdot}10)\sin 0°40'$$
$$= +1{\cdot}80$$

Difference in level from instrument axis	$+1{\cdot}80$
	$-6{\cdot}26$
	$-4{\cdot}46$
mean	$-4{\cdot}465$

To B,

$$V = 50(10{\cdot}20 - 3{\cdot}00)\sin 0°36'$$
$$= -3{\cdot}76$$

Difference in level from instrument axis	$-\ 3{\cdot}76$
	$-\ 6{\cdot}60$
	$-10{\cdot}36$
Check level	$-10{\cdot}37$
mean	$-10{\cdot}365$
Difference in level $A - B$	$-10{\cdot}365$
	$-\ 4{\cdot}465$
	$-\ 5{\cdot}900$ ft

Example 7.4 The readings below were obtained from an instrument station B using an anallatic tacheometer having the following constants: focal length of the object glass 8 in., focal length of the anallatic lens 4·5 in., distance between object glass and anallatic lens 7 in., spacing of outer cross hairs 0·0655 in.

Instrument at	Height of Instrument	To	Bearing	Vertical Angle	Stadia Readings	Remarks
B	4·93 ft	A	69°30′	+5°	2·16/3·46/ 4·76	Staff held vertical for both observations
		C	159°30′	0°	7·32/9·34/11·36	

Boreholes were sunk at A,B and C to expose a plane bed of rock, the ground surface being respectively 39·10, 33·68 and 18·45 ft above the rock plane. Given that the reduced level of B was 120·02 ft., determine the line of steepest rock slope relative to the direction AB.

(L.U.)

By Eq. (4.36),
$$f = 8\text{ in}$$
$$f_1 = 4{\cdot}5\text{ in}$$
$$x = 7\text{ in}$$
$$i = 0{\cdot}0655\text{ in}$$

Then the multiplying factor $m = \dfrac{ff_1}{i(f + f_1 - x)}$

$$= \frac{8 \times 4{\cdot}5}{0{\cdot}0655(8 + 4{\cdot}5 - 7)}$$

$$= 99{\cdot}93 \text{ (say 100)}$$

At station *B:*

To *A*, $H = 100 \times 2{\cdot}60 \cos^2 5° = 258{\cdot}02\text{ ft}$

$V = 258{\cdot}02 \tan 5° = +22{\cdot}57\text{ ft}$

$\therefore$ Level of $A = 120{\cdot}02 + 22{\cdot}57 + 4{\cdot}93 - 3{\cdot}46 = 144{\cdot}06\text{ ft}$

To *C*, $H = 100 \times 4{\cdot}04 = 404{\cdot}00\text{ ft}$

$V = 0$

$\therefore$ Level of $C = 120{\cdot}02 + 0 + 4{\cdot}93 - 9{\cdot}34 = 115{\cdot}61\text{ ft}$

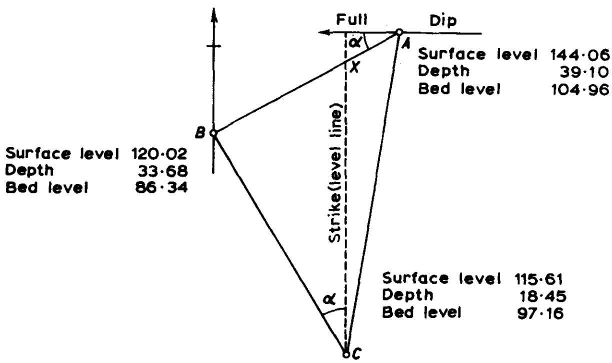


Fig. 7.4

Gradient *AB* is $(104{\cdot}96 - 86{\cdot}34)$ in $258{\cdot}02$ ft
$18{\cdot}62$ in $258{\cdot}02$ ft

At point *X* in Fig. 7.4, i.e. on line *AB* where the bed level is that of *C*,

Difference in level $AC = 104{\cdot}96 - 97{\cdot}16 = 7{\cdot}80$

$$\therefore \quad \text{Length } AX = 7{\cdot}80 \times \frac{258{\cdot}02}{18{\cdot}62}$$

$$= \underline{108{\cdot}09\text{ ft}}$$

$$BX = 258{\cdot}02 - 108{\cdot}09 = 149{\cdot}93\text{ ft}$$
$$\text{Angle } B = 159°30' - 69°30' = 90°00'$$

In triangle BXC, Angle $BCX(a) = \tan^{-1} BX/BC$
$$= \tan^{-1} 149{\cdot}93/404{\cdot}0$$
$$= 20°22'$$

Therefore the bearing of full dip is perpendicular to the level line CX, i.e.

$$= \text{Bearing } AB + a$$
$$= 69°30' + 180° + 20°22'$$
$$= \underline{269°52'}$$

7.4 The Effect of Errors in Stadia Tacheometry

7.41 Staff tilted from the normal (Fig. 7.5)

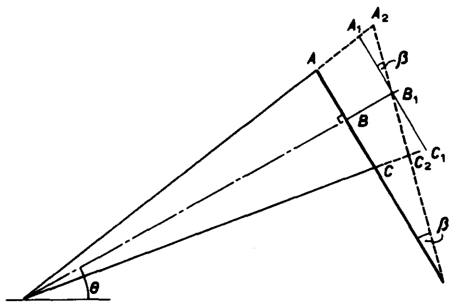


Fig. 7.5 Staff tilted from the normal

If the angle of tilt β is small

then $A_1C_1 \simeq AC = s$

$$A_1C_1 = A_2C_2 \cos\beta$$

i.e. $s = s_1 \cos\beta$

Thus the ratio of error $e = \dfrac{ms_1 - ms}{ms_1}$

$$= 1 - \frac{s}{s_1}$$

$$= \underline{1 - \cos\beta} \qquad (7.17)$$

Thus the error e is independent of the inclination θ.

7.42 Error in the angle of elevation θ with the staff normal

$$H = D\cos\theta + BE\sin\theta$$

Differentiating gives $\dfrac{\delta H}{\delta \theta} = -D \sin\theta + BE \cos\theta$

$$\underline{\delta H = (-D \sin\theta + BE \cos\theta)\,\delta\theta} \qquad (7.18)$$

7.43 **Staff tilted from the vertical** (Fig. 7.6)

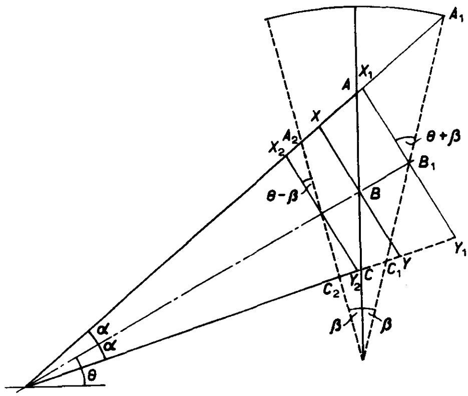


Fig. 7.6 Staff tilted from the vertical

Consider the staff readings on the vertical staff at A, B and C, Fig. 7.6. If the staff is inclined at an angle β away from the observer, the position of the staff normal to the line of collimation will be at XY when vertical and X_1Y_1 when normal to the collimation at the intersection with the inclined staff.

Assuming that

$$BXX_1 = BYC = B_1X_1A_1 = B_1Y_1Y \simeq 90°$$

then, with angle θ an *elevation*,

$$XY = AC\cos\theta = s\cos\theta$$

$$X_1Y_1 = A_1C_1\cos(\theta+\beta) = s_1\cos(\theta+\beta)$$

Assuming that $XY \simeq X_1Y_1$, then

$$s\cos\theta \simeq s_1\cos(\theta+\beta)$$

$$\therefore \quad s = \frac{s_1\cos(\theta+\beta)}{\cos\theta} \qquad (7.19)$$

i.e. the reading s on the staff if it had been held vertically compared with the actual reading s_1 taken on to the inclined staff.

Similarly; if the staff is inclined towards the observer,

$$s = \frac{s_1\cos(\theta-\beta)}{\cos\theta} \qquad (7.20)$$

If the angle θ is a *depression* the equations have the opposite sense, i.e.

Away from the observer $$s = \frac{s_1 \cos(\theta - \beta)}{\cos\theta} \qquad (7.21)$$

Towards the observer $$s = \frac{s_1 \cos(\theta + \beta)}{\cos\theta} \qquad (7.22)$$

Thus the general expression may be written as

$$s = \frac{s_1 \cos(\theta \pm \beta)}{\cos\theta} \qquad (7.23)$$

The error e in the horizontal length due to reading s_1 instead of s is thus shown as

$$\text{True length} = H_T = ms\cos^2\theta$$

$$= \frac{ms_1 \cos(\theta \pm \beta)}{\cos\theta}\cos^2\theta \qquad (7.24)$$

$$\text{Apparent length} = H_A = ms_1\cos^2\theta$$

$$\text{Error } e = H_T - H_A = ms_1\cos^2\theta\left[\frac{\cos(\theta \pm \beta)}{\cos\theta} - 1\right] \qquad (7.25)$$

$$\text{The error expressed as a ratio} = \frac{H_T - H_A}{H_A}$$

$$= \frac{ms_1\cos^2\theta\left[\frac{\cos(\theta \pm \beta)}{\cos\theta} - 1\right]}{ms_1\cos^2\theta}$$

$$= \frac{\cos(\theta \pm \beta)}{\cos\theta} - 1 \qquad (7.26)$$

$$= \frac{\cos\theta\cos\beta \mp \sin\theta\sin\beta - \cos\theta}{\cos\theta}$$

$$= \cos\beta \pm \tan\theta\sin\beta - 1$$

If β is small, $<5°$, then $e = \beta\tan\theta$.

Example 7.5 In a tacheometric survey an intercept of 2·47 ft. was recorded on a staff which was believed to be vertical and the vertical angle measured on the theodolite was 15°. Actually the staff which was 12 ft long was 5 in out of plumb and leaning backwards away from the instrument position.

Assuming it was an anallatic instrument with a multiplying constant of 100, what would have been the error in the computed horizontal distance?

In what conditions will the effect of not holding the staff vertical but at the same time assuming it to be vertical be most serious? What alternative procedure can be adopted in such conditions?

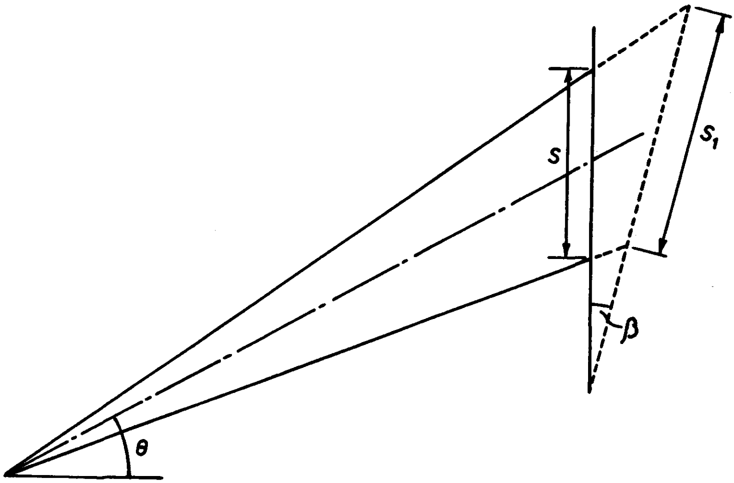


Fig. 7.7

By Eq. (7.19),
$$s = \frac{s_1 \cos(\theta + \beta)}{\cos\theta}$$

$$\beta = \tan^{-1} 5''/12 \times 12$$
$$= 1^\circ 59' 20''$$

Thus
$$s = \frac{2{\cdot}47 \cos(15^\circ + 1^\circ 59' 20'')}{\cos 15^\circ}$$
$$= 2{\cdot}445\,6$$

By Eq. (7.12),
$$H = ms \cos^2\theta$$
$$\delta H = m \cos^2\theta\, \delta s$$
$$= 100 \times \cos^2 15^\circ \times (2{\cdot}47 - 2{\cdot}445\,6)$$
$$= 2{\cdot}44 \cos^2 15$$
$$= 2{\cdot}28 \text{ ft}$$

Alternatively,
By Eq.(7.25),
$$\delta H = ms_1 \cos^2\theta \left[\frac{\cos(\theta + \beta)}{\cos\theta} - 1\right]$$
$$= 247 \cos^2 15 \left[\frac{\cos 16^\circ 59' 20''}{\cos 15^\circ} - 1\right]$$
$$= 230{\cdot}130\,3\ [0{\cdot}990\,09 - 1]$$
$$= \underline{2{\cdot}28 \text{ ft}}$$

7.44 Accuracy of the vertical angle θ to conform to the overall accuracy (Assuming an accuracy of 1/1000)

From $\quad H = ms\cos^2\theta$

differentiation gives $\quad \delta H = -2ms\cos\theta\sin\theta\,\delta\theta$

For the ratio
$$\frac{\delta H}{H} = \frac{1}{1000} = \frac{2ms\cos\theta\sin\theta\,\delta\theta}{ms\cos^2\theta}$$

$$\delta\theta = \frac{\cos\theta}{2\sin\theta\times 1000}$$

$$= \frac{1}{2000}\cot\theta$$

If $\theta = 30°$,
$$\delta\theta = \frac{206\,265\cot 30°}{2000}\text{ seconds}$$

$$= 178\text{ seconds; i.e.} \simeq \underline{3\text{ minutes}}$$

N.B. 1 in 1000 represents 0·1 in 100 ft. The staff is graduated to 0·01 ft but as the multiplying factor is usually 100 this would represent 1 ft.

If estimating to the nearest 0·01 ft the maximum error = ±0·005 ft.

Thus taking the average error as ±0·0025 for sighting the two stadii,

$$\text{Average error} = 0{\cdot}0025\sqrt{2}$$
$$= \pm 0{\cdot}0035$$

$\therefore$ Error in distance (H) due to reading

$$= \pm 0{\cdot}0035\, m_1\cos^2\theta$$

The effect is greater as $\theta \to 0$

Thus, if $m = 100$,
$$\delta H = \pm 0{\cdot}35\text{ ft}$$

If
$$H = 100\text{ ft}$$

$$\frac{\delta H}{H} \simeq \frac{1}{300}$$

From
$$V = D\sin\theta$$
$$\delta V = D\cos\theta\,\delta\theta$$
$$\frac{\delta V}{V} = \frac{D\cos\theta\,\delta\theta}{D\sin\theta} = \cot\theta\,\delta\theta$$

If $\dfrac{\delta V}{V} = \dfrac{1}{1000} = \cot\theta\,\delta\theta$:

when $\theta = 45°$,
$$\delta\theta = \frac{206\,265}{1000} = 206\text{ sec} = 3\text{ min }26\text{ sec}$$

when $\theta = 10°$,

$$\delta\theta = \frac{206\,265 \tan 10°}{1000} = \frac{206\,265 \times 0{\cdot}176\,3}{1000}$$

$$= \underline{36 \text{ sec}}$$

7.45 The effect of the stadia intercept assumption

(i.e. assuming $B\widehat{A_1}A = B\widehat{C_1}C = 90°$, Fig. 7.8)

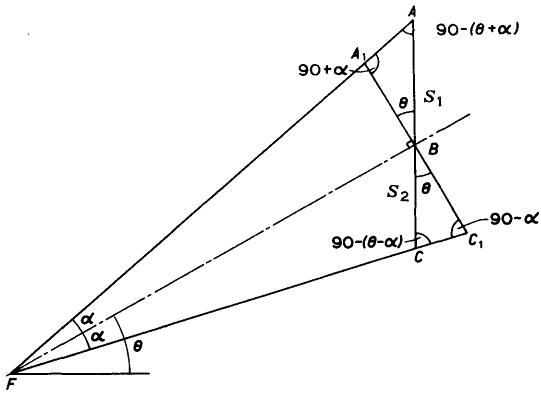


Fig. 7.8

Let the multiplying factor $m = 100$

Then $\alpha = \tan^{-1}\dfrac{1}{200} = \dfrac{206265}{200}$ sec

$$= 0°17'11{\cdot}35''$$

$$2\alpha = \underline{0°34'23''}$$

In triangle BA_1A $\quad A_1B = \dfrac{s_1 \sin[90 - (\theta + \alpha)]}{\sin(90 + \alpha)}$

$$= \frac{s_1 \cos(\theta + \alpha)}{\cos\alpha} = \frac{s_1 (\cos\theta\cos\alpha - \sin\theta\sin\alpha)}{\cos\alpha}$$

In triangle BC_1C $\quad BC_1 = \dfrac{s_2 \sin[90 - (\theta - \alpha)]}{\sin(90 - \alpha)}$

$$= \frac{s_2 \cos(\theta - \alpha)}{\cos\alpha} = \frac{s_2(\cos\theta\cos\alpha + \sin\theta\sin\alpha)}{\cos\alpha}$$

$$A_1B + BC_1 = \frac{s_1(\cos\theta\cos\alpha - \sin\theta\sin\alpha)}{\cos\alpha} + \frac{s_2(\cos\theta\cos\alpha + \sin\theta\sin\alpha)}{\cos\alpha}$$

$$= s_1(\cos\theta - \sin\theta\tan\alpha) + s_2(\cos\theta + \sin\theta\tan\alpha)$$

$$A_1C_1 = (s_1 + s_2)\cos\theta + (s_2 - s_1)(\sin\theta\tan\alpha) \quad (7.27)$$

Thus the accuracy of assuming $A_1C_1 = AC\cos\theta$ depends on the second term $(s_2 - s_1)(\sin\theta\tan\alpha)$.

Example 7.6 (see Fig. 7.8)

If $\theta = 30°$, $FB = 1000$ ft., $m = 100$ and $K = 0$,

$$A_1B = BC_1 = \frac{1000}{200} = 5{\cdot}0\text{ ft}$$

$$s_1 = \frac{A_1B}{\cos\theta - \sin\theta\tan\alpha}$$

$$= \frac{5{\cdot}0}{0{\cdot}866 - 0{\cdot}5 \times 0{\cdot}005}$$

$$= \frac{5{\cdot}0}{0{\cdot}866 - 0{\cdot}0025}$$

$$= \underline{5{\cdot}7904}$$

Similarly

$$s_2 = \frac{BC_1}{\cos\theta + \sin\theta\tan\alpha}$$

$$= \frac{5{\cdot}0}{0{\cdot}866 + 0{\cdot}0025}$$

$$= \underline{5{\cdot}7571}$$

Therefore the effect of ignoring the second term

$$(s_2 - s_1)(\sin\theta\tan\alpha) = (5{\cdot}7904 - 5{\cdot}7571)(0{\cdot}0025)$$

$$= -0{\cdot}0333 \times 0{\cdot}0025$$

$$= -8{\cdot}325 \times 10^{-5}$$

The inaccuracy in the measurement FB thus

$$= -8{\cdot}325 \times 10^{-2}$$

$$\simeq 0{\cdot}1\text{ ft in }1000\text{ ft}$$

and the effect is negligible.

Thus the relative accuracy is very dependent on the ability to estimate the stadia readings. For very short distances the staff must be read to 0·001 ft, whilst as the distances increase beyond clear reading distance the accuracy will again diminish.

Example 7.7

A theodolite has a tacheometric multiplying constant of 100 and an additive constant of zero. The centre reading on a vertical staff held at a point B was 7·64 ft when sighted from A. If the vertical

angle was $+25°$ and the horizontal distance AB 634·42 ft, calculate the other staff readings and show that the two intercept intervals are not equal.

Using these values, calculate the level of B if A is 126·50 ft A.O.D. and the height of the instrument 4·50 ft.

(L.U.)

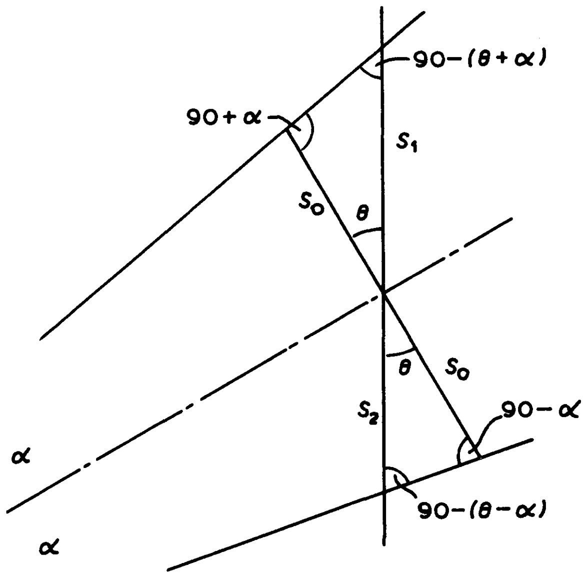


Fig. 7.9

$$\text{Horizontal distance} = ms\cos^2\theta$$

$$\therefore \quad s = \frac{HD}{m\cos^2\theta} = \frac{634{\cdot}42}{100\cos^2 25°}$$

$$= \underline{7{\cdot}72\text{ ft}}$$

$$\text{Inclined distance} = HD\sec\theta$$

$$= 634{\cdot}42\sec 25°$$

$$= \underline{700{\cdot}00\text{ ft}}$$

$$\therefore \quad s_0 = 3{\cdot}50$$

$$s_1 = \frac{s_0\cos\alpha}{\cos(\theta+\alpha)} \quad \text{(sine rule)}$$

$$= \frac{3{\cdot}50\cos 0°17'11''}{\cos 25°17'11''}$$

$$= \underline{3{\cdot}87}$$

Similarly, $s_2 = \dfrac{s_0 \cos a}{\cos(\theta - a)}$ (sine rule)

$$= \frac{3{\cdot}50 \cos 0^\circ 17'11''}{\cos 24^\circ 42'49''}$$

$$= \underline{3{\cdot}85}$$

Check $3{\cdot}87 + 3{\cdot}85 = 7{\cdot}72$ ft

$\therefore$ Staff readings are

$$\begin{array}{rr} 7{\cdot}64 & 7{\cdot}64 \\ +3{\cdot}87 & \text{and} \quad -3{\cdot}85 \\ \hline \text{Upper} \quad 11{\cdot}51 & \text{Lower} \quad 3{\cdot}79 \\ -3{\cdot}79 & \\ \hline \end{array}$$

Check s $\quad \underline{7{\cdot}72}$

Vertical difference $= HD \tan\theta$

$= 634{\cdot}42 \tan 25^\circ$

$= +295{\cdot}84$ ft

Level of $A = 126{\cdot}50$

$+295{\cdot}84$

$+\ \ 4{\cdot}50$

$+426{\cdot}84$

$-\ \ 7{\cdot}64$

Level of $B = \underline{+419{\cdot}20}$

Example 7.8 Two sets of tacheometric readings were taken from an instrument station A, the reduced level of which was 15·05 ft., to a staff station B.

(a) Instrument P – multiplying constant 100, additive constant 14·4 in, staff held vertical.

(b) Instrument Q – multiplying constant 95, additive constant 15·0 in, staff held normal to line of sight.

Inst	At	To	Height of Inst.	Vertical Angle	Stadia Readings
P	A	B	4·52	30°	2·37/3·31/4·27
Q	A	B	4·47	30°	

What should be the stadia readings with instrument Q? (L.U.)

To find level of B (using instrument P)

By Eq. (7.10),

$$V = ms \sin\theta \cos\theta + K \sin\theta$$

$$\therefore \qquad V_1 = 100 \times (4{\cdot}27 - 2{\cdot}37) \sin 30^\circ \cos 30^\circ + 1{\cdot}2 \sin 30$$
$$= 190 \times 0{\cdot}5 \times 0{\cdot}8660 + 1{\cdot}2 \times 0{\cdot}5$$
$$= 82{\cdot}27 + 0{\cdot}60 = 82{\cdot}87 \text{ ft}$$

By Eq. (7.9),
$$H_1 = ms \cos^2\theta + K \cos\theta$$
$$= 190 \times 0{\cdot}866\,03^2 + 1{\cdot}2 \times 0{\cdot}866\,03:$$
$$= 142{\cdot}49 + 1{\cdot}03 = 143{\cdot}52 \text{ ft}$$

Also by Eq. (7.11),
$$V_1 = H_1 \tan\theta$$
$$= 143{\cdot}52 \times 0{\cdot}57735 = 82{\cdot}86 \text{ ft} \qquad \textit{(Check)}$$

Level of B $= 15{\cdot}05 + \text{Ht of inst} + V - \text{middle staff reading}$
$$= 15{\cdot}05 + 4{\cdot}52 + 82{\cdot}87 - 3{\cdot}31 = \underline{99{\cdot}13 \text{ ft}}$$

Using instrument Q

In Fig 7.2,
$$V = (H - BE \sin\theta) \tan\theta$$
$$\therefore \qquad V_2 = (143{\cdot}52 - BE \sin 30^\circ) \tan 30^\circ$$
$$= 143{\cdot}52 \times 0{\cdot}577\,35 - BE \times 0{\cdot}5 \times 0{\cdot}577\,35$$
$$= 82{\cdot}86 - 0{\cdot}288\,68\, BE$$

Level of B $= 15{\cdot}05 + 4{\cdot}47 + V_2 - BE \cos\theta = 99{\cdot}13$
$$= (82{\cdot}86 - 0{\cdot}288\,68\, BE) - 0{\cdot}866\,03\, BE = 79{\cdot}61$$
$$-1{\cdot}154\,71\, BE = -3{\cdot}25$$
$$BE = \frac{-3{\cdot}25}{1{\cdot}154\,71}$$

$$\underline{\text{middle reading}} = \underline{2{\cdot}81}$$

By Eq. (7.5),
$$H_2 = ms \cos\theta + BE \sin\theta$$
$$= 95 \times 0{\cdot}866\,03\, s + 2{\cdot}81 \times 0{\cdot}5 = 143{\cdot}52$$
$$= 82{\cdot}27\, s = 143{\cdot}52 - 1{\cdot}40$$
$$\therefore \qquad s = \frac{142{\cdot}12}{82{\cdot}27} = 1{\cdot}727$$
$$\tfrac{1}{2}s = 0{\cdot}86$$

$\therefore$ Readings are $2{\cdot}81 \pm 0{\cdot}86 = \underline{1{\cdot}95/2{\cdot}81/3{\cdot}67}$

Example 7.9 Three points A, B and C lie on the centre line of an existing mine roadway. A theodolite is set up at B and the following observations were taken on to a vertical staff.

Staff at	Horizontal Circle	Vertical Circle	Staff Readings Stadia	Collimation
A	002°10′20″	+2°10′	6·83/4·43	5·63
C	135°24′40″	−1°24′	7·46/4·12	5·79

If the multiplying constant is 100 and the additive constant zero calculate:

(a) the radius of the circular curve which will pass through A, B and C.

(b) the gradient of the track laid from A to C if the instrument height is 5·16.

(R.I.C.S.)

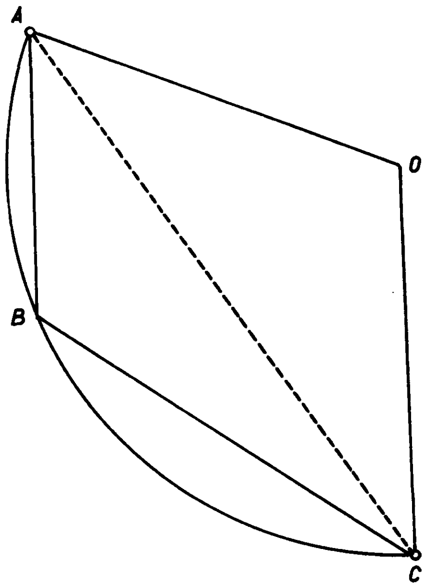


Fig. 7.10

Assumed bearing BA = 002°10′20″

BC = 135°24′40″

Angle ABC = $\underline{133°14'20''}$

Angle AOC = 360 − 2(133°14′20″)

= $\underline{93°31'20''}$

Line AB

$$\text{Horizontal length } (H) = ms\cos^2\theta$$

$$= 100(6{\cdot}83 - 4{\cdot}43)\cos^2 2°10'$$

$$= 240\cos^2 2°10'$$

$$= \underline{239{\cdot}66\text{ ft}}$$

$$\text{Vertical difference } (V) = H \tan\theta$$
$$= 239{\cdot}66 \tan 2°10'$$
$$= \underline{+9{\cdot}07 \text{ ft}}$$

Line BC

$$H = 100\,(7{\cdot}46 - 4{\cdot}12) \cos^2 1°24'$$
$$= \underline{333{\cdot}80 \text{ ft}}$$
$$V = 333{\cdot}80 \tan 1°24'$$
$$= \underline{8{\cdot}16 \text{ ft}}$$

In triangle ABC

$$\text{Tan}\,\frac{A-C}{2} = \frac{a-c}{a+c}\tan\frac{A+C}{2}$$
$$= \frac{333{\cdot}80 - 239{\cdot}66}{333{\cdot}80 + 239{\cdot}66}\tan\frac{180 - 133°14'20''}{2}$$
$$= \frac{94{\cdot}14}{573{\cdot}46}\tan 23°22'50''$$
$$\frac{A-C}{2} = 4°03'35''$$
$$\frac{A+C}{2} = 23°22'50''$$
$$\therefore \quad \hat{A} = 27°26'25''$$
$$\hat{C} = 19°19'15''$$
$$\frac{AB}{\sin C} = 2R \qquad \text{(sine rule)}$$
$$\therefore \quad R = \frac{239{\cdot}66}{2\sin 19°19'15''}$$
$$= \underline{362{\cdot}18 \text{ ft}}$$

Differences in level

$$BA = 5{\cdot}16 + 9{\cdot}07 - 5{\cdot}63$$
$$= \underline{+8{\cdot}60}$$
$$BC = 5{\cdot}16 - 8{\cdot}16 - 5{\cdot}79$$
$$= \underline{-8{\cdot}79}$$
$$\underline{AC = 17{\cdot}39}$$

Length of arc

$$AC = 362{\cdot}18 \times 93°31'20'' \text{ rad}$$
$$= 362{\cdot}18 \times 1{\cdot}632\,271$$
$$= \underline{591{\cdot}18 \text{ ft}}$$

$$\text{Gradient} = 17{\cdot}39\text{ ft in } 591{\cdot}18\text{ ft}$$
$$= \underline{1 \text{ in } 34}$$

Example 7.10 The following observations were taken during a tacheometric survey using the stadia lines of a theodolite (multiplying constant 100, no additive constant.)

Station Set at	Station Observed	Staff Readings U	M	L	Vertical Angle	Bearing
B	*A*	5·62	6·92	8·22	+5°32′	026°36′
	C	3·14	4·45	5·76	−6°46′	174°18′

Calculate (a) the horizontal lengths *AB* and *BC*.
(b) the difference in level between *A* and *C*.
(c) the horizontal length *AC*.

Line BA

$$s = 8{\cdot}22 - 5{\cdot}62 = 2{\cdot}60$$

$$\text{Horizontal length} = 100\, s \cos^2\theta$$
$$= 100 \times 2{\cdot}60 \cos^2 5°32'$$
$$= \underline{257{\cdot}58\text{ ft}}$$

$$\text{Vertical difference} = H \tan\theta$$
$$= 257{\cdot}58 \tan 5°32'$$
$$= \underline{+24{\cdot}95\text{ ft}}$$

Line BC

$$s = 5{\cdot}76 - 3{\cdot}14 = 2{\cdot}62$$
$$H = 100 \times 2{\cdot}62 \cos^2 6°46'$$
$$= \underline{258{\cdot}36\text{ ft}}$$
$$V = 258{\cdot}36 \tan 6°46'$$
$$= \underline{-30{\cdot}66\text{ ft}}$$

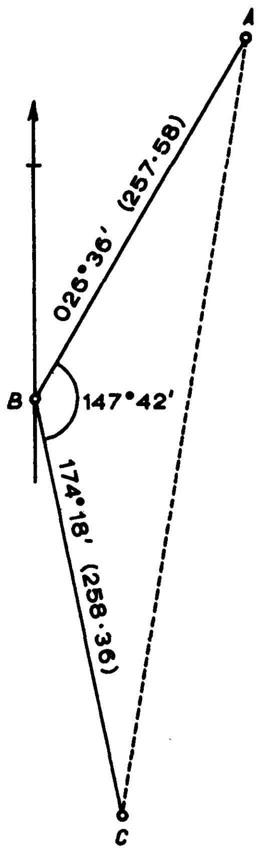


Fig. 7.11

Relative levels

A	+24·95	
	− 6·92	
A	+18·03	above *B*
	−30·66	
	− 4·45	
C	−35·11	below *B*

Difference in level *A-C* 53·14 ft

In triangle ABC,

$$\text{Tan}\ \frac{A-C}{2} = \frac{258{\cdot}36 - 257{\cdot}58}{258{\cdot}36 + 257{\cdot}58}\ \tan\frac{180 - 147°42'}{2}$$

$$\frac{A-C}{2} = 0°01'$$

$$\frac{A+C}{2} = 16°09'$$

$$\therefore \quad A = 16°10'$$

$$\text{Length } AC = 258{\cdot}36 \sin 147°42' \operatorname{cosec} 16°10' = \underline{495{\cdot}82}$$

Exercises 7(a)

1. P and Q are two points on opposite banks of a river about 100 yd wide. A level with an anallatic telescope and a constant of 100 is set up at A on the line QP produced, then at B on the line PQ produced and the following readings taken on to a graduated staff held vertically at P and Q.

What is the true difference in level between P and Q and what is the collimation error of the level expressed in seconds of arc, there being 206 265 seconds in a radian.

		Staff readings in feet		
From	To	Upper Stadia	Collimation	Lower Stadia
A	P	5·14	4·67	4·20
	Q	3·27	1·21	below ground
B	P	10·63	8·51	6·39
	Q	5·26	4·73	4·20

(I.C.E. Ans. 3·62 ft; 104″ above horizontal)

2. Readings taken with a tacheometer that has a multiplying constant of 100 and an additive constant of 2·0 ft were recorded as follows:

Instrument at	Staff at	Vertical Angle	Stadia Readings	Remarks
P	Q	30°00′ elevation	5·73, 6·65, 7·57	Vertical staff

Although the calculations were made on the assumption that the staff was vertical, it was in fact made at right angles to the collimation.

Compute the errors, caused by the mistake, in the calculation of horizontal and vertical distances from the instrument to the foot of the staff. Give the sign of each error.

If the collimation is not horizontal, is it preferable to have the staff vertical or at right angles to the collimation? Give reasons for

your preference.

(I.C.E. Ans. Horizontal error −24·7 ft, Vertical error −13·2 ft)

3. The following readings were taken with an anallatic tacheometer set up at each station in turn and a staff held vertically on the forward station, the forward station from D being A.

Station	Height of Instrument	Stadia Readings	Inclination (elevation +ve)
A	4·43	4·93 3·54 2·15	+0°54′
B	4·61	5·96 4·75 3·54	−2°54′
C	4·74	5·15 3·72 2·29	+2°48′
D	4·59	6·07 4·64 3·21	−1°48′

The reduced level of A is 172·0 ft and the constant of the tacheometer is 100.

Determine the reduced levels of B, C and D, adjusted to close on A, indicating and justifying your method of adjustment.

(I.C.E. Ans. 177·5; 165·4; 180·7)

4. The focal lengths of the object glass and anallatic lens are 5 in and 4½ in respectively. The stadia interval was 0·1 in.

A field test with vertical staffing yielded the following:

Instrument Station	Staff Station	Staff Intercept	Vertical Angle	Measured Horizontal Distance (ft)
P	Q	2·30	+7°24′	224·7
	R	6·11	−4°42′	602·3

Find the distance between the object glass and anallatic lens. How far and in what direction must the latter be moved so that the multiplying constant of the instrument is to be 100 exactly.

(L.U. 0·02 in away from objective)

5. Sighted horizontally a tacheometer reads $r_1 = 6{\cdot}71$ and $r_3 = 8{\cdot}71$ on a vertical staff 361·25 ft away. The focal length of the object glass is 9 in. and the distance from the object glass to the trunnion axis 6 in.
Calculate the stadia interval. (I.C.E. Ans. 0·05 in)

6. With a tacheometer stationed at X sights were taken on three points, A, B, and C as follows:

Instrument at	To	Vertical Angle	Stadia Readings	Remarks
X	A	−4°30′	7·93/6·94/5·95	R.L. of A = 357·09 (Staff normal to line of sight
	B	0°00′	4·55/3·54/2·54	R.L. of B = 375·95 (Staff vertical)
	C	+2°30′	8·85/5·62/2·39	Staff vertical

The telescope was of the draw-tube type and the focal length of the object glass was 10 in. For the sights to A and B, which were of equal length, the distance of the object glass from the vertical axis was 4·65 in.

Derive any formulae you use. Calculate (a) the spacing of the cross hairs in the diaphragm and (b) the reduced level of C.

(L.U. Ans. 0·102 in.; 401·7 ft)

7. The following readings were taken on a vertical staff with a tacheometer fitted with an anallatic lens and having a constant of 100:

Staff Station	Bearing	Stadia Readings	Vertical Angle
A	27°30′	2·82 4·50 6·18	+8°00′
B	207°30′	2·54 6·00 9·46	−5°00′

Calculate the reduced levels of the ground at A and B, and the mean slope between A and B.

(L.U. Ans. +41·81; −66·08; 1 in 9·42)

8. Tacheometric readings were taken from a survey station S to a staff held vertically at two pegs A and B, and the following readings were recorded:

Point	Horizontal Circle	Vertical Circle	Stadia Readings
A	62°00′	+4°10′30″	4·10/6·17/8·24
B	152°00′	−5°05′00″	2·89/6·17/9·45

The multiplying constant of the instrument was 100 and the additive constant zero. Calculate the horizontal distance from A to B and the height of peg A above the axis level of the instrument.

(I.C.E. Ans. 770·1 ft; 23·89 ft)

9. In a tacheometric survey made with an instrument whose constants were $f/i = 100$, $(f+d) = 1·5$, the staff was held inclined so as to be normal to the line of sight for each reading. How is the correct inclination assured in the field?

Two sets of readings were as given below. Calculate the gradient between the staff stations C and D and the reduced level of each. The reduced level of station A was 125·40 ft.

Instrument at	Staff at	Height of Instrument	Azimuth	Vertical Angle	Stadia Readings
A	C	4·80	44°	+4°30′	3·00/4·25/5·50
	D		97°	−4°00′	3·00/4·97/6·94

(L.U. Ans. 1 in 6·57)

10. (a) A telescope with tacheometric constants m and c is set up at A and sighted on a staff held vertically at B. Assuming the usual relationship $D = ms + c$ derive expressions for the horizontal and vertical distances between A and B.

(b) An instrument at A, sighted on to a vertical staff held at B and C, in turn gave the following readings:

Sight	Horizontal Circle	Vertical Circle	Staff Readings (ft)
B	05°20′	+4°29′00″	1·45/2·44/3·43
C	95°20′	−0°11′40″	2·15/3·15/4·15

If the instrument constants are $m = 100$, $c = 0$, calculate the gradient of the straight line BC.

(N.U. Ans. 1 in 16·63)

7.5 Subtense systems

7.51 Tangential method (with fixed intercept s and variable vertical angles α and β)

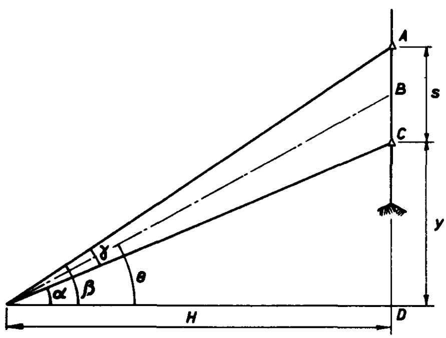


Fig. 7.12 Tangential method

From Fig. 7.12

$$DC = y = H \tan \alpha$$
$$AD = s + y = H \tan \beta$$
$$AC = s = H(\tan \beta - \tan \alpha)$$

$$\therefore \quad H = \frac{s}{\tan \beta - \tan \alpha} \tag{7.28}$$

Alternatively, as $\gamma = \beta - \alpha$,

$$\frac{s}{\sin \gamma} = \frac{H/\cos \alpha}{\sin(90 - \beta)}$$

$$\therefore \quad \underline{H = s \cos \alpha \cos \beta \operatorname{cosec} \gamma} \tag{7.29}$$

This equation (7.29) was modified by M. Geisler (*Survey Review*, Oct. 1964) as follows:

$$H = \frac{s}{\sin\gamma}\cos\alpha\cos(\gamma+\alpha)$$

$$= \frac{s}{\sin\gamma}\,\frac{\cos(2\alpha+\gamma)+\cos\gamma}{2}$$

$$= \frac{s}{\sin\gamma}\left[\frac{1+\cos\gamma)-\{1-\cos(2\alpha+\gamma)\}}{2}\right]$$

$$= \frac{s}{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}\left[\cos^2\frac{\gamma}{2} - \sin^2\left(\alpha+\frac{\gamma}{2}\right)\right]$$

$$= \frac{1}{2}s\cot\frac{\gamma}{2} - \frac{1}{2}s\cot\frac{\gamma}{2}\left\{\frac{\sin^2\left(\alpha+\frac{\gamma}{2}\right)}{\cos^2\frac{\gamma}{2}}\right\}$$

As γ is small, $\cos^2\frac{\gamma}{2} \simeq 1$.

Also $\quad \alpha+\frac{\gamma}{2} = \theta$

$\therefore$

$$H = \frac{1}{2}s\cot\frac{\gamma}{2} - \frac{1}{2}s\cot\frac{\gamma}{2}\sin^2\theta$$

$$= \frac{1}{2}s\cot\frac{\gamma}{2}\,(1-\sin^2\theta)$$

$$= \frac{1}{2}s\cot\frac{\gamma}{2}\cos^2\theta \qquad (7.30)$$

Alternatively, the above equation may be derived by reference to Fig. 7.13.

$$H_1 = \frac{1}{2}s_1\cot\frac{\gamma}{2} \qquad \text{where } s_1 = A_1C_1$$

but $\quad s_1 \simeq s\cos\theta$ (assuming A_1AB and BC_1C are similar figures)

and $\quad H = H_1\cos\theta$

$\therefore$ $\quad H = \frac{1}{2}s\cot\frac{\gamma}{2}\cos^2\theta$ as above

N.B. In the term $-\frac{1}{2}s\cot\frac{\gamma}{2}\,\frac{\sin^2\left(\alpha+\frac{\gamma}{2}\right)}{\cos^2\frac{\gamma}{2}}$ $\cot\frac{\gamma}{2}$ is very large,

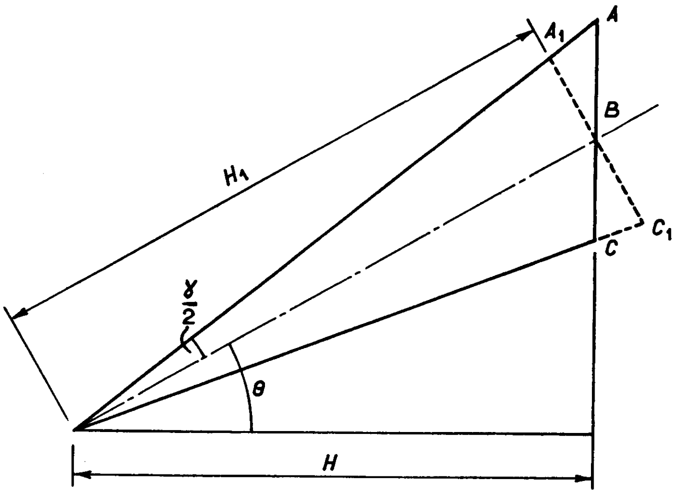


Fig. 7.13

so that any approximation to $\cos^2\frac{\gamma}{2}$ is greatly magnified and the following approximation is preferred:

As before $$H = \frac{s}{\sin\gamma}\cos\alpha\,\cos(\gamma+\alpha)$$

$$= \frac{s}{\sin\gamma}\cos\left(\theta-\frac{\gamma}{2}\right)\cos\left(\theta+\frac{\gamma}{2}\right) \qquad \left(\alpha = \theta-\frac{\gamma}{2}\right)$$

$$= \frac{s}{\sin\gamma}\left[\frac{\cos 2\theta + \cos\gamma}{2}\right]$$

but $$\cos\gamma \simeq 1-\frac{\gamma^2}{2} \simeq 1$$

$$\therefore \quad H \simeq \frac{s}{\sin\gamma}\left[\frac{\cos 2\theta + 1}{2}\right]$$

$$\simeq \frac{s\cos^2\theta}{\sin\gamma}$$

$$\underline{H \simeq s\,\mathrm{cosec}\,\gamma\cos^2\theta} \qquad (7.31)$$

Geisler suggests that by using special targets on the staff, thus ensuring the accuracy of the value of s, and the use of a $1''$ theodolite, a relative accuracy up to 1/5000 may be attained. He improved the efficiency of the operation by using prepared tables and graphs relative to his equation.

The accuracy of the method is affected by:

(1) An error in the length of the intercept s.

(2) An error in the vertical angle.

(3) Tilt of the staff from the vertical.

(1) *Error in the intercept s*

This depends on (a) error in the graduation, (b) the degree of precision of the target attachment.

$$\delta H = \frac{H\,\delta s}{s} \tag{7.32}$$

(2) *Error in the vertical angle*

From Eq. (7.28), $$H = \frac{s}{\tan\beta - \tan\alpha}$$

$$\therefore \quad \delta H_\alpha = \frac{+s\sec^2\alpha\,\delta\alpha}{(\tan\beta - \tan\alpha)^2}$$

$$\delta H_\beta = \frac{-s\sec^2\beta\,\delta\beta}{(\tan\beta - \tan\alpha)^2} \tag{7.33}$$

$\therefore$ total r.m.s. error δH

$$= \sqrt{\delta H_\alpha^2 + \delta H_\beta^2} = \frac{s}{(\tan\beta - \tan\alpha)^2}[\sec^4\alpha\,\delta\alpha^2 + \sec^4\beta\,\delta\beta^2]^{\frac{1}{2}}$$

If $\delta\alpha = \delta\beta$,

$$\delta H = \frac{s\,\delta\alpha}{(\tan\beta - \tan\alpha)^2}[\sec^4\alpha + \sec^4\beta]^{\frac{1}{2}}$$

$$= \frac{H^2\,\delta\alpha}{s}[\sec^4\alpha + \sec^4\beta]^{\frac{1}{2}} \tag{7.34}$$

If α and β are small,

$$\delta H = \frac{\sqrt{2}H^2\,\delta\alpha}{s} \tag{7.35}$$

(3) *Tilt of the staff*

The effect here is the same as that described in Section 7.43, i.e.

$$s = \frac{s_1\cos(\theta \pm \beta)}{\cos\theta}$$

where β = tilt of the staff from the vertical.

Example 7.11 A theodolite was set over station A, with a reduced level of 148·73 ft A.O.D., the instrument height being 4·74 ft. Observations were taken to the 10 ft and 2 ft marks on a staff held vertical at three stations with the following results:

Instrument Station	*Station Observed*	*Vertical Angles*	
		Top	Bottom
A	B	+9°10′	+3°30′
A	C	+1°54′	−2°24′
A	D	−5°15′	−12°10′

Find the distance from A to each station and also their reduced levels.

(E.M.E.U.)

By Eq. (7.28),

$$\text{Horizontal distance} \quad AB = \frac{s}{\tan\beta - \tan\alpha}$$

$$= \frac{8}{\tan 9°10' - \tan 3°30'}$$

$$= \frac{8}{0{\cdot}100\,21} = \underline{79{\cdot}84 \text{ ft}}$$

$$\text{Vertical distance} = AB \tan 3°30' = 4{\cdot}883 \text{ ft}$$

$$\therefore \quad \text{Level of } B = 4{\cdot}88 + 4{\cdot}74 + 148{\cdot}73 - 2{\cdot}00 \text{ ft}$$

$$= 156{\cdot}35 \text{ ft} \quad \text{A.O.D.}$$

$$\text{Horizontal distance } AC = \frac{8}{\tan 1°54' + \tan 2°24'}$$

$$= \frac{8}{0{\cdot}075\,09} = \underline{106{\cdot}55 \text{ ft}}$$

$$\text{Vertical distance} = -106{\cdot}55 \tan 2°24' = -4{\cdot}47 \text{ ft}$$

$$\text{Level of } C = 148{\cdot}73 + 4{\cdot}74 - 4{\cdot}47 - 2{\cdot}00$$

$$\underline{= 147{\cdot}00 \text{ ft} \quad \text{A.O.D.}}$$

$$\text{Horizontal distance} \quad AD = \frac{8}{\tan 12°10' - \tan 5°15'}$$

$$= \frac{8}{0{\cdot}123\,71} = \underline{64{\cdot}53 \text{ ft}}$$

$$\text{Vertical distance} = -64{\cdot}53 \tan 12°10' = -13{\cdot}91 \text{ ft}$$

$$\text{Level of } D = 148{\cdot}73 + 4{\cdot}74 - 13{\cdot}91 - 2{\cdot}00 \text{ ft}$$

$$\underline{= 137{\cdot}56 \text{ ft} \quad \text{A.O.D.}}$$

Alternative solutions

By Eq. (7.29),

$$\text{Horizontal distance} \quad AD = \frac{8 \cos 9°10' \cos 3°30'}{\sin(9°10' - 3°30')}$$

$$= \frac{8 \times 0{\cdot}987\,23 \times 0{\cdot}998\,14}{0{\cdot}098\,74}$$

$$\underline{= 79{\cdot}84 \text{ ft}}$$

or by Eq. (7.30),

$$AB = \tfrac{1}{2} \times 8 \times \cot 2°50' \times \cos^2 \tfrac{1}{2}(12°40')$$

$$= 4 \cot 2°50' \cos^2 6°20'$$

$$= 4 \times 20{\cdot}2056 \times 0{\cdot}99\,390$$
$$= \underline{79{\cdot}84 \text{ ft}}$$

7.52 Horizontal subtense bar system (Fig. 7.14)

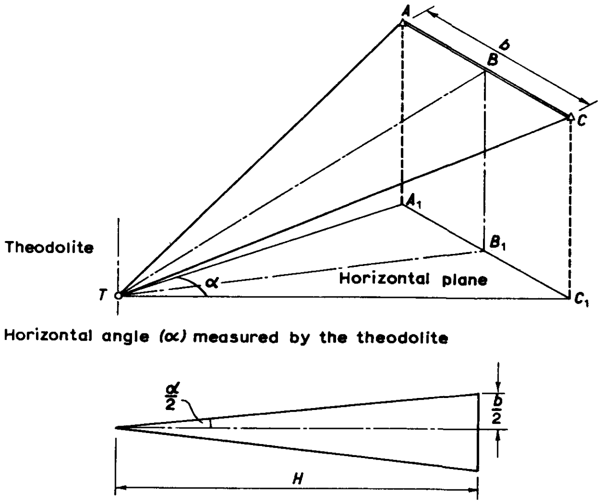


Fig. 7.14 Horizontal subtense bar system

The horizontal bar of known length b, usually 2 metres, is set perpendicular to the line of sight TB. Targets at A and C are successively sighted and the angle α, which is measured in the horizontal plane, recorded.

The horizontal distance $TB_1 = H$ is then obtained

$$H = \frac{b}{2} \cot \frac{\alpha}{2} \tag{7.36}$$

If the bar is 2 metres long,

$$H = \cot \frac{\alpha}{2} \text{ metres} \tag{7.37}$$

The horizontal angle α is not dependent on the altitude of the bar relative to the theodolite.

N.B. As the bar is horizontal, readings on one face only are necessary.

Factors affecting the accuracy of the result are

(1) *The effect of an error in the subtended angle* α

By Eq. (7.36),

$$H = \frac{b}{2} \cot \frac{\alpha}{2}$$

$$= \frac{b}{2\tan\frac{\alpha}{2}}$$

If α is small, then $\tan\frac{\alpha}{2} \simeq \frac{\alpha}{2}$ radians.

$$\therefore \qquad H = \frac{b}{\alpha} \tag{7.38}$$

Differentiating with respect to α,

$$\delta H_\alpha = \frac{-b\,\delta\alpha}{\alpha^2} \tag{7.39}$$

but

$$\alpha = \frac{b}{H}$$

$$\therefore \qquad \delta H_\alpha = \frac{-H^2\,\delta\alpha}{b} \tag{7.40}$$

The error ratio

$$\frac{\delta H_\alpha}{H} = \frac{-\delta\alpha}{\alpha} \tag{7.41}$$

where $\delta\alpha$ and α are expressed in the same units.

If $\delta\alpha = \pm 1''$ and $b = 2\text{m}$,

then by Eq. (7.40),

$$\delta H = \pm \frac{H^2 \times 1}{2 \times 206\,265} \text{ metres}$$

$$\delta H = \pm \frac{H^2}{412\,530} \text{ metres} \tag{7.42}$$

$$\simeq \pm \frac{H^2}{400\,000} \text{ metres} \tag{7.43}$$

Example 7.12 To what accuracy should the subtense angle α be measured to a bar 2 metres long if the length of sight is approximately 50 metres and a fractional error of 1/10 000 must not be exceeded?

By Eq. (7.41),

$$\frac{\delta H}{H} = \frac{\delta\alpha}{\alpha} = \frac{1}{10\,000}$$

$$\therefore \qquad \delta\alpha = \frac{\alpha}{10\,000}$$

but

$$\alpha = \frac{b}{H}$$

$$\therefore \qquad \delta\alpha = \frac{b}{10\,000\,H}$$

$$= \pm \frac{2 \times 206\,265}{10\,000 \times 50} \text{ seconds}$$

$$= \underline{\pm\ 0{\cdot}825''}$$

Example 7.13 If the measured angle a is approximately 2°, how accurately must it be recorded if the fractional linear error must not exceed 1/10 000?

$$\frac{\delta H}{H} = \frac{\delta a}{a} = \frac{1}{10\,000}$$

$$\delta a = \frac{2 \times 3600}{10\,000}$$

$$= \underline{\pm 0{\cdot}72''}$$

(2) *The effect of an error in the length of the bar*

From Eq. (7.38),

$$H = \frac{b}{a}$$

$$\therefore \qquad \delta H_b = \frac{\delta b}{a} \tag{7.44}$$

$$\frac{\delta H_b}{H} = \frac{\delta b}{b} \tag{7.45}$$

If the error ratio is not to exceed 1/10 000, then

$$\frac{\delta H_b}{H} = \frac{\delta b}{b} = \frac{1}{10\,000}$$

As the bar = 2 metres = 2000 mm, δb is limited to 0·2 mm.

The bar is usually made of invar steel and guaranteed by the manufacturer to ± 0·05 mm. It requires a change of 20°C for these limits and thus for most uses the bar is considered constant.

(3) *The effect of an error in the orientation of the bar*

Consider half the bar, Fig. 7.15.

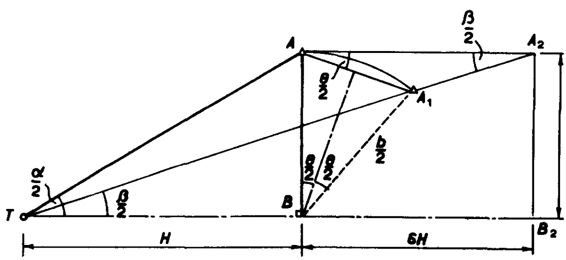


Fig. 7.15 Orientation of the bar

Let the half bar AB be rotated through an angle θ to A_1B. The line of sight will thus be assumed to be at A_2.

$$AA_1 = 2\times\frac{b}{2}\sin\frac{\theta}{2} = b\sin\frac{\theta}{2}$$

$$AA_2 = \delta H = \frac{AA_1\sin\left(\frac{\theta}{2}+\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{b\sin\frac{\theta}{2}\left\{\sin\frac{\theta}{2}\cos\frac{\beta}{2}+\cos\frac{\theta}{2}\sin\frac{\beta}{2}\right\}}{\sin\frac{\beta}{2}}$$

$$= b\sin^2\frac{\theta}{2}\cot\frac{\beta}{2}+b\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

but $$b\cot\frac{\beta}{2} \simeq 2H$$

$$\therefore \quad \delta H = 2H\sin^2\frac{\theta}{2}+\frac{b}{2}\sin\theta \tag{7.46}$$

Neglecting the second term as both $\frac{b}{2}$ and θ are small,

$$\frac{\delta H}{H} = 2\sin^2\frac{\theta}{2} \tag{7.47}$$

If the relative accuracy is limited to 1/10 000, then

$$\frac{\delta H}{H} = \frac{1}{10\,000} = 2\sin^2\frac{\theta}{2}$$
$$\sin^2\theta = 5\times 10^{-4}$$
$$\theta = 1°17'$$

As the bar usually has a sighting device, it can be oriented far more accurately than to the above limit, and this non-rigorous analysis shows that this effect can be ignored.

The accuracy of the whole system is thus entirely dependent on the angle α.

Assuming the angle α can be measured to $\pm 1''$ and the bar is 2 metres,

$$\frac{\delta\alpha}{\alpha} = \frac{1}{10\,000}$$

$$\therefore \quad \alpha = 10\,000''$$

$$\therefore \qquad H = \frac{b}{\alpha} = \frac{2 \times 206\,265}{10\,000} \text{ metres}$$

$$= \underline{41 \cdot 25 \text{ m}}$$

For most practical purposes, for an accuracy of ± 1 cm, the distance can be increased to 75 metres.

To increase the range of the instrument various processes may be used, and these are described in the next section.

7.6 Methods used in the field

7.61 Serial measurement (Fig. 7.16)

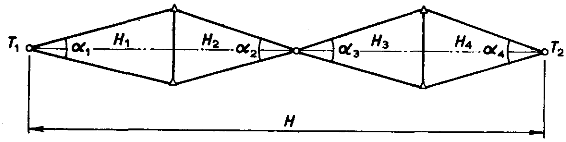


Fig. 7.16 Serial measurement

$$T_1 T_2 = H_1 + H_2 + \dots = H$$

$$= \frac{b}{2}\left[\cot\frac{\alpha_1}{2} + \cot\frac{\alpha_2}{2} + \dots\right]$$

By Eq. (7.40),

$$\delta H_1 = \frac{H_1^2\,\delta\alpha_1}{b}$$

$$\text{Total r.m.s. error} = \sqrt{\Sigma(\delta H)^2} = \left[\left(\frac{H_1^2\,\delta\alpha_1}{b}\right)^2 + \left(\frac{H_2^2\,\delta\alpha_2}{b}\right)^2 + \dots\right]^{\frac{1}{2}} \qquad (7.48)$$

If $H_1 = H_2 = H_n = \dfrac{H}{n}$, $\alpha_1 = \alpha_2 = \alpha_n$ and $\delta\alpha_1 = \delta\alpha_2 = \delta_n$,

$$\Sigma\,\delta H = \pm\sqrt{n}\,\frac{H^2\,\delta\alpha}{n^2\,b}$$

$$= \pm\frac{H^2\,\delta\alpha}{n^{3/2}\,b} \qquad (7.49)$$

If $b = 2$ metres, $\delta\alpha = \pm 1''$ and $H = nH_1$

$$\text{Total} \quad \delta H = \pm\frac{H^2}{2 \times 206\,265 \times n^{3/2}}$$

$$= \pm\frac{H^2}{412\,530\,n^{3/2}} \qquad (7.50)$$

$$\simeq \pm \frac{H^2}{400\,000\, n^{3/2}} \quad (7.51)$$

The error ratio $$\frac{\delta H}{H} \simeq \frac{H}{400\,000\, n^{3/2}} \quad (7.52)$$

If $\delta\alpha = \pm 1''$, $b = 2$ metres, $n = 2$ and $\frac{\delta H}{H} = 1/10\,000$

then $$\frac{H}{400\,000\; 2^{3/2}} = \frac{1}{10\,000}$$

$$H = \frac{400\,000 \times 2{\cdot}83}{10\,000} = 113 \text{ metres}$$

7.62 Auxiliary base measurement (Fig. 7.17)

For lines in excess of 150 m, an auxiliary base of 20 – 30 m may be set out at right angles to the traverse line, Fig. 7.17.

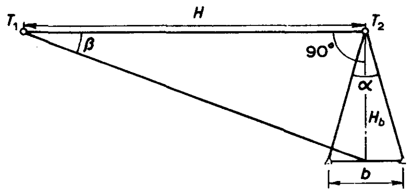


Fig. 7.17 Auxiliary base measurement

Angles α and β are measured

$$H_b = \frac{b}{2} \cot \frac{\alpha}{2}$$

$$\therefore \qquad H = H_b \cot \beta$$

$$= \frac{b}{2} \cot \frac{\alpha}{2} \cot \beta \quad (7.53)$$

$$= \frac{b}{\alpha} \cot \beta \quad (7.54)$$

Differentiating,

$$\delta H_\alpha = -\frac{b \cot \beta}{\alpha^2}\, \delta\alpha$$

$$\delta H_\beta = -\frac{b \operatorname{cosec}^2 \beta\; \delta\beta}{\alpha}$$

Total r.m.s. error $\delta H = \left[\dfrac{b^2 \cot^2 \beta \, \delta\alpha^2}{\alpha^4} + \dfrac{b^2 \operatorname{cosec}^4 \beta \, \delta\beta^2}{\alpha^2}\right]^{\frac{1}{2}}$ (7.55)

If α and β are both small,

$$\delta H = \left[\frac{b^2 \, \delta\alpha^2}{\alpha^4 \beta^2} + \frac{b^2 \, \delta\beta^2}{\alpha^2 \beta^4}\right]^{\frac{1}{2}} \tag{7.56}$$

but $$H = \frac{b}{\alpha\beta}$$

$$\therefore \quad \delta H = \left[\frac{H^2 \, \delta\alpha^2}{\alpha^2} + \frac{H^2 \, \delta\beta^2}{\beta^2}\right]^{\frac{1}{2}} \tag{7.57}$$

If $\alpha = \beta$ and $\delta\alpha = \delta\beta$,

$$\delta H = \frac{\sqrt{2}\, H \, \delta\alpha}{\alpha} \tag{7.58}$$

As $H_b = \dfrac{b}{\alpha}$ and $H = \dfrac{H_b}{\beta}$

$\alpha = \dfrac{b}{H_b}$, and $\beta = \dfrac{H_b}{H}$

and as $\alpha = \beta$,

$$\frac{H_b}{H} = \frac{b}{H_b}$$

$$H_b = \sqrt{(bH)} \tag{7.59}$$

and $$\alpha = \frac{b}{\sqrt{(bH)}} = \sqrt{\frac{b}{H}}$$

$$\therefore \quad \delta H = \frac{\sqrt{2}\, H^{3/2} \delta\alpha}{\sqrt{b}} \tag{7.60}$$

If $b = 2$ metres and $\delta\alpha = \pm 1''$,

$$\delta H = \frac{H^{3/2}}{206\,265} \tag{7.61}$$

and the fractional error $\dfrac{\delta H}{H} = \dfrac{\sqrt{H}}{206\,265}$ (7.62)

If $\dfrac{\delta H}{H} = 1/10\,000$,

then $\sqrt{H} = 20{\cdot}6265$

$H = \underline{410 \text{ metres}}$

the sub-base $H_b = \sqrt{(2H)}$

$= \sqrt{(2 \times 410)}$

$= \underline{28{\cdot}7 \text{ metres}}$

7.63 Central auxiliary base (Fig. 7.18)

For lines in excess of 400 metres, a double bay system may be adopted with the auxiliary base in the middle.

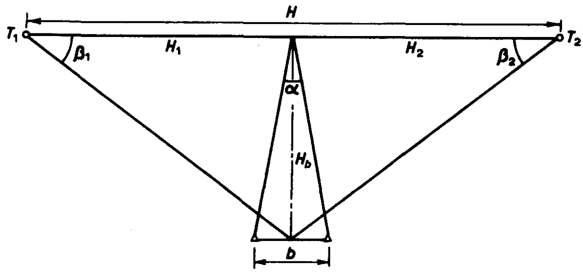


Fig. 7.18 Central auxiliary base

Length $T_1T_2 = H = H_1 + H_2$

$$= \frac{b}{2}\cot\frac{\alpha}{2}\cot\beta_1 + \frac{b}{2}\cot\frac{\alpha}{2}\cot\beta_2$$

$$= \frac{b}{2}\cot\frac{\alpha}{2}[\cot\beta_1 + \cot\beta_2] \tag{7.63}$$

$$= \frac{b}{a}[\cot\beta_1 + \cot\beta_2] \tag{7.64}$$

If α, β_1 and β_2 are each small,

$$H = \frac{b}{a}\left[\frac{1}{\beta_1} + \frac{1}{\beta_2}\right] \tag{7.65}$$

Differentiating,

$$\delta H_\alpha = -\frac{b}{a^2}\left[\frac{1}{\beta_1} + \frac{1}{\beta_2}\right]\delta\alpha$$

$$\delta H_{\beta_1} = -\frac{b}{a\beta_1^2}\,\delta\beta_1$$

$$\delta H_{\beta_2} = -\frac{b}{a\beta_2^2}\,\delta\beta_2$$

Total r.m.s. error $\delta H = \sqrt{\{\delta H_a^2 + \delta H_{\beta_1}^2 + \delta H_{\beta_2}^2\}}$

$$= \frac{b}{a}\sqrt{\left\{\frac{\delta a^2}{a^2}\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)^2+\frac{\delta\beta_1^2}{\beta_1^4}+\frac{\delta\beta_2^2}{\beta_2^4}\right\}} \quad (7.66)$$

If $a = \beta_1 = \beta_2$ and $\delta a = \delta\beta_1 = \delta\beta_2$, then

$$\delta H = \frac{b}{a}\sqrt{\left\{\frac{\delta a^2}{a^2}\left(\frac{2^2}{a^2}+\frac{1}{a^2}+\frac{1}{a^2}\right)\right\}}$$

$$= \frac{\sqrt{6}\, b\,\delta a}{a^3} \quad (7.67)$$

but by Eq. (7.65) $\quad H = \dfrac{2b}{a^2} \qquad (H_1 = H_2)$

$$\therefore \quad a = \sqrt{\frac{2b}{H}}$$

$$\delta H = \frac{\sqrt{6}\, H^{3/2}\, b\,\delta a}{2^{3/2}\, b^{3/2}}$$

$$= \frac{\sqrt{3}\, H^{3/2}\,\delta a}{2\sqrt{b}} \quad (7.68)$$

If $b = 2\,\text{m}$ and $\delta a = \pm 1''$,

$$\delta H = \frac{\sqrt{3}\, H^{3/2}}{2\sqrt{2}\times 206\,265}$$

$$= \frac{H^{3/2}}{336\,818} \quad (7.69)$$

$$\delta H \simeq \frac{H^{3/2}}{350\,000} \quad (7.70)$$

If $\delta H/H = 1/10\,000$,

$$\frac{\delta H}{H} = \frac{1}{10\,000} = \frac{\sqrt{H}}{350\,000}$$

$$\therefore \quad \sqrt{H} \simeq 35$$

$$\underline{H \simeq 1225 \text{ metres}}$$

The auxiliary base $\quad H_b = H_b = \sqrt{H}$

$$= \underline{35 \text{ metres.}}$$

7.64 Auxiliary base perpendicularly bisected by the traverse line (Fig. 7.19)

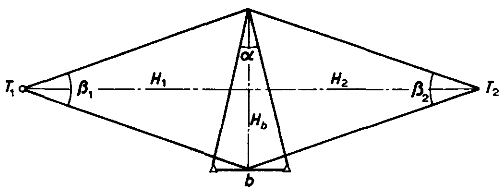


Fig. 7.19 Auxiliary base bisected by the traverse line

Here
$$H_b = \frac{b}{2}\cot\frac{\alpha}{2}$$

$$H_1 = \frac{H_b}{2}\cot\frac{\beta_1}{2}$$

$$H_2 = \frac{H_b}{2}\cot\frac{\beta_2}{2}$$

$\therefore$
$$H = H_1 + H_2$$

$$= \frac{H_b}{2}\left[\cot\frac{\beta_1}{2} + \cot\frac{\beta_2}{2}\right]$$

$$= \frac{b}{4}\cot\frac{\alpha}{2}\left[\cot\frac{\beta_1}{2} + \cot\frac{\beta_2}{2}\right] \tag{7.71}$$

If α, β_1 and β_2 are all small,

$$H = \frac{b}{2\alpha}\left[\frac{2}{\beta_1} + \frac{2}{\beta_2}\right]$$

$$= \frac{b}{\alpha}\left[\frac{1}{\beta_1} + \frac{1}{\beta_2}\right] \tag{7.72}$$

$$\delta H_\alpha = -\frac{b}{\alpha^2}\left[\frac{1}{\beta_1} + \frac{1}{\beta_2}\right]\delta\alpha$$

$$\delta H_{\beta_1} = \frac{-b}{\alpha\beta_1^2}\,\delta\beta_1$$

$$\delta H_{\beta_2} = \frac{-b}{\alpha\beta_2^2}\,\delta\beta_2$$

Total r.m.s. error $\delta H = \sqrt{\{\delta H_\alpha^2 + \delta H_{\beta_1}^2 + \delta H_{\beta_2}^2\}}$

$$= \frac{b}{a}\sqrt{\left\{\frac{\delta a^2}{a^2}\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)^2 + \frac{\delta\beta_1^2}{\beta_1^4} + \frac{\delta\beta_2^2}{\beta_2^4}\right\}} \qquad (7.73)$$

If $a = \beta_1 = \beta_2$ and $\delta a = \delta\beta_1 = \delta\beta_2$,

$$\delta H = \frac{b}{\alpha}\sqrt{\left\{\frac{\delta a^2}{a^2}\frac{4}{a^2} + \frac{\delta a^2}{a^4} + \frac{\delta a^2}{a^4}\right\}}$$

$$= \frac{\sqrt{6}\, b\delta a}{a^3}$$

but $$H = \frac{2b}{a^2}$$

$\therefore$ $$a = \sqrt{\frac{2b}{H}}$$

$$\delta H = \frac{\sqrt{6}\, bH^{3/2}\delta a}{2^{3/2}\, b^{3/2}}$$

$$= \frac{\sqrt{3}\, H^{3/2}\delta a}{2\sqrt{b}} \qquad (7.74)$$

N.B. This is the same value as for the central auxiliary base (7.68).

7.65 With two auxiliary bases (Fig. 7.20)

The auxiliary base H_b is extended twice to H.

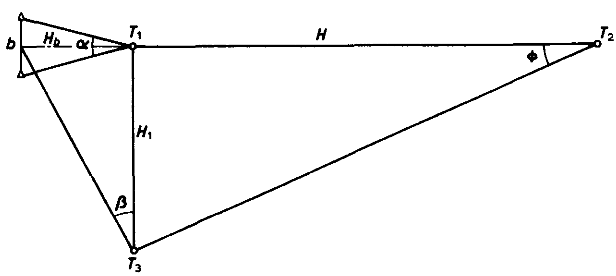


Fig. 7.20 Two auxiliary bases

Here $$H_b = \frac{b}{2}\cot\frac{\alpha}{2}$$

$$H_1 = H_b \cot\beta$$

$$H = H_1 \cot\phi$$

$$\therefore \qquad H = \frac{b}{2}\cot\frac{\alpha}{2}\cot\beta\cot\phi \tag{7.75}$$

$$H = \frac{b}{\alpha}\cot\beta\cot\phi \tag{7.76}$$

If α, β and ϕ are all small,

$$H = \frac{b}{\alpha\beta\phi}$$

$$\delta H = -\frac{b\delta\alpha}{\alpha^2\beta\phi}$$

$$\delta H_\beta = -\frac{b\delta\beta}{\alpha\beta^2\phi}$$

$$\delta H_\phi = -\frac{b\delta\phi}{\alpha\beta\phi^2}$$

Total r.m.s. error $\delta H = \sqrt{\delta H_\alpha^2 + \delta H_\beta^2 + \delta H_\phi^2}$

$$= \frac{b}{\alpha\beta\phi}\sqrt{\left\{\frac{\delta\alpha^2}{\alpha^2} + \frac{\delta\beta^2}{\beta^2} + \frac{\delta\phi^2}{\phi^2}\right\}} \tag{7.77}$$

If $\alpha = \beta = \phi$ and $\delta\alpha = \delta\beta = \delta\phi$,

i.e. $$\frac{H_b}{b} = \frac{H_1}{H_b} = \frac{H}{H_1}$$

then $$H_1 = \sqrt{H_b H}$$

$$H_b = \frac{H_1^2}{H} = \sqrt{bH_1}$$

$$\delta H = \frac{b}{\alpha^3}\sqrt{\frac{3\delta\alpha^2}{\alpha^2}}$$

$$= \frac{\sqrt{3}\,b\delta\alpha}{\alpha^4}$$

but $$\alpha = \sqrt[3]{\frac{b}{H}}$$

$$\therefore \qquad \delta H = \frac{\sqrt{3}\,b\delta\alpha H^{4/3}}{b^{4/3}}$$

$$= \frac{\sqrt{3}\,H^{4/3}\delta\alpha}{b^{1/3}} \tag{7.78}$$

If $b = 2\text{m}$ and $\delta\alpha = \pm 1''$,

$$\delta H = \frac{\sqrt{3}\,H^{4/3}}{206\,265 \times \;^{1/3}\!\sqrt[3]{2}}$$

$$\simeq \frac{H^{4/3}}{150\,000} \tag{7.79}$$

If $\dfrac{\delta H}{H} = 1/10\,000$,

$$\frac{\delta H}{H} = \frac{1}{10\,000} = \frac{\sqrt[3]{H}}{150\,000}$$

$$\therefore \quad \sqrt[3]{H} = 15$$

$$H = \underline{3375 \text{ metres}}$$

$$H_b = \frac{b}{\alpha}$$

and $$\alpha = \frac{b^{1/3}}{H^{1/3}}$$

$$\therefore \quad H_b = b^{2/3}\,H^{1/3}$$

$$= 2^{2/3}\,H^{1/3}$$

$$= 1{\cdot}5866 \times 15$$

$$= \underline{23{\cdot}8 \text{ metres}}$$

7.66 The auxiliary base used in between two traverse lines (Fig. 7.21)

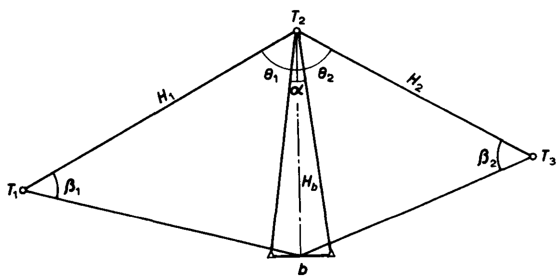


Fig. 7.21 Auxiliary base between two traverse lines

$$H_b = \frac{b}{2}\cot\frac{\alpha}{2}$$

$$H_1 = \frac{H_b \sin(\beta_1 + \theta_1)}{\sin\beta_1}$$

$$= \frac{\frac{b}{2}\cot\frac{a}{2}\sin(\beta_1+\theta_1)}{\sin\beta_1} \quad (7.80)$$

$$= \frac{b\sin(\beta_1+\theta_1)}{a\sin\beta_1} \quad (7.81)$$

Similarly $$H_2 = \frac{\frac{b}{2}\cot\frac{a}{2}\sin(\beta_2+\theta_2)}{\sin\beta_2}$$

$$= \frac{b\sin(\beta_2+\theta_2)}{a\sin\beta_2}$$

Here the errors are not analysed as the lengths and angles are variable.

Example 7.14 A colliery base line *AB* is unavoidably situated on ground where there are numerous obstructions which prevent direct measurement .It was decided to determine the length of *AB* by the method illustrated in Fig. 7.22, where *DE* is a 50 metre band hung in catenary with light targets attached at the zero and 50 metre marks.

From the approximate angular values shown, determine the maximum allowable error in the measurements of the angles such that the projection of error due to these measurements does not exceed:

(a) 1/200 000 of the actual length *CD* when computing *CD* from the length *DE* and the angle *DCE* and

(b) 1/100 000 of the actual length *AB* when computing *AB* from the angles *ACD*, *CDA*, *BDC*, and *DCB* and the length *DC*. For this calculation, assume that the length *DC* is free from error. (R.I.C.S.)

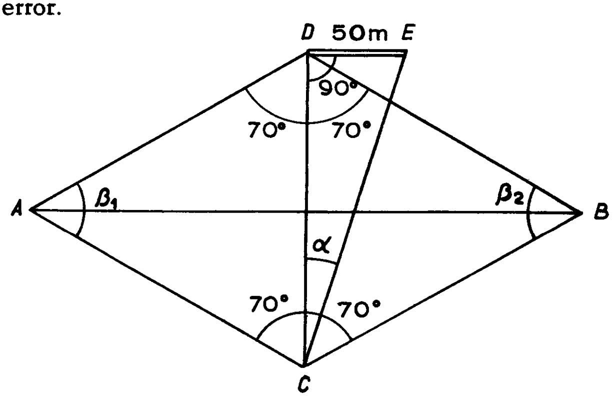


Fig. 7.22

Assuming Angle $DCE(\alpha)$ = Angle $DAB\left(\frac{\beta}{2}\right) = \frac{1}{2}(180-2\times70) = 20°$,

(a)
$$DC = DE\cot\alpha$$

The error $$\delta DC = DE\operatorname{cosec}^2\alpha\,\delta\alpha$$

The error ratio $$\frac{\delta DC}{DC} = \frac{DE\operatorname{cosec}^2\alpha\,\delta\alpha}{DE\cot\alpha}$$

$$= \frac{\delta\alpha}{\sin\alpha\cos\alpha} = \frac{1}{200\,000}$$

$\therefore$ $$\delta\alpha = \frac{206\,265\times\frac{1}{2}\sin 2\alpha}{200\,000}$$

$$= 0{\cdot}51566\sin 40° = \underline{0{\cdot}33 \text{ seconds}}$$

(say 1/3″)

(b)
$$AB = \frac{DC}{2}\left[\cot\frac{\beta_1}{2}+\cot\frac{\beta_2}{2}\right]$$

$$\delta AB_{\beta_1} = \pm\frac{DC}{2}\operatorname{cosec}^2\frac{\beta_1}{2}\delta\beta_1$$

$$\delta AB_{\beta_2} = \pm\frac{DC}{2}\operatorname{cosec}^2\frac{\beta_2}{2}\delta\beta_2$$

Total error $$\delta AB = \pm\frac{DC}{2}\left[\operatorname{cosec}^4\frac{\beta_1}{2}\,\delta\beta_1^2+\operatorname{cosec}^4\frac{\beta_2}{2}\,\delta\beta_2^2\right]^{\frac{1}{2}}$$

but $\beta_1 = \beta_2$ and assuming $\delta\beta_1 = \delta\beta_2$.

$$\delta AB = \pm\frac{DC}{2}\sqrt{2}\operatorname{cosec}^2\frac{\beta}{2}\,\delta\beta$$

The error ratio $$\frac{\delta AB}{AB} = \frac{DC/2\sqrt{2}\operatorname{cosec}^2\beta/2\,\delta\beta}{DC/2\times 2\cot\beta/2}$$

$$= \frac{\sqrt{2}\,\delta\beta}{2\sin\beta/2\cos\beta/2}$$

$$= \frac{\sqrt{2}\,\delta\beta}{\sin\beta} = \frac{1}{100\,000}$$

$\therefore$ $$\delta\beta = \frac{206\,265\sin 40°}{\sqrt{2}\times 100\,000}$$

$$= \underline{0{\cdot}94 \text{ seconds} \quad (\text{say } \pm 1'')}$$

Exercises 7(b)

11. (i) What do you understand by systematic and accidental errors in linear measurement, and how do they affect the assessment of the probable error?

Does the error in the measurement of a particular distance vary in proportion to the distance or to the square root of the distance?

(ii) Assume you have a subtense bar the length of which is known to be exactly 2 metres (6·562 ft) and a theodolite with which horizontal angles can be measured to within a second of arc. In measuring a length of 2000 ft., what error in distance would you get from an angular error of 1 second?

(iii) With the same equipment, how would you measure the distance of 2000 ft in order to achieve an accuracy of about 1/5000?

(Aide memoire: 1 second of arc = 1/206 265 radians.)

(I.C.E. Ans. (ii) $\pm 2{\cdot}95$ ft)

12. (a) When traversing with a 2 metre subtense bar, discuss the methods which can be adopted to measure lines of varying length. Include comments on the relative methods of angular measurement by repetition and reiteration.

(b) A bay length AB is measured with a subtense bar 2 metres in length, approximately midway between and in line with AB.

The mean angle subtended at $A = 1°27'00''$

at $B = 1°35'00''$

Calculate the length AB.

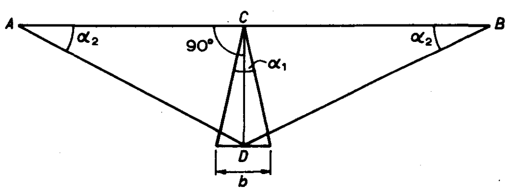
(E.M.E.U. Ans. 151·393 m)

13. The base AB is to be measured using a subtense bar of length b and the double extension layout shown in the figure.

If the standard error of each of the two measured angles is $\pm\,\delta\alpha$ develop a formula for the proportional standard error of the base length.

Find the ratio $a_1 : a_2$ which will give the minimum proportional standard error of the base length.

What assumptions have you made in arriving at your answers?



(R.I.C.S.)

14. (a) Describe, with the aid of sketches, the principles of subtense bar tacheometry.

(b) The sketch shows two adjacent lines of a traverse AB and BD with a common sub-base BC.

Calculate the lengths of the traverse lines from the following data:

Angles	BAC	=	$5°10'30''$
	CBA	=	$68°56'10''$
	YBX	=	$1°56'00''$
	CDB	=	$12°54'20''$
	DBC	=	$73°18'40''$

Length of bar 2 metres.

(E.M.E.U. Ans. AB, 631·96 m; BD, 264·78 m)

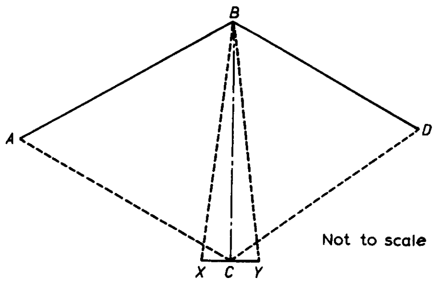


Fig. 7.24

15. Describe the 'single bay' and 'double bay' methods of measuring linear distance by use of the subtense bar.

Show that for a subtense bar

$$L = \frac{S}{2} \cot \phi/2$$

where L = the horizontal distance between stations,

S = length of subtense bar,

ϕ = angle subtended by targets at the theodolite.

Thereafter show that if $S = 2$ metres and an error of $\pm \Delta\phi$ is made in the measurement of the subtended angle, then

$$\frac{\Delta L}{L} = \pm \frac{\Delta\phi L}{2}$$

where ΔL is the corresponding error in the computed length.

Assuming an error of $\pm 1 \sec (\Delta\phi)$ in the measurement of the subtended angle what will be the fractional error at the following lengths?

(a) 50 metres (b) 100 metres (c) 500 metres.

(N.U. Ans. (a) 1/8250; (b) 1/4125; (c) 1/825.)

Exercises 7(c) (General)

16. The following readings were taken with a theodolite set over a station A, on to a staff held vertically on two points B and C.

Inst. St.	Horizontal Circle Reading	Vertical Circle Reading	Stadia Readings U	M	L	Staff St.
A	33°59′55″	+10°48′	8·44	6·25	4·06	B
A	209°55′21″	− 4°05′	7·78	6·95	6·12	C

If the instrumental constant is 100 and there is no additive constant; calculate the horizontal distance BC and the difference in elevation between B and C. (E.M.E.U. Ans. 587·48 ft; 93·11 ft)

17. Readings were taken on a vertical staff held at points A, B, and C with a tacheometer whose constants are 100 and 0. If the horizontal distances from instrument to staff were respectively 153, 212, and 298 ft, and the vertical angles +5°, +6° and −5°, calculate the staff intercepts. If the middle-hair reading was 7·00 ft in each case what was the difference in level between A, B and C?
(L.U. Ans. 7·77/7·00/6·23; 8·07/7·00/5·93; 8·50/7·00/5·50; $A-B$. +8·88; $B-C$. −48·30)

18. A theodolite has a tacheometric multiplying constant of 100 and an additive constant of zero. When set 4·50 ft above a station B, the following readings were obtained:

Station at	Sight	Horizontal Circle	Vertical Circle	Stadia Readings Top	Middle	Bottom
B	A	028°21′00″				
B	C	082°03′00″	+20°30′	3·80	7·64	11·40

The co-ordinates of station A are E 546·2, N 0·0 and those of B are E 546·2 N −394·7.

Find the co-ordinates of C and its height above datum, if the height of station B above datum is 91·01 ft.

(L.U. Ans. 1083·6 E 0·1 N; +337·17 ft)

19. The following readings were obtained in a survey with a level fitted with tacheometric webs, the constant multiplier being 100 and the additive constant zero.

Inst. at	Point	Staff Readings		
A	B.M. 207·56	1·32	2·64	3·96
	B	2·37	3·81	5·25
C	*B*	5·84	7·95	10·06
	D	10·11	11·71	13·31
	E	8·75	9·80	10·85
F	*E*	11·16	13·17	15·18
	T.B.M.	3·78	5·34	6·90

Subsequently the level was tested and the following readings obtained:

Inst. at	Point	Staff Readings		
P	*Q*	4·61	5·36	6·11
	R	3·16	3·91	4·66
S	*Q*	4·12	4·95	5·78
	R	3·09	3·17	3·25

Find the level of the T.B.M.

(L.U. Ans. 212·98 ft)

20. A theodolite was set up at *P*, the end of a survey line on uniformly sloping ground and the readings taken at approximately 100 ft intervals along the line as follows:

At	Point	Elevation Angle	Stadia Readings		
P	*A*	4°16′	3·66	4·16	4·66
	B	4°16′	2·45	3·46	4·47
	C	5°06′	1·30	2·82	4·34
	D	5°06′	5·87	7·88	9·89
	E	5°06′	6·15	8·65	11·15

An error of booking was apparent when reducing the observations. Find this error, the levels of the points *A B C D E* and the gradient *PE*, if the ground level below the instrument was 104·20 O.D. and the height of the instrument 4·75. Instrument constants 100 and 0.
(L.U. Ans. *A* 112·21, *B* 120·48, *C* 128·68, *D* 136·66, *E* 144·57;
Grad. 1 in 12·28)

21. The following data were taken during a survey when stadia readings were taken. The levelling staff was held vertically on the stations. The height above datum of station *A* is 475·5 ft above Ordinance Datum. The multiplying factor of the instrument is 99·5 and the additive constant 1·3 ft. Assume station *A* to be the point of origin and calculate the level above Ordinance Datum of each station and the horizontal distance of each line.

Back Station	Instrument Station	Fore Station	Instrument Height	Horizontal Angle	Vertical Angle	Staff Readings Upper	Middle	Lower
-	*A*	*B*	4·95	-	+5°40′	9·90	8·00	6·10
A	*B*	*C*	5·00	164°55′	+7°00′	8·44	6·61	4·78
B	*C*	*D*	5·10	179°50′	−8°20′	9·20	7·57	5·94

(R.I.C.S. Ans. *A* 475·50, *B* 509·73, *C* 552·33, *D* 502·66; *AB* 375·70, *BC* 360·04, *CD* 322·23)

22. The undermentioned readings were taken with a fixed-hair tacheometer theodolite on a vertical staff. The instrument constant is 100. Calculate the horizontal distance and difference in elevation between the two staves.

Instrument Station	Horizontal Circle	Vertical Circle		Staff Station
X	33°59′55″	+10°48′00″	8·44, 6·25, 4·06	*A*
X	209°55′21″	− 4°05′00″	7·78, 6·95, 6·12	*B*

(MQB/S Ans. 587·5 ft; 93·1 ft)

23. The undernoted readings were taken at the commencement of a tacheometric survey, the multiplying factor of the tacheometer being 100 and the additive constant 1·3 ft.

Calculate the co-ordinates and reduced level of station *D* assuming *A* to be the point of origin and the reduced level there at 657·6 ft above datum. The azimuth of the line *AB* is 205°10′.

(MQB/S Ans. S 893·83 ft; W 469·90 ft; 718·54 ft)

24. The following readings were taken by a theodolite used for tacheometry from a station *B* on to stations *A*, *C* and *D*:

Sight	Horizontal Angle	Vertical Angle	Stadia Readings Top	Centre	Bottom
A	301°10′	-	-	-	-
C	152°36′	−5°00′	3·48	7·61	11·74
D	205°06′	+2°30′	2·15	7·92	13·70

The line *BA* has a bearing of N 28°46′ E and the instrument has a constant multiplier of 100 and an additive constant zero. Find the slope of the line *CD* and its quadrantal bearing.

(L.U. Ans. 1 in 7·57; N 22° 27′ 10″ W)

25. The following observations were taken with a tacheometer (multiplier 100, additive constant 0) from a point A, to B and C.

The distance BC was measured as 157 ft. Assuming the ground to be a plane within the triangle ABC, calculate the volume of filling required to make the area level with the highest point, assuming the sides to be supported by concrete walls.

Height of instrument 4·7 ft, staff held vertically.

At	To	Staff Readings	Vertical Angle
A	B	1·48 2·73 3·98	$+7°36'$
	C	2·08 2·82 3·56	$-5°24'$

(L.U. Ans. 297 600 ft^3)

26. Explain the principles underlying the measurement of a horizontal distance by means of stadia readings.

Using stadia readings, the horizontal distance between two stations A and B is found to be 301·7 ft.

The difference in height between the two stations is 3·17 ft. Calculate the appropriate stadia readings, stating clearly the assumptions you have made.

(R.I.C.S.)

27. (i) Derive expressions for the probable errors of determination of horizontal and vertical distances tacheometrically due to known probable errors ds in the measurement of stadia intercept and da in the measurement of the vertical angle.

It may be assumed that an anallactic instrument is used and the staff is held vertically.

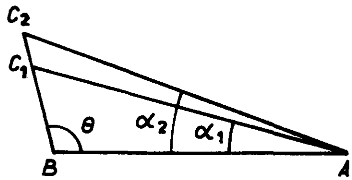
(ii) In a series of tacheometric observations a sight is taken to the top of a building where the vertical angle is about $11°30'$. If the stadia intercept is 2·52 ft and $ds = \pm 0{\cdot}0025$ ft and $da = \pm 1'$ what are the probable errors in determining the horizontal distances? The tacheometric constant is 100.

(iii) Assuming that using a staff graduated to 0·01 ft each stadia hair can be read correctly to the nearest 0·01 ft will such a staff be good enough to give an accuracy of 1 : 500 over distances from 100 to 500 ft, and if not what accuracy can be achieved? It may be assumed that the vertical angles will be small and errors in the vertical angle can be ignored.

(R.I.C.S. Ans. (ii) ±0·24 ft; ±0·03 ft)

28. A third order traverse line AB is measured by the following method: measure angles a_1, a_2, θ shown on the diagram; measure distances AC_1, AC_2, by the angles at A to a subtense bar at C_1 and C_2.

Two measures of AB are thus obtained, their mean being accepted.



A 2 m subtense bar is centred at C_1 and C_2 and oriented at right angles to AC_1 and AC_2.

Observed horizontal angles are as follows.

Subtense angles at A: to C_1 = $1°05'27{\cdot}1''$

to C_2 = $1°04'30{\cdot}0''$

$\theta = 100°\,35'\,33''$ $\alpha_1 = 2°51'27''$; $\alpha_2 = 2°53'55''$

Compute the horizontal distance AB.

(R.I.C.S. Ans. 104·70 m)

29. Describe in detail how you would determine the tacheometric constants of a theodolite in the field. Show how the most probable values could be derived by the method of least squares.

Sighted horizontally a tacheometer reads $r_1 = 6{\cdot}71$ and $r_3 = 8{\cdot}71$ on a vertical staff 361·25 ft away. The focal length of the object glass is 9 in and the distance from the object glass to the trunnion axis 6 in. Calculate the stadia interval.

$$\text{Given} \qquad D = \frac{f}{i}\,s + (f+c)$$

(N.U. Ans. 0·05 in.)

30. Describe the essential features of a subtense bar and show how it is used in the determination of distance by a single measurement. Allowing for a 1 second error in the measurement of the angle, calculate from first principles the accuracy of the measurement of 200 ft if a 2 metre subtense bar is used. Show how the accuracy of such a measurement varies with distance and outline the method by which max imum accuracy will be obtained if subtense tacheometry is used in the determination of the distance between points situated on opposite banks of a river about 600 ft wide.

(I.C.E. Ans. 1 in 6 800)

31. An area of ground was surveyed with a fixed stadia hair tacheometer (constants 100 and 0) set up in turn at each of four stations A, B, C and D. Observations were made with the staff held vertically. $ABCDA$ formed a closed traverse and it was found that the difference in level between these four stations as calculated from the tacheometric readings would not balance.

Later it was realised that a new altitude bubble had recently been fitted to the instrument but unfortunately had not been correctly adjusted. In order to determine the true differences of level between the four stations, a level known to be in perfect order was used.

Fieldbook observations from both surveys are as follows:

Instrument Station	Height of Instrument (ft)	Staff Station	Vertical Angle	Stadia Readings (ft)		
A	4·55	*B*	$+3°$	4·05	5·66	7·27
B	4·60	*C*	$+2°20'$	8·71	11·02	13·33
C	4·70	*D*	$-2°30'$	3·74	5·67	7·60
D	4·50	*A*	0	2·38	4·99	7·60

Backsight	Intermediate sight	Foresight	Remarks
11·78	-	-	Station *A*
10·30	-	5·04	Change point
-	3·19	-	Station *B*
9·84	-	1·65	Change point
3·27	-	1·69	Station *C*
2·83	-	14·78	Change point
-	11·34	-	Station *D*
6·41	-	9·70	Change point
-	-	11·57	Station *A*

Calculate the vertical angles that would have been observed from stations *A* and *C* if the altitude bubble of the tacheometer had been in correct adjustment.

Describe the procedure which should be adopted in correcting the adjustment of the altitude bubble, identifying the type of instrument for which your procedure is appropriate.

(I.C.E. Ans. $+2°40'$; $-2°50'$)

Bibliography

THOMAS, W.N., *Surveying* (Edward Arnold).
BANNISTER, A. and RAYMOND, S., *Surveying* (Pitman).
REDMOND, F.A., *Tacheometry* (Technical Press).
HIGGINS, A.L., *Higher Surveying* (Macmillan).
HODGES, D.J., Articles in *Colliery Engineering*, Dec. 1965 - Oct. 1966.
GEISLER, M., 'Precision tacheometry with vertical subtense bar applied to topographical surveying', *Survey Review*, Oct. 1964.
WATSON and SHADBOLT, 'The subtense bar applied to mine surveying', *Chartered Surveyor*, Vol. 90, No. 11, May 1958.

8 DIP AND FAULT PROBLEMS

Problems on gradients take a number of different forms and may be solved graphically or trigonometrically according to the accuracy required.

8.1 Definitions

Let $ABDE$ represent a plane inclined to the horizontal at $\alpha°$, Fig. 8.1.

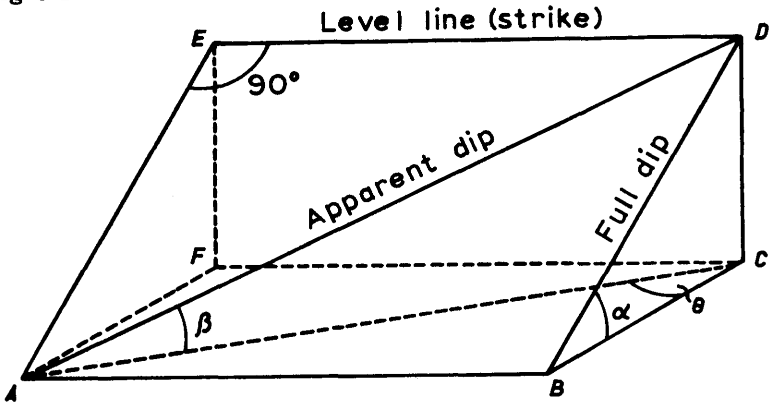


Fig. 8.1

Dip. The dip of a bed, seam or road in any direction is the angle of inclination from the horizontal plane.

It may be expressed as:

(a) An angle from the horizontal, e.g. $6°03'$,

(b) A gradient, 1 vertical in n horizontal (the fraction $1/n$ represents the tangent of the angle of inclination, whilst n represents its cotangent) or

(c) A vertical fall of x inches per horizontal yard, e.g. 3 inches per yard.

N.B. The term *rise* denotes the opposite of dip.

Full Dip (or true dip) is the maximum inclination of any plane from the horizontal and its direction is always at right-angles to the minimum inclination (i.e. nil) or level line known as *strike*.

In Fig. 8.1, lines AE and BD are lines of full dip, whilst ED and AB are level lines or strike lines.

Apparent Dip is the dip observed in any other direction. It is always less than full dip and more than strike. In Fig. 8.1 the line AD is an

apparent dip inclined at an angle of $\beta°$ in a direction $\theta°$ from full dip.

Depth of Strata. The depth of a stratum is generally measured relative to the surface, to Ordnance Mean Sea Level (as used in levelling) or, in order to obtain positive values, may be expressed relative to some arbitrary datum, e.g. the N.C.B. (National Coal Board) datum, which is 10 000 ft below M.S.L.

Thickness of Strata. The true thickness is measured at right-angles to the bedding plane. For inclined strata penetrated by vertical boreholes, an apparent thickness would be derived from the borehole core.

Example 8.1 A vertical borehole passes through a seam which is known to dip at 40°. If the apparent thickness of the seam as shown by the borehole core is 5 ft calculate: (a) the true thickness of the seam; (b) the gradient of the seam.

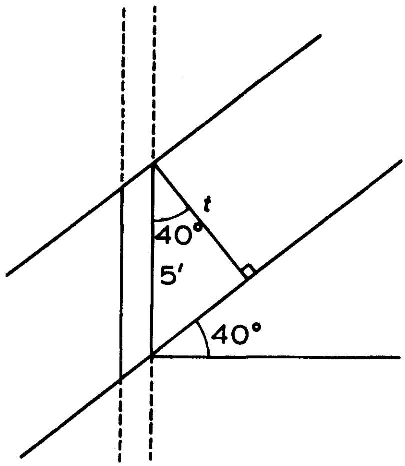


Fig. 8.2

(a) True thickness $t = 5 \cos 40° = \underline{3{\cdot}83\text{ ft}}$

$= \underline{3\text{ ft } 10\text{ in.}}$

(b) Gradient of seam $\cot 40° = 1{\cdot}192$

$\therefore$ Gradient is 1 in 1·192

Example 8.2 If a seam dips at 1 in 4 what is the true area of one square mile in plan

$$\text{Cot}\,\theta = 4 \quad \therefore \quad \theta = 14°\,02'$$

$\therefore$ True length of one dipping side

$$= \frac{1760}{\cos\theta}$$

$$= \frac{1760}{0{\cdot}970\,15}\ \text{yd}$$

True area

$$= \frac{1760^2}{0{\cdot}97015} = \underline{3\,192\,908\ \text{sq. yd}}$$

compared with 3 097 600 sq. yd in plan.

8.2 Dip Problems

8.21 Given the rate and direction of full dip, to find the apparent dip in any other direction

Let Fig. 8.3 represent the plan.

Graphical Solution

Draw AB and AC representing strike and full dip. Let the length AC be x units long. As the line AC dips at 1 in x, C will be 1 unit vertically below A. Draw the strike line CE through C, which is now 1 unit below strike line AB.

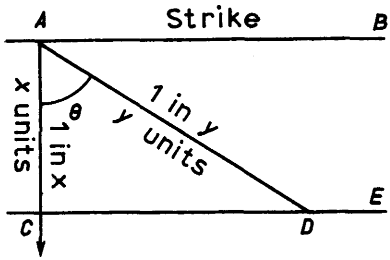


Fig. 8.3

Any line starting from A will then be 1 unit vertically below A when it it cuts the line CE, and its gradient will be 1 in y, where y is the length measured in the same units.

Trigonometrical solution

Triangle ADC is right-angled at C

$$\therefore \quad AD = \frac{AC}{\cos\theta}$$

$$\text{i.e.} \quad y = \frac{x}{\cos\theta} \qquad (8.1)$$

$$\text{or} \quad x = y\cos\theta \qquad (8.2)$$

i.e. the gradient value of full dip x = the gradient value of apparent dip y × cosine of the angle between.

8.22 Given the direction of full dip and the rate and direction of an apparent dip, to find the rate of full dip (Fig. 8.4)

Graphical solution

Draw directions AB and AC of full dip and apparent dip respectively.

Let $AC = y$ units.

Draw CD perpendicular to AB through C cutting the full dip

direction at B.

Length $AB = x$ units is the gradient equivalent of full dip.

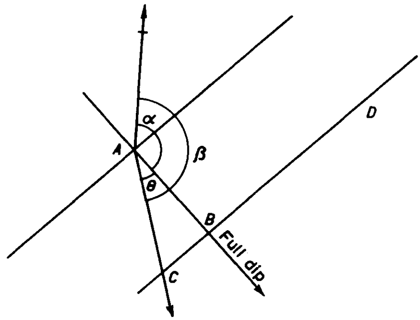


Fig. 8.4

Trigonometrical solution

$$AB = AC\cos\theta$$

$$\text{i.e.} \quad x = y\cos(\beta - \alpha) \tag{8.3}$$

i.e. cot full dip = cot apparent dip cosine angle between (8.4)
or tan apparent dip = tan full dip cosine angle between (8.5)

Example 8.3 The full dip of a seam is 1 in 4 N 30° E. Find the gradients of roadways driven in the seam

(a) due N, (b) N 75° E, (c) due E.

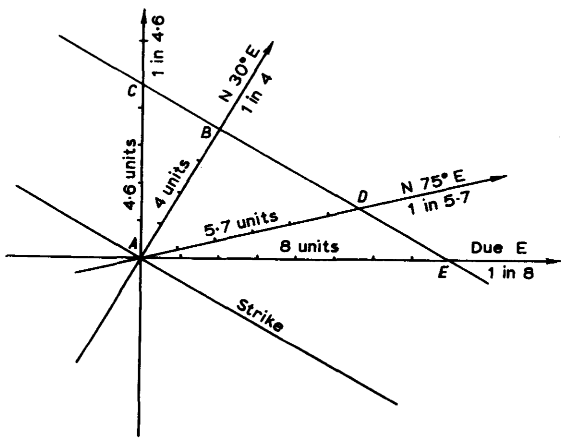


Fig. 8.5

Graphically,

(a) AC due N 1 in 4·6
(b) AD N 75° E 1 in 5·7
(c) AE due E 1 in 8

Trigonometrically,

$$\text{(a) } AC = \frac{4}{\cos 30^\circ} = \frac{4}{0{\cdot}8660} = 4{\cdot}618 \quad (12^\circ\ 13')$$

$$\text{(b) } AD = \frac{4}{\cos 45^\circ} = \frac{4}{0{\cdot}7071} = 5{\cdot}657 \quad (10^\circ\ 02')$$

$$\text{(c) } AE = \frac{4}{\cos 60^\circ} = \frac{4}{0{\cdot}5} = 8{\cdot}0 \quad (7^\circ\ 08')$$

N.B. (a) Lines at 45° to full dip have gradients 1 in $\sqrt{2}x$,

e.g. 1 in $\sqrt{2} \times 4$ = $\underline{1 \text{ in } 5{\cdot}657}$

(b) Lines at 60° to full dip have double the gradient value, i.e. 1 in $2x$,

e.g. 1 in 2×4 = $\underline{1 \text{ in } 8}$

8.23 Given the rate and direction of full dip, to find the bearing of an apparent dip (Fig. 8.6)

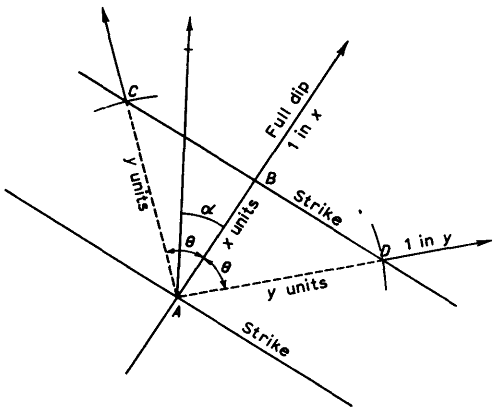


Fig. 8.6

This is the converse of 8.22 but it should be noted that there are two directions in which a given apparent dip occurs.

Graphical solution

Plot full dip, direction α°, of length x units as before.

Draw strike lines through A and B.

With centre A draw arcs of length y to cut strike line through B, giving θ° on either side of AB as at AC and AD.

Trigonometrical solution

$$\cos\theta = \frac{x}{y}$$

$$\text{Bearing } AC = \alpha - \theta$$
$$AD = \alpha + \theta$$

Example 8.4 A seam dips 1 in 5 in a direction 208° 00′. In what direction will a gradient of 1 in 8 occur?

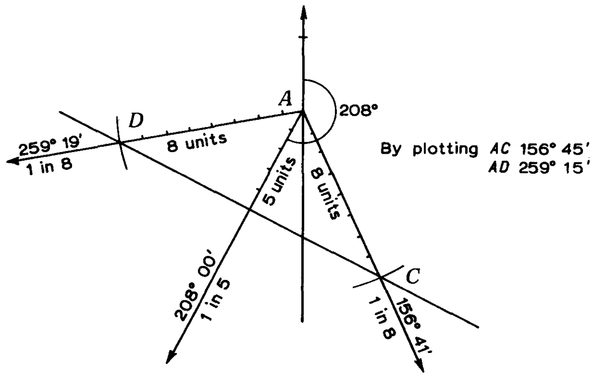


Fig. 8.7

Trigonometrically

$$\theta = \cos^{-1} 5/8$$
$$= 51°19'$$

$$\therefore \text{ Bearing } AC = 208°\,00' - 51°\,19' = \underline{156°\,41'}$$
$$AD = 208°\,00' + 51°\,19' = \underline{259°\,19'}$$

8.24 Given two apparent dips, to find the rate and direction of full dip (Fig. 8.8)

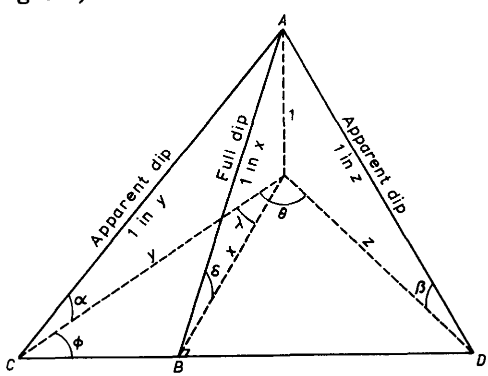


Fig. 8.8

Graphical Solution

Plot direction of apparent dips AC and AD of length y and z units respectively.

Join CD.

Draw AB perpendicular to CD. Measure AB in the same units as y and z.

Trigonometrical solution

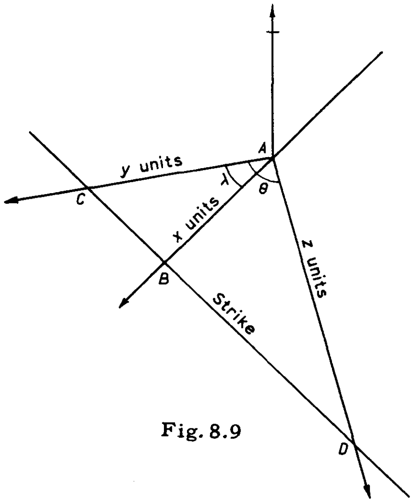


Fig. 8.9

In triangle ADC, Fig. 8.9, $AC = y \quad AD = z \quad D\hat{A}C = \theta$

$$\operatorname{Tan}\frac{C-D}{2} = \frac{z-y}{z+y} \quad \frac{\tan 180-\theta}{2}$$

From this, angles C and D are known and thus λ.

Triangle ABC may now be solved

$$AB = AC\cos\lambda$$

$$\text{Bearing } AB = \text{Bearing } AC - \lambda$$

Alternative solution

$$x = y\cos\lambda$$

also $$x = z\cos(\theta-\lambda)$$

$$\therefore \quad y\cos\lambda = z\cos(\theta-\lambda)$$

$$= z(\cos\theta\cos\lambda + \sin\theta\sin\lambda)$$

Dividing by $\cos\lambda$ $\quad y = z(\cos\theta + \sin\theta\tan\lambda)$

$$\text{Thus} \quad \tan\lambda = \frac{y}{z\sin\theta} - \frac{\cos\theta}{\sin\theta}$$

$$= \frac{y\operatorname{cosec}\theta}{z} - \cot\theta$$

If y and z are given as angles of inclination,

$$y = \cot\alpha$$

$$z = \cot\beta$$

then $\tan\lambda = \tan\beta \operatorname{cosec}\theta \cot\alpha - \cot\theta$ (8.6)

This gives the direction of full dip.

The amount $x = \cot\alpha \cos\lambda$

$$\text{or} \quad \cot\delta = \cot\alpha \cos\lambda \tag{8.7}$$

$$\text{or} \quad \tan\alpha = \tan\delta \cos\lambda \tag{8.8}$$

N.B. If $\theta = 90°$, by Eq.(8.6),

$$\tan\lambda = \tan\beta \cot\alpha \tag{8.9}$$

Alternative solution given angles of inclination, Fig. 8.10

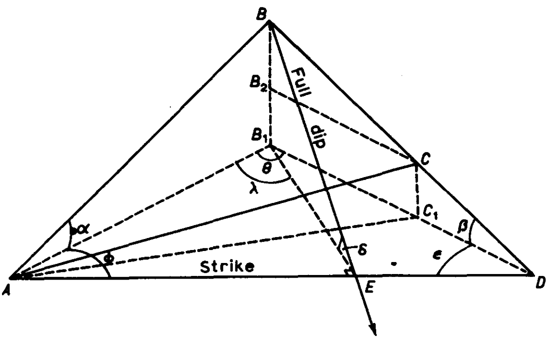


Fig. 8.10

Let A, B and C be three points in a plane with B at the highest level and A at the lowest.

Let AB_1C_1D be a horizontal plane through A, with B_1 and C_1 vertically below B and C respectively and D the intersection on this plane of BC and B_1C_1 each produced.

AB_1C_1 represents the plan of points A, B and C.

In the plane $ABCD$, AD is in the horizontal plane and is therefore a line of strike. BE is perpendicular to AD and is therefore the line of full dip.

In the right-angled triangle ABB_1, $BB_1 = AB_1 \tan\alpha$.

In the right-angled triangle DB_1B, $BB_1 = DB_1 \tan\beta$

$$\therefore \quad AB_1 \tan\alpha = DB_1 \tan\beta$$

i.e. $$\frac{AB_1}{DB_1} = \frac{\tan\beta}{\tan\alpha}$$

Also in triangle AB_1D

$$\frac{AB_1}{DB_1} = \frac{\sin\epsilon}{\sin\phi}$$

$$\therefore \quad \frac{\sin\epsilon}{\sin\phi} = \frac{\tan\beta}{\tan\alpha} = K$$

but $$\epsilon + \phi = 180 - \theta = P$$

$$\therefore \quad \epsilon = P - \phi$$

$$\therefore \quad \frac{\sin(P - \phi)}{\sin\phi} = K$$

i.e. $$\begin{aligned} K\sin\phi &= \sin P\cos\phi - \cos P\sin\phi \\ K &= \sin P\cot\phi - \cos P \\ &= \sin(180 - \theta)\cot\phi - \cos(180 - \theta) \\ &= \sin\theta\cot\phi + \cos\theta \end{aligned}$$

$$\therefore \quad \cot\phi = \frac{K - \cos\theta}{\sin\theta} = \frac{\tan\beta}{\tan\alpha\sin\theta} - \cot\theta \qquad (8.10)$$

The value of ϕ will then give the direction of the strike with full dip at 90° to this.

The inclination of full dip (δ) is now required.

Let $AB_1 = \cot\alpha$

then $B_1E = \cot\delta$

as before therefore $\cot\delta = \cot\alpha\sin\phi$ (8.11)

i.e. Eq.(8.7) $\cot\delta = \cot\alpha\cos\lambda \qquad (\lambda = 90 - \phi)$

Example 8.5 A roadway dips 1 in 4 in a direction 085° 30′, intersects another dipping 1 in 6, 354° 30′. Find the rate and direction of full dip.

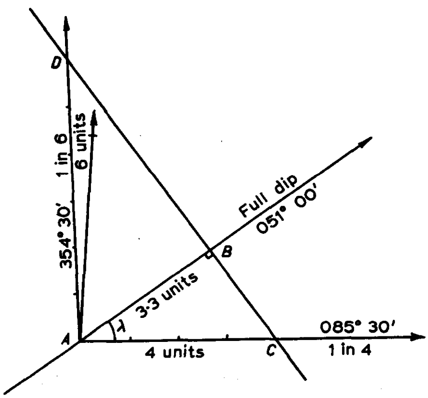


Fig. 8.11

Graphically Full dip 051° 00′ 1 in 3·3

Trigonometrically

$$\tan\frac{C-D}{2} = \frac{6-4}{6+4}\tan\frac{180-(360°-354°\,30' + 085°\,30')}{2}$$

$$= \frac{2}{10}\tan\frac{89°\,00'}{2}$$

$$= \frac{1}{5}\tan 44°\,30'$$

$$\frac{C-D}{2} = 11°\,07'\,10''$$

$$\frac{C+D}{2} = 44°\,30'\,00''$$

$$\therefore \quad C = 55°\,37'\,10''$$

$$\lambda = 34°\,22'\,50''$$

Bearing of full dip $= 085°\,30' - (90° - 55°\,37'\,10'')$

$= \underline{051°\,07'\,10''}$

$(AB) = 4 \sin 55°\,37'\,10''$

$= 4 \times 0{\cdot}825\,31$

$= 3{\cdot}301\,24$

Gradient of full dip = $\underline{1 \text{ in } 3{\cdot}3}$

Alternative solution

Gradient 1 in 4 = 14° 02′ 10″

Gradient 1 in 6 = 9° 27′ 40″

From Eq. (8.6),

$$\tan\lambda = \frac{\tan 9°\,27'\,40''}{\tan 14°\,02'\,10''\,\sin 91°\,00''} - \cot 91°\,00''$$

$$= \frac{0{\cdot}166\,64}{0{\cdot}250\,00 \times 0{\cdot}999\,85} + 0{\cdot}017\,46$$

$$= 0{\cdot}666\,6\dot{6} + 0{\cdot}017\,46$$

$$= 0{\cdot}684\,13$$

$$\lambda = 34°\,22'\,40''$$

Bearing of full dip $= 085°\,30'\,00'' - 34°\,22'\,40''$

$= \underline{051°\,07'\,20''}$

Rate of full dip

$$x = \cot\delta = \cot 14°\,02'\,10''\,\cos 34°\,22'\,40''$$

$$= 4 \times 0{\cdot}825\,33$$

$$= 3{\cdot}301 \quad \text{(Gradient 1 in } 3{\cdot}3)$$
$$\delta = \underline{16°\,51'\,10''}$$

8.25 Given the rate of full dip and the rate and direction of an apparent dip,to find the direction of full dip (Fig. 8.12)

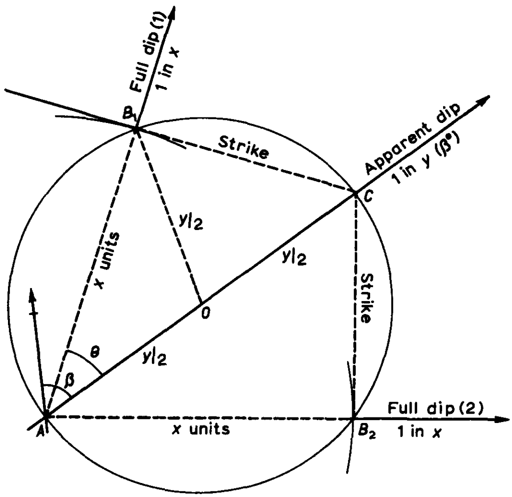


Fig. 8.12

There are two possible solutions.

Graphical solution

Draw apparent dip AC in direction and of y units.
Bisect AC and draw arcs of radius $\frac{1}{2}y$.
Through A draw arcs of length x to cut circle at B_1 and B_2
This gives the two possible solutions AB_1 and AB_2

Trigonometrical solution

In triangle AB_1O

$$AB_1 = x$$
$$AO = OB_1 = \tfrac{1}{2}y$$
$$\tan\frac{\theta}{2} = \sqrt{\frac{(s-x)(s-\frac{1}{2}y)}{s(s-\frac{1}{2}y)}} = \sqrt{\frac{s-x}{s}}$$

where $s = \frac{1}{2}[x + \frac{1}{2}y + \frac{1}{2}y] = \frac{1}{2}(x+y)$

$$\tan\frac{\theta}{2} = \sqrt{\frac{(y-x)}{(x+y)}} \tag{8.12}$$

Bearing of full dip $= \beta \pm \theta$

Example 8.6 A roadway in a seam of coal dips at 1 in 8, 125° 30′. Full dip is known to be 1 in 3. Find its direction.

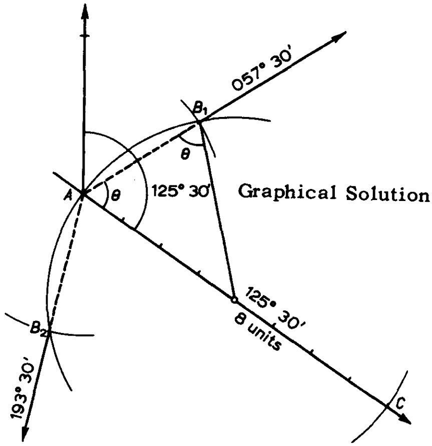


Fig. 8.13

By Eq. (8.12),
$$\tan\frac{\theta}{2} = \sqrt{\frac{(8-3)}{(8+3)}} = \sqrt{\frac{5}{11}}$$
$$= 0{\cdot}67420$$
$$\frac{\theta}{2} = 33°\,59'\,20''$$
$$\theta = 67°\,58'\,40''$$

∴ Bearing of full dip

$$= 125°\,30'\,00'' - 67°\,58'\,40''$$
$$= \underline{57°\,31'\,20''}$$
or
$$= 125°\,30'\,00'' + 67°\,58'\,40''$$
$$= \underline{193°\,28'\,40''}$$

8.26 Given the levels and relative positions of three points in a plane (bed or seam), to find the direction and rate of full dip

This type of problem is similar to 8.24 but the apparent dips have to be obtained from the information given.

To illustrate the methods

Draw an equilateral triangle ABC of sides 600 ft each.

Example 8.7 If the levels of A, B and C are 200, 300 and 275 ft respectively, find the rate and direction of full dip.

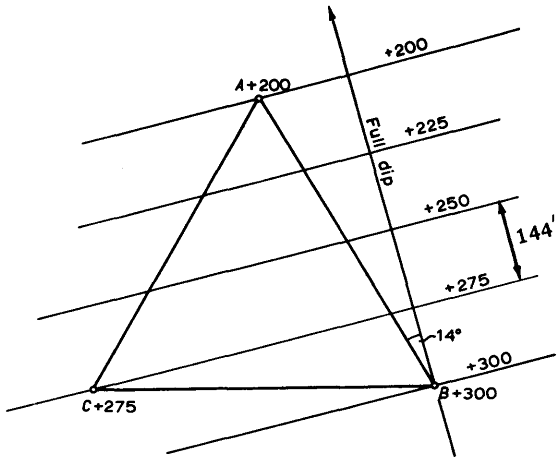


Fig. 8.14

Select the highest point, i.e. B – gradients will then dip away from B.

Semi-graphic method

As B is the highest point and A is the lowest the level of C must be between them.

Difference in level $AB = 300 - 200 = 100$ ft

Difference in level $CB = 300 - 275 = 25$ ft

$\therefore$ Level of C must be $\frac{25}{100}$ of distance AB from B.

$\therefore$ Gradient of full dip = 25 ft in 144 ft (scaled value)

= 1 in 5·76

Direction scaled from plan 14° E of line AB.

N.B. Strike or contour lines in the plane may be drawn parallel as shown.

Alternative method

Gradient $B - A$ = (300 – 200) ft in 600 ft

i.e. = 1 in 6

Gradient $B - C$ = (300 – 275) ft in 600 ft

= 1 in 24.

Lay off units of 6 and 24 as in previous method to give x and y, Fig. 8.15. Line XY is now the strike line and BZ perpendicular to XY produced is the direction of full dip. Length BZ represents the relative full dip gradient value.

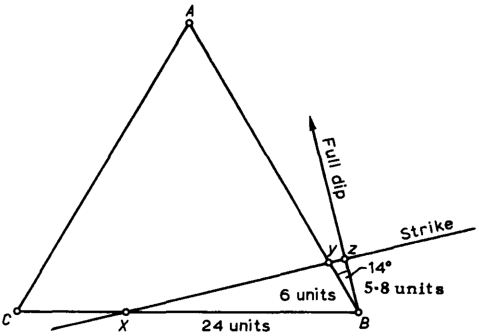


Fig. 8.15

By calculation:

In triangle XYB,

$$\tan\frac{Y-X}{2} = \frac{24-6}{24+6}\tan 60^\circ$$
$$= \frac{18\times 1{\cdot}732}{30} = 1{\cdot}0392$$
$$\frac{Y-X}{2} = 43^\circ 54'$$
$$\frac{Y+X}{2} = 60^\circ 00'$$
$$\therefore \quad Y = 103^\circ 54'$$

In triangle YZB,

$$ZB = 6\sin 103^\circ 54'$$
$$= 6\times 0{\cdot}970\,72 = 5{\cdot}82 \text{ (Gradient of full dip).}$$

Angle $YZB = 103^\circ 54' - 90^\circ = \underline{13^\circ 54'}$ E of line AB.

Example 8.8 From the following information, as proved by boreholes, calculate the direction and rate of full dip of a seam, assuming it to be regular.

Borehole	Level of surface above Ordnance Datum (ft)	Depth from surface to floor of seam (ft)
A	370	1050
B	225	405
C	255	185

Borehole B is 1414 yards from A in the direction 340°

Borehole C is 1750 yards from A in the direction 264°

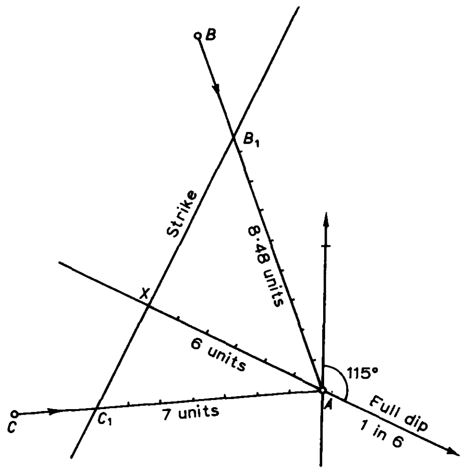


Fig. 8.16

At A,	Surface	+	370
	Depth	−	1050
	Seam level	−	680 ft
At B,	Surface	+	225
	Depth	−	405
	Seam level	−	180 ft
At C,	Surface	+	225
	Depth	−	185
	Seam level	+	70 ft

Gradient

CA = (680 + 70) ft in 3 × 1750 ft i.e. 1 in 7.

BA = (680 − 180) ft in 3 × 1414 ft i.e. 1 in 8·484

∴ Let C_1A = 7 units and B_1A = 8·484 units.

In triangle AB_1C_1

$$\text{Angle } A = 340 - 264 = 76^\circ$$

$$\tan\frac{C_1 - B_1}{2} = \frac{8{\cdot}484 - 7}{8{\cdot}484 + 7}\tan\frac{180 - 76}{2}$$

$$= \frac{1{\cdot}484\tan 52^\circ}{15{\cdot}484} = 0{\cdot}12267$$

$$\frac{C_1 - B_1}{2} = 7^\circ\,00', \quad \frac{C_1 + B_1}{2} = 52^\circ\,00'$$

$$\therefore \quad C_1 = 59^\circ\,00'$$

In triangle AC_1X, XA is full dip perpendicular to C_1B_1.

$$\therefore \quad XA = C_1A \sin C_1$$
$$= 7 \sin 59^\circ = 6$$

$\therefore$ full dip is 1 in 6.

$$\text{Bearing } AC_1 = 264^\circ$$
$$\text{Bearing } AX = 264^\circ + (90 - 59) = 295^\circ$$

$\therefore$ Bearing of full dip is $XA = 115^\circ$

Example 8.9 Three boreholes A, B and C are put down to prove a coal seam. The depths from a level surface are 735 ft, 1050 ft, and 900 ft respectively. The line AB is N 10° E a distance of 1200 ft, whilst AC is N 55° W, 900 ft.

Calculate the amount and direction of full dip.

B is the lowest and A is the highest.

$$\theta = 55 + 10 = 65^\circ$$

$$\cot \alpha = \frac{1200}{1050 - 735} = \frac{1200}{315}$$
$$= 3{\cdot}809\,52$$

$$\alpha = 14^\circ 42' 30''$$

$$\cot \beta = \frac{900}{900 - 735} = \frac{900}{165}$$
$$= 5{\cdot}454\,54$$

$$\beta = 10^\circ 23' 20''$$

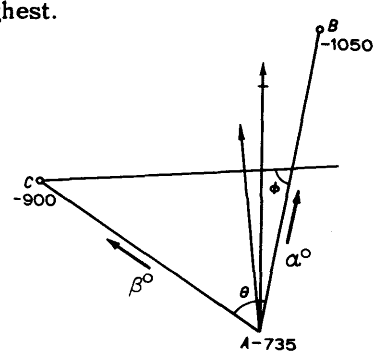


Fig. 8.17

Then by Eq.(8.10)

$$\cot \phi = \frac{\tan \beta}{\tan \alpha \sin \theta} - \cot \theta$$
$$= \frac{0{\cdot}183\,33}{0{\cdot}262\,50 \times 0{\cdot}906\,31} - 0{\cdot}466\,31$$
$$= 0{\cdot}770\,62 - 0{\cdot}466\,31 = 0{\cdot}304\,31$$
$$\phi = 73^\circ 04' 30''$$

$$\text{Bearing of full dip} = 010^\circ - (90 - 73^\circ 04' 30'')$$
$$= 353^\circ 04' 30'' = \text{N } 6^\circ 55' 30'' \text{ W}$$

Amount of full dip $\cot\delta = \cot\alpha \sin\phi$

$$= 3{\cdot}809\,52 \times 0{\cdot}956\,69$$
$$= 3{\cdot}644\,53$$

i.e. Gradient 1 in 3·64

Inclination $(\delta) = 15^\circ 20' 40''$

8.3 Problems in which the Inclinations are Expressed as Angles and a Graphical Solution is Required

To illustrate the processes graphical solutions of the previous examples are given.

8.31 Given the inclination and direction of full dip, to find the rate of apparent dip in a given direction

Example 8.10 Full dip N 30° E 1 in 4 (14° 02′)

Apparent dip (a) Due N, (b) N 75° E.

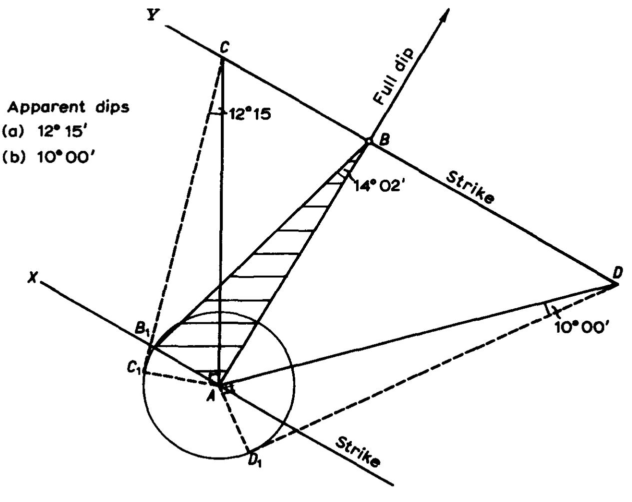


Fig. 8.18

Draw AB (full dip) N 30° E of convenient length say 3 in. Through A and B draw strike lines AX and BY and assume that AX is 1 unit vertically above BY. At B set off the inclination of full dip (i.e. 14° 02′) to cut AX at B_1. AB_1B may now be considered as a vertical section with AB_1 of length 1 unit. Draw a circle of centre A, radius AB_1 (1 unit). Now draw the direction of apparent dips

AC (due N) to cut strike BY at C

and AD (N 75°E) to cut strike BY at D.

Also draw AC_1 and AD_1 perpendicular to AC and AD respectfully, cutting the circle at C_1 and D_1.

N.B. $AB_1 = AC_1 = AD_1 = 1$ unit.

Then $\dfrac{AC_1}{AC}$ represents the gradient of AC (1 in 4·6). The angle ACC_1 is the angle of dip 12°15'.

Similarly $\dfrac{AD_1}{AD}$ represents the gradient of AD (1 in 5·7). The angle ADD_1 is the angle of dip 10°00'.

8.32 Given the inclination and direction of full dip, to find the direction of a given apparent dip

Example 8.11 Full dip 1 in 5 (11°18') 208°00'.
Apparent dip 1 in 8 (7°07').

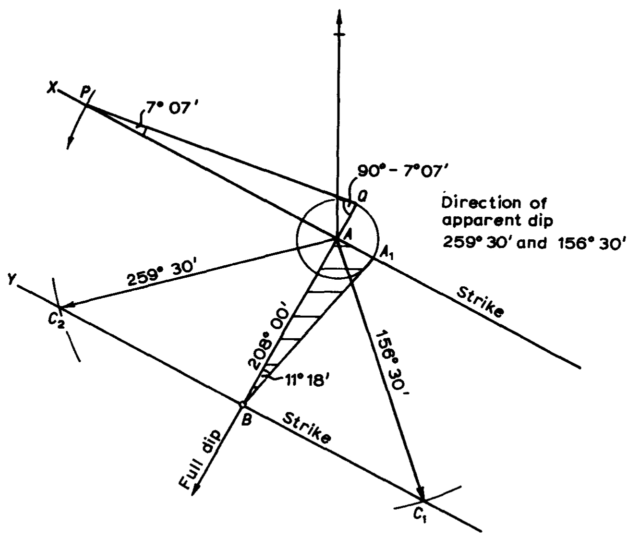


Fig. 8.19

Draw AB in direction as before of say 3 in. At A and B draw the strike lines AX and BY. At B set off the angle of full dip 11°18' to cut AX produced at A_1. Draw a circle of centre A and radius AA_1

Produce BA to cut the circle at Q and set off the angle (90° − 7°07') to cut AX at P— this represents a section of the apparent dip, AP being the length of the section proportional to the vertical fall of 1 unit (AQ). With centre A and radius AP draw an arc to cut BY at C_1 and C_2, i.e. $AC_1 = AC_2 = AP$. These represent the direction of the apparent dips required.

8.33 Given the inclination and direction of two apparent dips to find the inclination and direction of full dip

Example 8.12 1 in 4 (14°) 085° 30′
1 in 6 (9°30′) 354° 30′

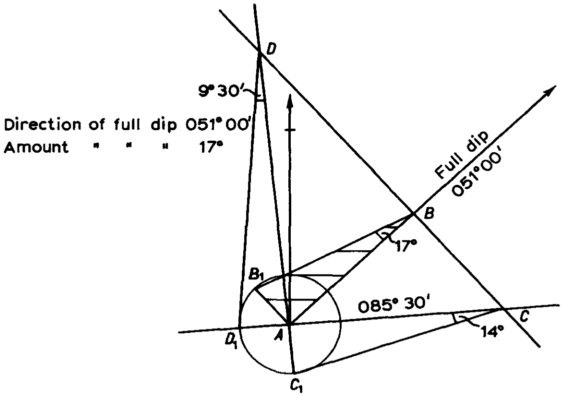


Fig. 8.20

Draw AC and AD in the direction of the apparent dips. With A as centre draw a circle of unit radius. Draw AC_1 and AD_1 perpendicular to AC and AD respectively. Set off at C_1 (90° – 14°) and at D (90° – 9°30′). This will give 14° at C and 9°30′ at D. Join CD, the strike line. Draw AB perpendicular to CD through A and AB_1 perpendicular to AB. Join B_1B and measure the direction of AB and the angle of inclination B_1BA.

Graphical solution:

Full dip 17° – 051°00′.

Exercises 8(a)

1. The full dip of a seam is 4 inches in the yard. Calculate the angle included between full dip and an apparent dip of 3 inches in the yard.

(Ans. 41° 24′)

2. The angle included between the directions of full dip and apparent dip is 60°. If the apparent dip is 9°10′, calculate the full dip, expressing the answer in angular measure, and also as a gradient.

(Ans. 17° 50′; 1 in 3·1)

3. On a hill sloping at 18° runs a track at an angle of 50° with the line of greatest slope. Calculate the inclination of the track and also its length, if the height of the hill is 1500 ft.

(Ans. 11° 48′; 7335 ft)

4. The full dip of a seam is 1 in 3 in a direction N 85° 14′ W. A roadway is to be driven in the seam in a southerly direction dipping at 1 in 10. Calculate the quadrant and azimuth bearings of the roadway.

(Ans. S 22° 14′ W; 202° 14′)

5. The full dip of a seam is 1 in 5, N 4° W. A roadway is to be set off rising at 1 in 8. Calculate the alternative quadrant bearings of the roadway.

(Ans. S 47° 19′ W; S 55°19′ E.)

6. The following are the particulars of 3 boreholes.

Borehole	Surface Level A.O.D.	Depth of Borehole
A	600 ft	500 ft
B	400 ft	600 ft
C	1000 ft	600 ft

If the distance from *A* to *B* is 1200 ft and from *B* to *C* 1800 ft, calculate the gradiants of the lines *AB* and *BC*.

(Ans. 1 in 4; 1 in 3)

7. *A* and *B* are two boreholes which have been put down to prove a seam. They are on the line of full dip of the seam, the direction of line *BA* being N 50°E and its plan length 1000 ft.

Borehole	Surface Level	Depth
A	600 ft	750 ft
B	800 ft	700 ft

A shaft is to be sunk at a point *C*, the surface level of *C* being 1000 ft, the length *BC* 800 ft, and the direction of *BC* due East.

Calculate (a) the dip of the seam from *B* to *C*,

(b) the depth of the shaft at *C*.

(Ans. (a) 1 in 5·23; (b) 1053 ft)

8. A seam dips at 1 in 12·75, S 17° W and at 1 in 12·41, S 20°15′ E Calculate the magnitude and direction of full dip.

(Ans. 1 in 11·64; S 6°46′ E).

9. The co-ordinates and level values of points *A*, *B* and *C* respectively, in a mine, are as follows:

	Departure (ft)	Latitude (ft)	Levels in ft above a datum 10 000 ft below O.D.
A	+119·0	+ 74·0	9872
B	−250·0	+ 787·5	9703
C	+812·0	+1011·0	9805

(a) Plot the positions of the points to a convenient scale and graphically determine the direction and amount of full dip.

(b) Calculate the direction and amount of full dip.

(Ans. (a) N 38° W; 1 in 4·66.
(b) N 37°56′ W; 1 in 4·672)

10. In a steep seam a roadway AB has an azimuth of 190° dipping at 22° and a roadway AC has an azimuth of 351° rising at 16°.

Calculate the direction and rate of full dip of the seam.

(Ans. 225° 54′ 50″, 26° 31′)

11. A cross-measures drift, driven due South and dipping at 16°, passes through a bed of shale dipping due North at 27°. The distance between the roof and floor of the bed of shale measured on the floor of the drift is 56 ft.

Calculate the true and vertical thickness of the bed.

(Ans. 38·19 ft; 42·86 ft)

12. A small colliery leasehold was proved by 3 bores. The surface level at each bore and the depth to the seam were as follows:

Bore	Surface Level	Depth
A	30 ft above O.D.	190 ft
B	20 ft above O.D.	220 ft
C	10 ft above O.D.	240 ft

Bore B is 560 yds from A N 30° E and bore C is 420 yd from B S 60° E. Find graphically, or otherwise, the direction of strike and the rate of full dip in inches per yard.

If a shaft is sunk 800 yd from A in a direction S 45° E at what depth below datum will it reach the seam?

(Ans. N 15° W; 1¼ in. per yd; 200·4 ft)

13. An underground roadway driven on the strike of the seam has a bearing S 30° E. The seam has a full dip of 8 in to the yd in a northerly direction. At a point A on the roadway another road is to be set off rising at 1 in 5·8.

Calculate the alternative bearings on which this second road may be set off.

(Ans. N 80° 54′ W; S 20° 54′ E)

14. Three bores A, B and C have been put down to a coal seam. B is due north from A, 1000 ft, and C is N 76° W, 850 ft from A. The surface levels of the boreholes are the same. The depth of A is 700 ft, of B 1250 ft and of C 950 ft.

Calculate the direction and rate of full dip and the slope area in the triangle formed by the boreholes.

(Ans. N 16° 49′ W; 1 in 1·74; 475 610 sq ft)

8.4 The Rate of Approach Method for Convergent Lines

In Fig. 8.21, let AC rise from A at 1 in K.
AD dip from A at 1 in M.
AB be the horizontal line through A.

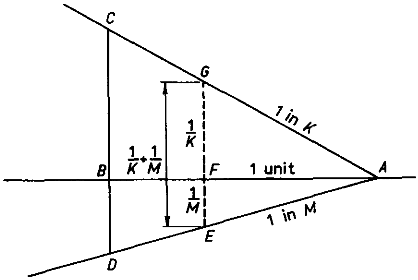


Fig. 8.21

$$AF = 1 \text{ unit}$$

$$\text{then} \quad GF = \frac{1}{K} \text{ units}$$

$$\text{and} \quad FE = \frac{1}{M} \text{ units}$$

Comparing similar triangles ADC and AEG,

$$\frac{CD}{BA} = \frac{GE}{FA} = \frac{\frac{1}{K}+\frac{1}{M}}{1}$$

$$\therefore \quad CD = BA\left(\frac{1}{K}+\frac{1}{M}\right)$$

$$\text{and} \quad BA = \frac{CD}{\frac{1}{K}+\frac{1}{M}}$$

Thus if 2 convergent lines CA and DA are CD vertically apart, the horizontal distance BA when they meet $= \dfrac{CD}{\frac{1}{K}+\frac{1}{M}}$ (8.13)

When the lines both dip from A (Fig. 8.22)

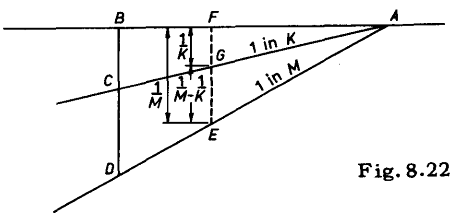


Fig. 8.22

Let AC dip from A at 1 in K
AD dip from A at 1 in M.

As before,

$$\frac{CD}{BA} = \frac{GE}{FA} = \frac{\frac{1}{M} - \frac{1}{K}}{1}$$

$$\therefore \quad CD = BA\left(\frac{1}{M} - \frac{1}{K}\right)$$

$$\text{and} \quad BA = \frac{CD}{\frac{1}{M} - \frac{1}{K}}$$

Therefore, if the two convergent lines CA and DA are CD vertically apart, the horizontal distance BA when they meet

$$= \frac{CD}{\frac{1}{M} - \frac{1}{K}} \qquad (8.14)$$

Thus if a rise is considered +ve and dip −ve, then the general expression applies.

$$\text{Horizontal distance} = \frac{\text{Vertical distance apart}}{\frac{1}{M} \sim \frac{1}{K}} \qquad (8.15)$$

Example 8.13 Two seams of coal, 100 ft vertically apart, dip at 1 in 6. Find the length of a (drift) roadway driven between the seams (a) at a rise of 1 in 4 from the lower to the upper and (b) at a fall of 1 in 2 from upper to lower.

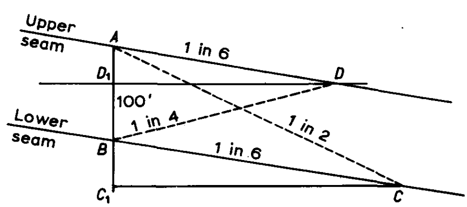


Fig. 8.23

(a) In triangle ADB,

AD falls at 1 in 6.
BD rises at 1 in 4.
AB = 100 ft

Then, by Eq. (8.15),

$$D_1D = \frac{AB}{\frac{1}{4} - \left(-\frac{1}{6}\right)} = \frac{100}{\frac{1}{4} + \frac{1}{6}}$$

$$= \frac{12 \times 100}{3 + 2}$$

i.e. plan length of drift $= 240$ ft

inclined length of drift $= \dfrac{240 \times \sqrt{4^2 + 1^2}}{4}$

$$= \frac{240 \times \sqrt{17}}{4}$$

$$= \underline{247{\cdot}2 \text{ ft}}$$

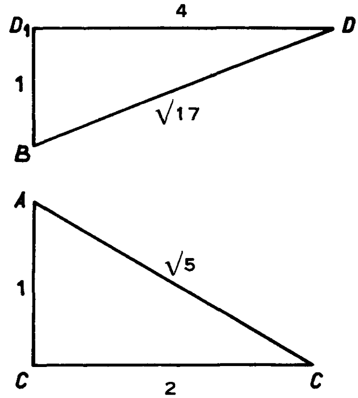


Fig. 8.24

(b) Similarly, by Eq.(8.15),

$$C_1 C = \frac{AB}{-\frac{1}{6} - \left(-\frac{1}{2}\right)} = \frac{100}{\frac{1}{2} - \frac{1}{6}}$$

plan length $\quad C_1 C = \dfrac{600}{2} = 300$ ft

inclined length $\quad AC = \dfrac{300 \times \sqrt{5}}{2} = \underline{336{\cdot}0 \text{ ft}}$

Example 8.14 Two parallel levels, 200 ft apart, run due East and West in a seam which dips due North at 3 in. to the yard. At a point A in the lower level a cross-measures drift rising at 6 in. to the yard and bearing N 30° E is driven to intersect another seam, situated 200 ft vertically above the seam first mentioned, at a point C. From a point

B in the upper level due South from A another cross-measures drift rising at 12 in. to the yard and bearing N 30° E is also driven to intersect the upper seam at a point D.

Calculate the length and azimuth of CD. (M.Q.B.)

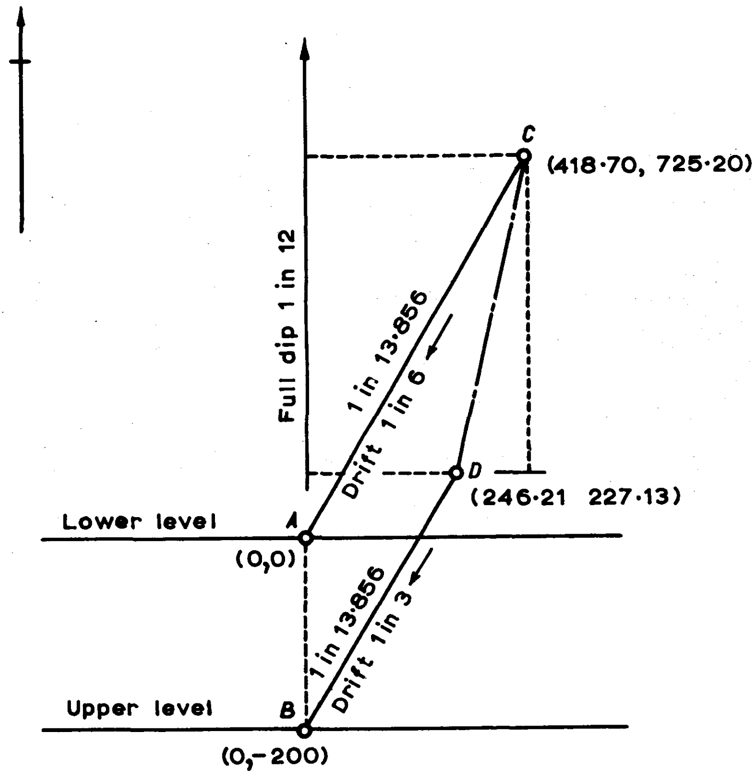


Fig. 8.25

Apparent dip of seam in direction of drift (Fig. 8.25)

By Eq. (8.4)

$$\text{Cot apparent dip } AC = \frac{\text{cot full dip}}{\text{cosine angle } \theta \text{ between}}$$

$$= \frac{12}{\cos 30^\circ} = 13{\cdot}856.$$

To find length of drift AC, Fig. 8.26(a),

$$\text{Plan length} \quad AC = \frac{200}{\frac{1}{6} + \frac{1}{13{\cdot}856}} = \frac{200 \times 83{\cdot}136}{19{\cdot}856}$$

$$= \underline{837{\cdot}39 \text{ ft}}$$

To find length of drift BD, Fig. 8.26(b),

$$\text{Plan length } BD = \frac{200}{\frac{1}{3}+\frac{1}{13\cdot856}} = \frac{200 \times 41\cdot568}{16\cdot856}$$

$$= 493\cdot21 \text{ ft}$$

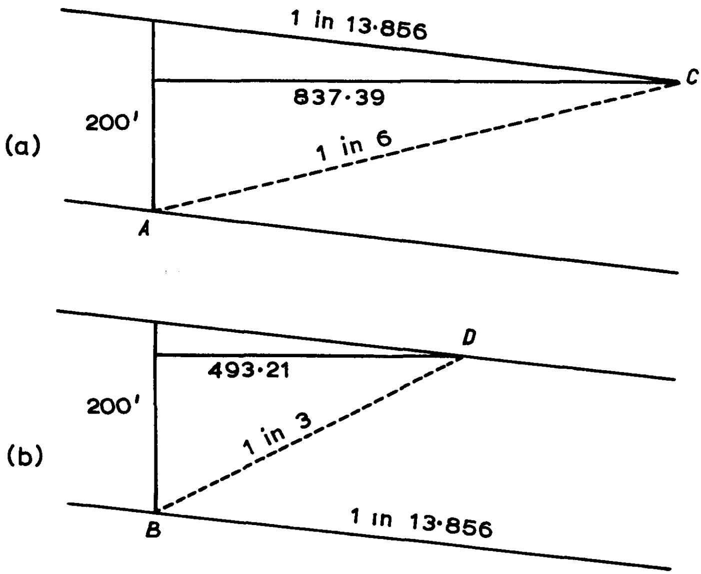


Fig. 8.26

Assuming the two levels are 200 ft apart in plan, the relative positions of A, B, C and D can now be found from the co-ordinates (see chapter 3).

<u>AC N 30° E 837·39 ft</u>

sin 30° = 0·500 00	∴ P.D. = +418·70	
cos 30° = 0·866 03	∴ P.L. = +725·20	
∴ (relative to A)	Total departure of C	= +418·70
	Total latitude of C	= +725·20

<u>AB due South 200 ft</u>

∴	Total departure of B	= 0·0
	Total latitude of B	= −200·0

<u>BD N 30° E 493·21 ft</u>

sin 30° = 0·500 00	∴ P.D. = +246·21	
cos 30° = 0·866 03	∴ P.L. = +427·13	
∴	Total departure of D	= <u>+246·21</u>

$$\text{Total latitude of } D = +427{\cdot}13 - 200$$
$$= \underline{+227{\cdot}13}$$

$$\text{Bearing of line } CD = \tan^{-1} \frac{246{\cdot}21 - 418{\cdot}70}{227{\cdot}13 - 725{\cdot}20}$$
$$= \tan^{-1} \frac{-172{\cdot}49}{-498{\cdot}07}$$
$$= \tan^{-1} 0{\cdot}346\,32$$
$$= \underline{\text{S } 19^\circ 06' 10'' \text{ W}}$$

$$\text{Length of line } CD = \frac{498{\cdot}07}{\cos 19^\circ 06' 10''} = \frac{498{\cdot}07}{0{\cdot}944\,93}$$
$$= \underline{527{\cdot}10 \text{ ft}} \text{ (horizontal length)}$$

If the inclined length is required,

AC rises at 1 in 6 $\therefore$ the difference in level

$$A - C = +\frac{837{\cdot}39}{6} = +139{\cdot}56 \text{ ft}$$

BD rises at 1 in 3 $\therefore$ the difference in level

$$B - D = +\frac{493{\cdot}21}{3} = +164{\cdot}40 \text{ ft}$$

$\therefore$ level of D relative to B = $+164{\cdot}40$.

AB rises at 1 in 12 $\therefore$ the difference in level

$$A - B = +\frac{200}{12} = +16{\cdot}67 \text{ ft}$$

$$\therefore \text{ Level of } D \text{ relative to } C = +164{\cdot}40 + 16{\cdot}67 - 139{\cdot}56$$
$$= \underline{+41{\cdot}51 \text{ ft.}}$$

$$\therefore \text{ Inclined length } CD = \sqrt{(527{\cdot}10^2 + 41{\cdot}51^2)}$$
$$= \sqrt{(277\,834 + 1723)}$$
$$= \underline{528{\cdot}7 \text{ ft}}$$

8.5 Fault Problems

8.51 Definitions

A geological fault is a fracture in the strata due to strains and stresses within the earth's crust, accompanied by dislocation of strata. The direction of movement decides the nature of the fault.

With simple displacement either the corresponding strata are forced apart giving a *normal* fault or movement in the opposite direction causes an overlap known as a *reverse* fault. Many variations are possible with these basic forms.

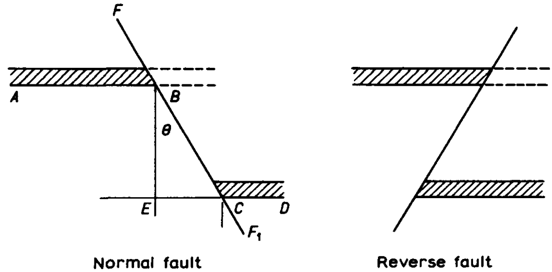


Fig. 8.27

Fig. 8.27 only illustrates the end view and no indication is given of movement in any other direction.

The following terms are used (*see* Fig. 8.27):

- FF_1 is known as the *fault plane*,
- B is the *upthrow* side of the fault,
- C is the *downthrow* side of the fault,
- θ is the angle of *hade* of the fault (measured from the vertical),
- BE is the vertical displacement or *throw* of the fault,
- EC is the horizontal displacement, lateral shift or *heave*, causing an area of *want* or barren ground in the normal fault.

Faults which strike parallel to the strike of the bed are known as *strike* faults.

Faults which strike parallel to the dip of the beds are known as *dip* faults.

Faults which strike parallel to neither dip nor strike are known as *oblique* faults and are probably the most common form, frequently with rotation along the fault plane, Fig. 8.28.

ab = strike slip

bc = dip slip

ac = net or resultant slip

bd = vertical throw

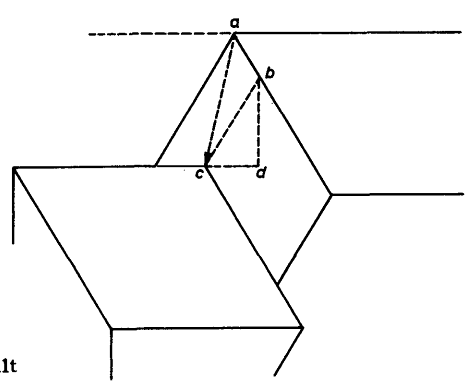


Fig. 8.28 Diagonal or oblique fault

Where the direction and amount of full dip remain the same on both sides of the fault, the fault is of the simple type and the lines of contact between seam and fault on both sides of the fault are parallel.

Where the direction and/or the amount of dip changes, rotation of the strata has taken place and the lines of contact will converge and diverge. The vertical throw diminishes towards the convergence until there is a change in the direction of the throw which then increases, Fig. 8.29.

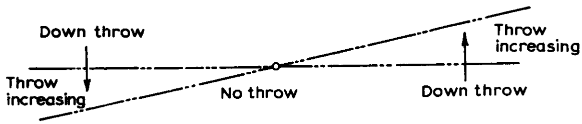


Fig. 8.29

N.B. The strike or level line of a fault is its true bearing, which will differ from the bearing of the line of contact between seam and fault plane.

Example 8.15 A vertical shaft, which is being sunk with an excavated diameter of 23 ft 6 in. passes through a well-defined fault of uniform direction and hade.

Depths to the fault plane below a convenient horizontal plane are taken vertically at the extremities of two diameters *AB* and *CD*, which bear north-south and east-west, respectively. The undernoted depths were measured:

at *A* (north point) 10′ 1″
at *B* (south point) 26′ 3″
at *C* (east point) 4′ 0″
at *D* (west point) 32′ 4″

Calculate the direction of the throw of the fault and the amount of hade. Express the latter to the nearest degree of inclination from the vertical. (M.Q.B./S)

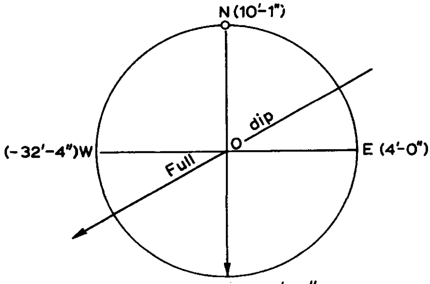


Fig. 8.30

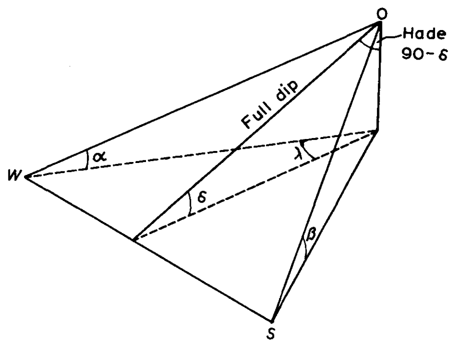


Fig. 8.31

Gradient NS line

$(26'3'' - 10'1'')$ in $23'6''$

i.e. $16'2''$ in $23'6''$ $= 1$ in $1{\cdot}453\,57$

$\therefore \quad \beta = 34°31'40''$

Gradient EW line

$(32'4'' - 4'{-}0'')$ in $23'{-}6''$

i.e. $28'{-}4''$ in $23'6''$ $= 1$ in $0{\cdot}829\,51$

$\alpha = 50°19'30''$

From Eq. (8.6),

$$\tan\lambda = \tan\beta \operatorname{cosec}\theta \cot\alpha - \cot\theta$$

but $\theta = 90°$ $\therefore$ $\tan\lambda = \tan\beta \cot\alpha$ (by Eq. 8.9)

$$= \tan 34°31'40'' \times 0{\cdot}829\,51$$
$$= 0{\cdot}570\,70$$
$$\delta = 29°42'50''$$

$\therefore$ Bearing of full dip

$$= 270° - 29°42'50''$$
$$= \underline{240°\,17'10''}$$

From Eq. (8.7),

$$\cot\delta = \cot\alpha \cos\lambda$$
$$= 0{\cdot}829\,51 \cos 29°42'50''$$
$$= 0{\cdot}720\,44$$
$$\delta = 54°13'50''$$

$\therefore$ Angle of hade $= 90 - \delta$

$= \underline{35°46'10''}$ (36° to nearest degree)

Example 8.16 A roadway advancing due West in a level seam encounters a normal fault running North and South, with a hade of 30°, which

throws the seam up by a vertical displacement of 25 ft.

A drift rising at 1 in 3·6 is driven from the point where the roadway meets the fault on the East side to intersect the seam on the West side of the fault.

Find, by drawing to scale, the inclined length of the drift and check your answer by calculation.

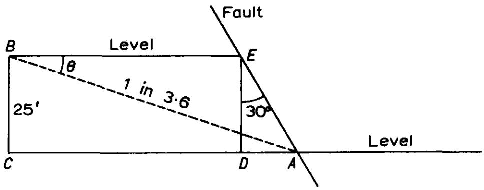


Fig. 8.32

Length of drift scaled from drawing = 93·5 ft.

$$\text{Length of drift} \quad AB = \frac{BC}{\sin\theta} = \frac{25}{\sin\theta}$$

$$\text{but} \quad \cot\theta = 3{\cdot}6$$

$$\therefore \quad \theta = 15°31'$$

$$\therefore \quad AB = \frac{25}{\sin 15°31} = \underline{93{\cdot}42 \text{ ft.}}$$

$$\therefore \quad \text{Length of drift} = 93{\cdot}4 \text{ ft.}$$

Example 8.17 A roadway dipping 1 in 8 in the direction of full dip of a seam strikes an upthrow fault, bearing at right-angles thereto. Following the fault plane a distance of 45 ft the seam is again located and the hade of the fault proved to be 30°.

Calculate the length of a cross-measures drift to win the seam, commencing at the lower side of the fault and rising at 1 in 6 in the same direction as the roadway.

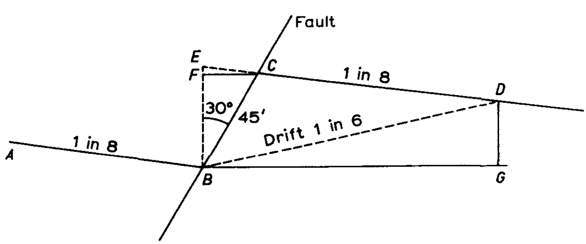


Fig. 8.33

$$\text{Vertical throw of fault } FB = 45\cos 30° = 38{\cdot}97 \text{ ft}$$

$$\text{Lateral displacement } FC = 45\sin 30 = 22{\cdot}50 \text{ ft}$$

In triangle EFC

$$EF = \frac{EC}{8} = \frac{22{\cdot}50}{8} = 2{\cdot}81 \text{ ft}$$

$$\therefore \quad EB = FB + EF = 38{\cdot}97 + 2{\cdot}81 = 41{\cdot}78 \text{ ft.}$$

To find the plan length of the drift by the rate of approach method,

$$BG = \frac{41{\cdot}78}{\frac{1}{8}+\frac{1}{6}} = \frac{41{\cdot}78 \times 48}{14}$$

$$= 143{\cdot}24 \text{ ft}$$

Inclined length of drift BD

$$= 143{\cdot}24 \times \frac{\sqrt{37}}{6} = \underline{145{\cdot}22 \text{ ft}}$$

Example 8.18 A roadway, advancing due East in the direction of full dip of 1 in 8, meets a downthrow fault bearing S 35° E, with a throw of 60 ft and a hade of 30°.

At a distance 150 ft along the roadway, back from the fault, a drift is to be driven in the same direction as the roadway in such a way that it meets the point of contact of seam and fault on the downthrow side.

Calculate the inclined length and gradient of the drift, assuming the seam is constant on both sides of the fault.

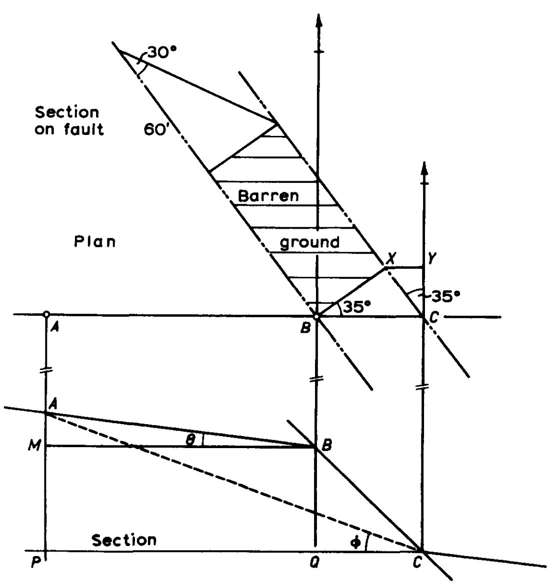


Fig. 8.34

In plan

Width of barren ground $= BX = 60 \tan 30° = 34{\cdot}64$ ft.

On the line of the roadway $BC = 34{\cdot}64 \sec 35° = 42{\cdot}29$ ft $(= QC)$.

As X is 60 ft vertically below B, it is necessary to obtain the relative level of C.

$$XC = BC \sin 35°$$

$$XY = XC \sin 35 = BC \sin^2 35° = 42{\cdot}29 \sin^2 35° = 13{\cdot}91 \text{ ft}$$

As the dip of XY is 1 in 8, the level of Y relative to X is

$$-\frac{13{\cdot}91}{8} = -1{\cdot}74 \text{ ft}$$

But YC is the line of strike

$$\begin{aligned} \therefore \quad \text{Level of } Y = \text{level of } C &= 1{\cdot}74 \text{ ft below } X \\ &= 60 + 1{\cdot}74 \text{ ft below } A \\ &= 61{\cdot}74 \text{ ft below } A \\ &\qquad \text{(i.e. } BQ = MP). \end{aligned}$$

In section

Gradient of roadway $AB = 1$ in 8.

$$\begin{aligned} \therefore \quad \cot\theta &= 8 \\ \theta &= 7°08' \\ \text{Thus} \quad AM &= 150 \sin 7°08' = 18{\cdot}63 \text{ ft} \\ \text{and} \quad MB &= 150 \cos 7°08' = 148{\cdot}84 \text{ ft} \\ &\qquad (= PQ). \end{aligned}$$

$$\begin{aligned} \text{Difference in level } A - C &= AP = AM + MP \\ &= 18{\cdot}63 + 61{\cdot}74 = 80{\cdot}37 \text{ ft} \\ \text{Plan length of drift } AC &= PC = PQ + QC \\ &= 148{\cdot}84 + 42{\cdot}29 = 191{\cdot}13 \text{ ft} \\ \therefore \text{ Gradient of drift} &= 80{\cdot}37 \text{ ft in } 191{\cdot}13 \text{ ft} \\ &= 1 \text{ in } \frac{191{\cdot}13}{80{\cdot}37} \\ &= \underline{1 \text{ in } 2{\cdot}378} \\ \cot\phi &= 2{\cdot}378 \\ \phi &= \underline{22°49'} \\ \text{Length of drift} &= 191{\cdot}13 \sec 22°49' \\ &= \underline{207{\cdot}36 \text{ ft}} \end{aligned}$$

8.52 To find the relationship between the true and apparent bearings of a fault

The true bearing of a fault is the bearing of its strike or level line. The apparent bearing of a fault is the bearing of the line of con-

tact between the seam and the fault.

The following conditions may exist:

(1) If the seam is level, the contact line is the true bearing of the fault.

(2) The apparent bearings are alike, i.e. parallel, if the full dip of the seam is constant in direction and amount. N.B. The throw of the fault will also be constant throughout its length–this is unusual.

(3) The apparent bearings differ, due to variation in direction or amount of full dip on either side of the fault. The barren ground will thus diminish in one direction. N.B. The throw of the fault will vary and ultimately reduce to zero and then change from upthrow to downthrow. This is generally the result of rotation of the beds.

Two general cases will therefore be considered:

(1) When the throw of the fault opposes the dip of the seam.

(2) When the throw of the fault is in the same general direction as the dip of the seam.

Let the full dip on the downthrow side be 1 in x,
the full dip on the upthrow side be 1 in y,
the angle between the full dip and the line of contact be α,
the angle between the line of contact and the true bearing of the fault be β,
the angle of hade be $\theta°$ from the vertical,
the throw of the fault be t ft down in the direction NW,
the angle between the full dip 1 in y and the true bearing of the fault be ϕ.

8.53 To find the true bearing of a fault when the throw of the fault opposes the dip of the seam (Fig. 8.35)

If the throw of the fault is t ft, then D will be $+t$ ft above A, and for the true bearing of the fault DC, C must be at the same level as D. Angle $ADC = 90°$ with DC tangential to the arc of radius $t.\tan\theta$. The full dip 1 in x requires a horizontal length $AB = tx$ ft.

The same applies on the other side of the fault. E must be at the same level as A, and EA, the true bearing of the fault, must be parallel to DC; also for FE to be the strike in the seam DF must equal ty ft.

Referring to Fig. 8.35,

In triangle ABC,

$$AC = \frac{tx}{\cos\alpha_1}$$

In triangle ACD,

$$\sin\beta_1 = \frac{t\tan\theta}{\dfrac{tx}{\cos\alpha_1}} = \frac{\tan\theta\cos\alpha_1}{x} \tag{8.16}$$

In triangles ADE and EDF,

$$\sin\beta_2 = \frac{\tan\theta\cos\alpha_2}{y} \tag{8.17}$$

but

$$\alpha_2 = \phi + \beta_2$$

$$\therefore\quad \sin\beta_2 = \frac{\tan\theta}{y}\cos(\phi + \beta_2) \tag{8.18}$$

$$= \frac{\tan\theta}{y}(\cos\phi\cos\beta_2 - \sin\phi\sin\beta_2)$$

i.e. $$y\cot\theta = \cos\phi\cot\beta_2 - \sin\phi$$

$$\therefore\quad \cot\beta_2 = \frac{y\cot\theta + \sin\phi}{\cos\phi} \tag{8.19}$$

Hence the true bearing of the fault

= bearing of contact line $AC - \beta_1$ (8.20)

Bearing of contact line ED

= true bearing of fault + β_2

= bearing of $AC - \beta_1 + \beta_2$ (8.21)

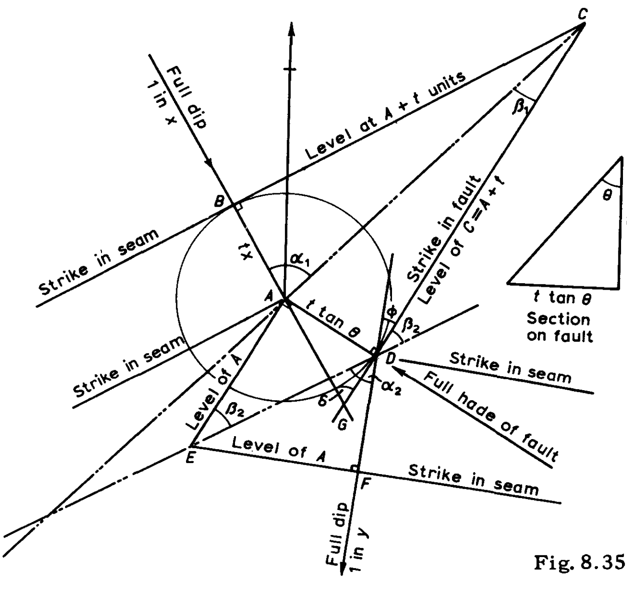


Fig. 8.35

8.54 **Given the angle δ between the full dip of the seam and the true bearing of the fault, to find the bearing of the line of contact (Fig. 8.35)**

$$\alpha_1 = \delta + \beta_1$$

From Eq. (8.16),

$$\sin\beta_1 = \frac{\tan\theta\cos\alpha_1}{x}$$

$$= \frac{\tan\theta}{x}\cos(\delta+\beta_1)$$

$$\therefore \quad x\cot\theta = \frac{\cos\delta\cos\beta_1 - \sin\delta\sin\beta_1}{\sin\beta_1}$$

$$= \cos\delta\cot\beta_1 - \sin\delta$$

$$\therefore \quad \cot\beta_1 = \frac{x\cot\theta + \sin\delta}{\cos\delta} \qquad (8.22)$$

Example 8.19 A plan of workings in a seam dipping at a gradient of 1 in 3 in a direction S 30° E shows a fault bearing $^{S}_{N}45^{\circ}{}^{W}_{E}$ in the seam which throws the measures down to the North-West. The hade of the fault is 30° to the vertical. Calculate the true bearing of the fault.

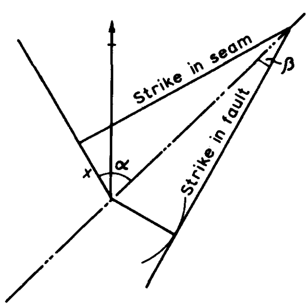


Fig. 8.36

From Eq. (8.16),

$$\sin\beta = \frac{\cos\alpha\tan\theta}{x} = \frac{\cos 75^{\circ}\tan 30^{\circ}}{3}$$

$$= \frac{0{\cdot}149\,430}{3}$$

$$= 0{\cdot}049\,810$$

$$\beta = 2^{\circ}51'20''$$

$$\therefore \quad \text{True bearing} = 045^{\circ}00'00'' - 2^{\circ}51'20''$$

$$= \underline{042^{\circ}\,08'\,40''}$$

Example 8.20 A seam dipping 1 in 5, S 60° E, is intersected by a fault the hade of which is 30° to the vertical and the bearing N 35° W. The fault is a downthrow to the South-West. Calculate the bearing of the fault as exposed by the seam.

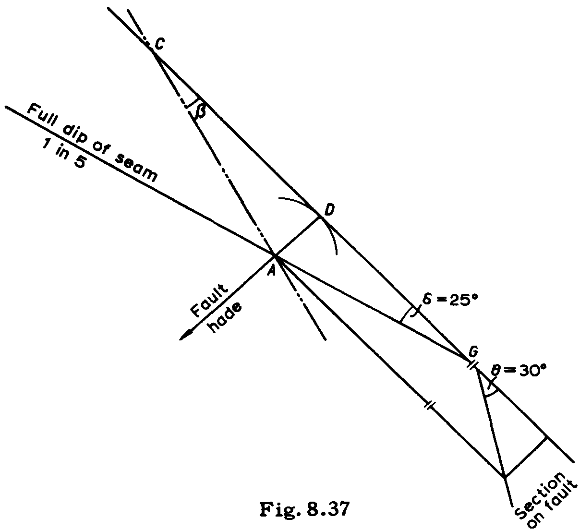


Fig. 8.37

As full dip and hade oppose each other, by Eq. (8.22)

$$\cot\beta = \frac{x\cot\theta + \sin\delta}{\cos\delta}$$

$$= \frac{5\cot 30^\circ + \sin 25^\circ}{\cos 25^\circ}$$

$$= 10{\cdot}02181$$

$$\beta = 5^\circ 42'$$

∴ Bearing of fault exposed in seam

$$= 325^\circ - 5^\circ 42'$$

$$= \underline{319^\circ 18' = \text{N}40^\circ 42'\,\text{W}}$$

Example 8.21 Headings in a seam at A and B have made contact with a previously unlocated fault which throws the measures up 100 ft to the south-east with a true hade of 40° from the vertical.

Full dip is known to be constant in direction, namely 202° 30′, but the amount of dip changes from 1 in 5 on the north side to 1 in 3 on the south side of the fault.

Given the co-ordinates of A and B as follows:

A	3672·46 ft E.	5873·59 ft N.
B	4965·24 ft E.	7274·38 ft N.

Calculate (a) the true bearing of the fault, (b) the bearings of the lines of contact between fault and seam.

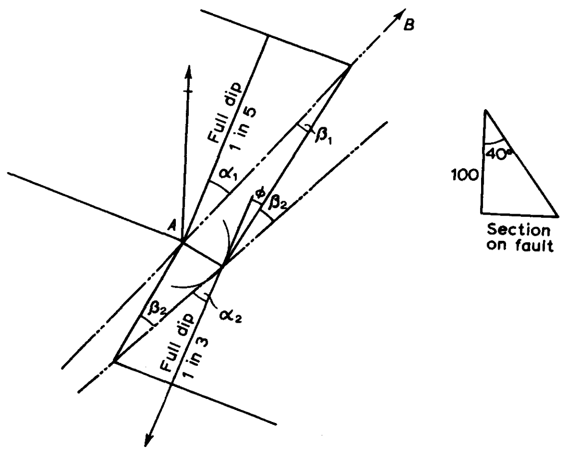


Fig. 8.38

Co-ordinates		E		N
	A	3672·46		5873·59
	B	4965·24		7274·38
	dE	+ 1292·78	dN	+ 1400·79

$$\tan \text{bearing } AB = \frac{dE}{dN} = \frac{+1292{\cdot}78}{+1400{\cdot}79} = 0{\cdot}922\,89$$

$$\text{bearing of contact line } AB = \underline{042°\,42'}$$

Then $\alpha_1 = 180° + 042°42' - 202°30' = 20°12'$

By Eq. (8.16),

$$\sin\beta_1 = \frac{\cos 20°12' \tan 40°}{5} = 0{\cdot}157\,50$$

$$\beta_1 = 9°04'$$

∴ Bearing of line of strike of fault,

i.e. true bearing of fault $= 042°42' - 9°04'$
$= \underline{033°38'}$

Now $\phi = \alpha_1 - \beta_1 = 20°12' - 9°04' = 11°08'$

By Eq. (8.19)

$$\cot\beta_2 = \frac{3\cot 40° + \sin 11°08'}{\cos 11°08'}$$

$$= 3{\cdot}840\,62$$

$$\therefore \quad \beta_2 = 14°34'$$

∴ Bearing of line of contact on upthrow side of fault
$= 033°38' + 14°34' = \underline{048°12'}$

8.55 To find the true bearing of a fault when the downthrow of the fault is in the same general direction as the dip of the seam (Fig. 8.39)

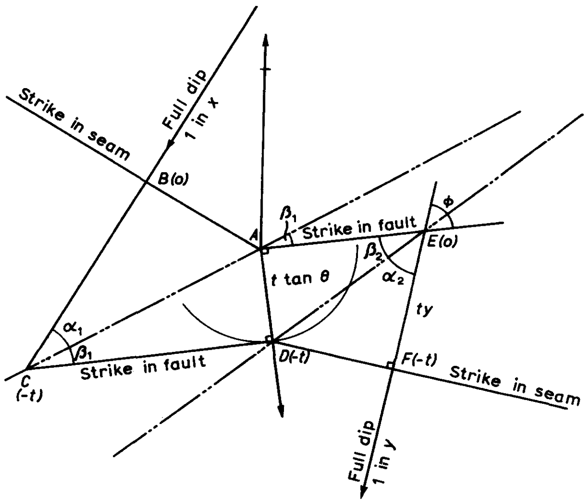


Fig. 8.39

$$\sin\beta_1 = \frac{\tan\theta\cos\alpha_1}{x} \quad \text{as before} \tag{8.23}$$

$$\text{and} \quad \sin\beta_2 = \frac{\tan\theta\cos\alpha_2}{y} \tag{8.24}$$

but $\phi = \alpha_2 + \beta_2$

$\therefore \quad \alpha_2 = \phi - \beta_2$

$$\sin\beta_2 = \frac{\tan\theta}{y}(\cos\phi\cos\beta_2 + \sin\phi\sin\beta_2)$$

$$y\cot\theta = \cos\phi\cot\beta_2 + \sin\phi$$

$$\cot\beta_2 = \frac{y\cot\theta - \sin\phi}{\cos\phi} \tag{8.25}$$

(8.26)

Hence true bearing of fault = $\underline{\text{bearing of contact line } CA + \beta_1}$

bearing of contact line DE = true bearing of fault $- \beta_2$

$= \underline{\text{bearing of } CA + \beta_1 - \beta_2}$ (8.27)

8.56 Given the angle δ between the full dip of the seam and the true bearing of the fault, to find the bearing of the line of contact

Here $\delta = \alpha_1 + \beta_1$

i.e. $\alpha_1 = \delta_1 - \beta_1$

then, from Eq. (8.23),

$$\sin\beta_1 = \frac{\tan\theta}{x}\cos(\delta - \beta_1)$$

giving

$$\cot\beta_1 = \frac{x\cot\theta - \sin\delta_1}{\cos\delta_1} \tag{8.28}$$

Example 8.22 A fault is known to be an upthrow in a NW direction with a full hade of 30° to the vertical.

The bearing of the fault as exposed in the seam is N 40° E, and the full dip of the seam is 1 in 5 S 35° W. Find the true bearing of the fault.

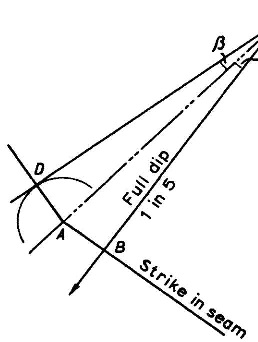


Fig. 8.40

As the downthrow of the fault is in general direction as the full dip, by Eq. (8.23),

$$\sin\beta = \frac{\tan\theta\cos\alpha}{x}$$

$$= \frac{\tan 30^\circ \cos 5^\circ}{5}$$

$$= 0{\cdot}11503$$

$$\beta = 6^\circ 36'$$

$\therefore$ Bearing of fault $=$ N 40° + 6°36′ E

$=$ <u>N 46°36′ E</u>

8.6 To Find the Bearing and Inclination of the line of Intersection (AB) of Two Inclined Planes

Let (1) the horizontal angle between the lines of full dip of the two planes inclined at α and β respectively be δ,

(2) the horizontal angle between the line of full dip α and the intersection line AB of inclination ϕ be θ.

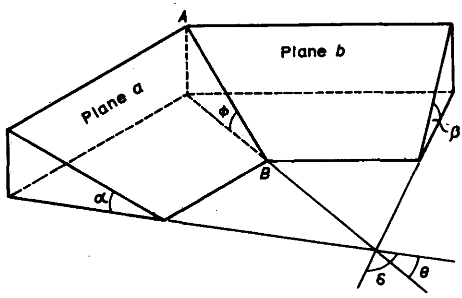


Fig. 8.41

From Eq. (8.5),

$$\tan\phi = \tan\alpha\cos\theta$$

also $$= \tan\beta\cos(\delta - \theta)$$

$$\therefore \quad \tan\alpha\cos\theta = \tan\beta(\cos\delta\cos\theta + \sin\delta\sin\theta)$$

i.e. $$\tan\alpha\cot\beta = \cos\delta + \sin\delta\tan\theta$$

$$\therefore \quad \tan\theta = \frac{\tan\alpha\cot\beta - \cos\delta}{\sin\delta} \tag{8.29}$$

If plane (a) is a plane in the form of a seam and plane (b) is a fault plane of hade $(90 - \beta)$

then $$\tan\theta = \frac{\tan\alpha\tan\beta - \cos\delta}{\sin\delta} \tag{8.30}$$

Example 8.23 A is a point on the line of intersection of two inclined planes. The full dip of one plane is 1 in 8 in a direction 222°15′ and the full dip of the other is 1 in 4 in a direction 145°25′.

If B is a point to the south of A on the line of intersection of the two planes, calculate the bearing and the inclination of the line AB.

Let the angle of the full dip of plane (1) be $\alpha = \cot^{-1}4 = 14°02'$

of plane (2) be β.

Then the horizontal angle δ between them $= 222°15' - 145°25'$
$= 76°50'$

From Eq. (8.29),

$$\tan\theta = \frac{\tan\alpha\cot\beta - \cos\delta}{\sin\delta}$$

i.e. $$\tan\theta = \frac{8\tan 14°02' - \cos 76°50'}{\sin 76°50'}$$

$$= \frac{1{\cdot}999\,60 - 0{\cdot}227\,78}{0{\cdot}973\,71}$$

$$= 1{\cdot}819\,66$$

$$\therefore \qquad \theta = 61°12'30''$$

$$\therefore \text{ Bearing of } AB = 145°25' + 61°12'30''$$
$$= \underline{206°37'30''}$$

Angle of inclination $AB(\phi)$

$$\tan\phi = \tan\alpha\cos\theta$$
$$= 0{\cdot}120\,38$$
$$\phi = 6°52'$$

i.e. 1 in 8·3

Exercises 8(b) (Faults)

15. A heading in a certain seam advancing due East and rising at 6 in. to the yd met a fault with a displacement up to the East and the seam has been recovered by a cross-measures drift rising at 18 in. to the yd in the same direction as the heading. The floor of the seam beyond the fault where the gradient is unchanged was met by the roof of the drift when it had advanced 345 ft, and the roof of the seam when it had advanced 365 ft

Calculate the thickness of the coal seam.

(M.Q.B./M Ans. 5 ft 10½ in.)

16. Two parallel seams 60 ft vertically apart dip due W at 1 in 6. A drift with a falling gradient of 1 in 12 is driven from the upper to the lower seam in a direction due E.

Calculate the length of the drift.

(Ans. Hor. 240 ft; Incl. 240·82 ft).

17. A roadway in a level seam advancing due N meets a normal fault with a hade of 30° from the vertical and bearing at right-angles to the roadway.

An exploring drift is set off due N and rising 1 in 1. At a distance of 41 ft, as measured on the slope of the drift, the seam on the north side of the fault is again intersected.

(a) Calculate the throw of the fault and the width of the barren ground.
(b) If the drift had been driven at 1 in 4 (in place of 1 in 1) what would be the throw of the fault and the width of barren ground?

(Ans. (a) 29 ft; 16·7 ft
(b) 9·9 ft; 5·7 ft)

18. A roadway AB, driven on the full rise of a seam at a gradient of 1 in 10, is intersected at B by an upthrow fault, the bearing of which is parallel with the direction of the level course of the seam, with a hade of 30° from the vertical.

From B a cross-measures drift has been driven in the line of AB produced, to intersect the seam at a point 190 ft above the level of B

and 386 ft from the upper side of the fault as measured in the seam.

Calculate the amount of vertical displacement of the fault and the length and gradient of the cross-measures drift. Assume that the direction and rate of dip of the seam is the same on each side of the fault.

(Ans. 151·6 ft; 507·78 ft; 1 in 2·5)

19. A roadway advancing due East in a level seam meets a fault bearing North and South, which hades at 30°. A drift, driven up the fault plane in the same direction as the roadway, meets the seam again at a distance of 120 ft.

Calculate the length and gradient of a drift rising from a point on the road 400 ft to the West of the fault which intersects the seam 100 ft East of the fault.

(Ans. 570 ft; 1 in 5·4)

20. The direction and rate of full dip of two seams 60 yd vertically apart from floor to floor are N 12° E and 4½ in. to the yd respectively.

Calculate the length of a cross-measures drift driven from the lower seam to intersect the upper, and bearing N 18° W. (a) if the drift is level; (b) if it rises at a gradient of 4 in. to the yd.

(M.Q.B./M Ans. (a) 1662·7 ft;
(b) 825·2 ft)

21. A level roadway *AB* bearing due West, in a seam 8 ft thick, strikes a normal fault at *B*, the point *B* being where the fault plane cuts off the seam at floor level.

The hade of the fault, as measured in the roadway is 35° from the vertical. A proving drift is driven on the same bearing as the roadway and dipping from the point *B* at 1 in 3. The floor of the drift intersects the floor of the seam, on the lower side of the fault at a point *C* and *BC* is 80 ft measured on the slope. At *C* the seam is found to be rising at 1 in 10 due East towards the fault.

Draw a section to a scale of 1 in. = 20 ft on the line *ABC* showing the seam on both sides of the fault, the drift *BC* and the fault plane and mark on your drawing the throw of the fault and the distance in the seam from *C* to the fault plane.

(M.Q.B./M Ans. 19 ft; 63 ft)

22. The direction of full dip is due North with a gradient of 1 in 6 on the upthrow side and 1 in 9 on the downthrow side. Workings show the line of contact of the seam and fault on the upthrow side as N 45° E with a fault hade of 30° and throw of 30 ft.

Calculate (a) the true bearing of the fault, (b) the bearing of the line of contact on the downthrow side.

(Ans. 041°06′; 043°45′)

Exercises 8(c) (General)

23. In a seam a roadway AB on a bearing 024°00′ dipping at 1 in 9 meets a second roadway AC bearing 323°00′ dipping at 1 in 5½.

Calculate the rate and direction of full dip of the seam.

(Ans. 331°13′; 1 in 5·44)

24. Two seams 60 ft vertically apart, dip at 1 in 8 due N. It is required to drive a cross-measures drift in a direction N 30° W and rising at 1 in 5 from the lower to the upper seam.

Calculate the length of the drivage.

(Ans. 198·5 ft)

25. A roadway driven to the full dip of 1 in 12 in a coal seam meets a 240 ft downthrow fault. A cross-measures drift is set off in the direction of the roadway at a gradient of 6 in. to the yd.

In what distance will it strike the seam again on the downthrow side if (a) the hade is 8° from the vertical, (b) the hade is vertical?

(Ans. (a) 2885·7 ft; (b) 2919·8 ft)

26. Three boreholes A, B and C intersect a seam at depths of 540 ft, 624 ft and 990 ft respectively. A is 1800 ft North of C and 2400 East of B.

Calculate the rate and direction of full dip.

(Ans. 1 in 3·96; S 7°58′ W)

27. The rate of full dip of a seam is 4½ in. to the yd and the direction is S 45° E.

Find, by calculation or by drawing to a suitable scale, the inclinations of two roadways driven in the seam of which the azimuths are 195° and 345° respectively.

(Ans. 1 in 16; 1 in 9·24)

28. Two parallel roadways AB and CD advancing due North on the strike of a seam are connected by a road BD in the seam, on a bearing N 60° E.

The plan length of BD is 150 yd and the rate of dip of the seam is 1 in 5 in the direction BD.

Another roadway is to be driven on the bearing N 45° E to connect the two roadways commencing at a point 200 yd out by B on the road AB, the first 20 yd to be level.

Calculate the total length of the new roadway and the gradient of the inclined portion.

(Ans. 186·44 yd; 1 in 5·46)

29. A roadway AB, 700 ft in length, has been driven in a seam of coal on an azimuth of 173°54′. It is required to drive a cross-measures drift from a point C in another seam at a uniform gradient to intersect at a point D, the road AB produced in direction and gradient. The

levels of A, B and C in feet below Ordnance Datum and the co-ordinates of A and C in feet are respectively as follows:

	Levels	Latitude	Departure
A	-1378	$+9209$	$+18041$
B	-1360		
C	-1307	$+6180$	$+17513$

Calculate the length AD, and the length and gradient of the proposed drift CD, assuming that the latter is to have an azimuth of 032°27′.

(Ans. 1892·9 ft; 1359·0 ft; 1 in 3·27)

30. A heading AB driven direct to the rise in a certain seam at a gradient of 6 in. to the yd and in the direction due N is intersected at B by an upthrow fault, bearing at right-angles thereto with a hade of 30° from the vertical. From the point B a cross-measures drift has been driven in the direction of AB produced, and intersects the seam at a point 420 ft from the upper side of the fault. The levels at the beginning and end of the cross-measures drift are 100 ft and 365 ft respectively above datum.

Calculate (a) the vertical displacement of the fault, (b) the length and gradient of the drift.

Assume the direction and rate of dip of the seam to be uniform on each side of the fault.

(M.Q.B./M Ans. (a) 195·9 ft (b) 590·2; 1 in 2)

31. A roadway in a seam dipping 1 in 7 on the line of the roadway meets a downthrow fault of 30 ft with a hade of 2 vertical to 1 horizontal.

Calculate the length of the drift, dipping at 1 in 4 in the line of the roadway, to win the seam, a plan distance of 50 ft from the dip side of the fault; also the plan distance from the rise side of the fault from which the drift must be set off.

Assume the gradient of the seam to be uniform and the line of the fault at right-angles to the roadway.

(Ans. 267·97 ft; 194·97 ft)

32. A roadway, bearing due East in a seam which dips due South at 1 in 11, has struck a fault at a point A. The fault which, on this side, runs in the seam at N 10° W is found to hade at 20° from the vertical and to throw the seam down 30 ft at the point A. The dip of the seam on the lower side of the fault is in the same direction as the upper side but the dip is 1 in 6.

From a point B in the roadway 140 yd West of A a slant road is driven in the seam on a bearing N 50° E and is continued in the same direction and at the same gradient until the seam on the East side of the fault is intersected at a point C.

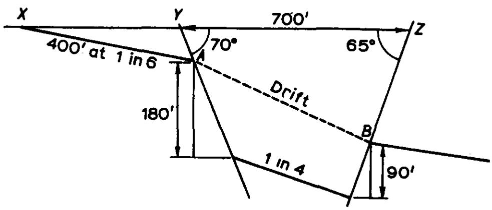
Draw to a scale of 1 in. to 100 ft a plan of the roads and fault and mark the point C. State the length of the slant road BC.

(M.Q.B./S Ans. 264 yd)

33. The sketch shows a seam of coal which has been subjected to displacement by a trough fault.

Calculate the length and gradient of a cross-measures drift to connect the seam between A and B from the details shown.

(Ans. 587·2 ft; 1 in 2·74)



Ex. 8.33

34. A seam dips 1 in 4, 208°30′. Headings at A and B have proved the bearing of the contact line AB to be 075°00′.

If the hade of the fault is 30°, what is the true bearing of the fault if (a) it is a downthrow to the South; (b) it is a downthrow to the North. (Ans. 080°42′; 069°18′)

35. (a) Define the true and apparent azimuth of a fault.

(b) A fault exposed in a certain seam has an azimuth of 086°10′, and a hade of 33°. It throws down to the North West. The full dip of the seam is 1 in 6·5 at 236°15′.

Calculate the true azimuth of the fault. Check by plotting.

(c) Two seams, separated by 86 yd of strata, dip at 1 in 13 in a direction S 36° W. They are to be connected by a drift falling at 1 in 5, N 74°30′E.

Calculate the plan and slope length of the drift.

(N.R.C.T. Ans. (b) 081°12′ (c) 330·51 yd 337·07 yd)

36. A mine plan shows an area of 3·6 acres in the form of a square. Measured on the line of full dip underground the length is 632·4 links.

Calculate the rate of full dip. (Ans. 1 in 3 or 18°24′)

Bibliography

M.H. HADDOCK, *The Location of Mineral Fields* (Crosby Lockwood).
R. McADAM, *Colliery Surveying* (Oliver & Boyd).
FORREST, S.N. *Mining Mathematics* (Senior Course) (Edward Arnold).
METCALFE, J.E., *A Mining Engineer's Survey Manual* (Electrical Press, London).
M.H. HADDOCK, *The Basis of Mine Surveying* (Chapman & Hall).

9 AREAS

9.1 Areas of Regular Figures

The following is a summary of the most important formulae.

9.11 Areas bounded by straight lines

Triangle (Fig. 9.1)

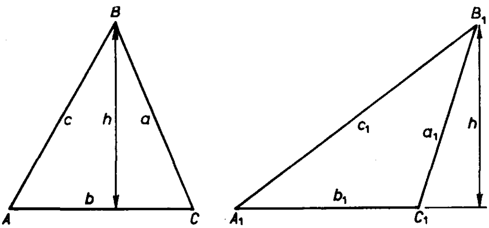


Fig. 9.1 Triangle

(a) (Area) A = half the base $\times$ the perpendicular height

i.e. $A = 1/2\, bh$ (9.1)

(b) A = half the product of any two sides $\times$ the sine of the included angle

i.e. $A = 1/2\, ab \sin C = 1/2\, ac \sin B = 1/2\, bc \sin A$ (9.2)

(c) $A = \sqrt{s(s-a)(s-b)(s-c)}$ (9.3)

where $s = 1/2(a+b+c)$.

Quadrilateral

(a) Square, $A = \text{side}^2$ or $1/2\,(\text{diagonal}^2)$ (9.4)

(b) Rectangle, A = length $\times$ breadth (9.5)

(c) Parallelogram (Fig. 9.2), (i) $A = a \times h$ (9.6)

(ii) $A = ab \sin a$ (9.7)

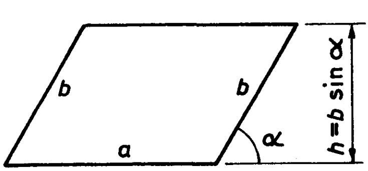


Fig. 9.2 Parallelogram

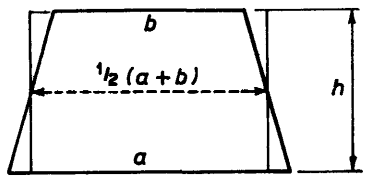


Fig. 9.3 Trapezium

(d) Trapezium (Fig. 9.3)

A = half the sum of the parallel sides $\times$ the perpendicular height

i.e. $A = 1/2(a+b)h$ (9.8)

(e) Irregular quadrilateral (Fig. 9.4)

(i) The figure is subdivided into 2 triangles,

$$A = 1/2(AC \times Bb) + 1/2(AC \times Dd)$$
$$= 1/2\, AC\,(Bb + Dd) \quad (9.9)$$

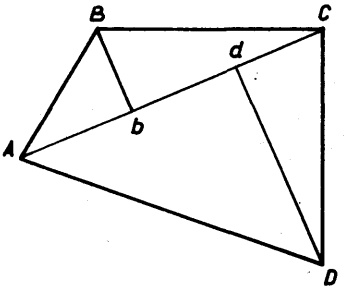


Fig. 9.4 Irregular quadrilateral

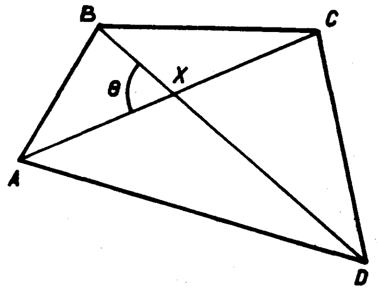


Fig. 9.5

(ii) $A = 1/2(AC \times BD)\sin\theta$ (9.10)

This formula is obtained as follows, Fig. 9.5:

$$A = 1/2\, AX.BX \sin\theta + 1/2\, BX.CX \sin(180-\theta)$$
$$+ 1/2\, CX.DX \sin\theta + 1/2\, DX.AX \sin(180-\theta)$$

As $\sin\theta = \sin(180-\theta)$,

$$A = 1/2 \sin\theta\ [AX(BX+DX) + CX(BX+DX)]$$
$$= 1/2 \sin\theta\ [(AX+CX)(BX+DX)]$$
$$A = \underline{1/2 \sin\theta\ AC \times BD}$$

Regular polygon (Fig. 9.6)

(i) $A = 1/2\, ar \times n$ (9.11)

(ii) As $a = 2r \tan\dfrac{\theta}{2}$

Area $A = 1/2\, rn \times 2r\tan\dfrac{\theta}{2}$

$$= nr^2 \tan\frac{\theta}{2}$$
$$= nr^2 \tan\frac{360}{2n} \quad (9.12)$$

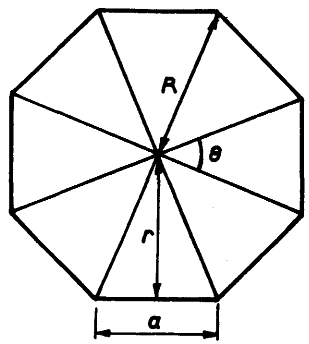


Fig. 9.6 Regular polygon (n sides)

(iii) $A = 1/2\, R^2 \sin\theta \times n$

$$= \frac{n}{2} R^2 \sin\frac{360}{n} \quad (9.13)$$

9.12 Areas involving circular curves

Circle

(i) $A = \pi r^2$ (9.14)

(ii) $A = 1/4\pi d^2$ (9.15)

where $\pi \simeq 3{\cdot}1416$, $1/4\pi \simeq 0{\cdot}7854$.

Sector of a circle (Fig. 9.7)

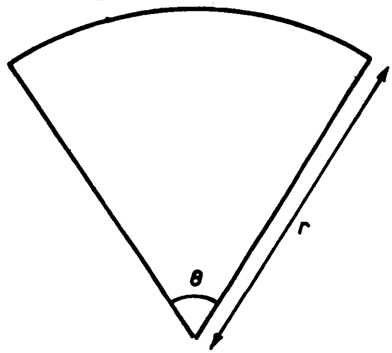


Fig. 9.7 Sector of a circle

(i) $A = \pi r^2 \dfrac{\theta}{360}$ (9.16)

(ii) $A = \frac{1}{2} r^2 \theta_{rad}$ (9.17)

N.B. (ii) is generally the better formula to use as the radian value of θ is readily available in maths. tables or may be derived from first principles (see Section 2.22).

Segment of a circle (Fig. 9.8)

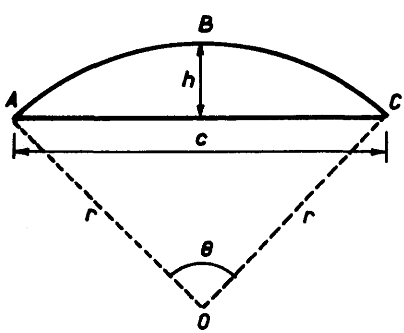


Fig. 9.8 Segment of a circle

(i) A = area of sector – area of triangle

i.e. $$A = 1/2\, r^2\theta_{rad} - 1/2\, r^2 \sin\theta$$

$$= 1/2\, r^2(\theta - \sin\theta) \qquad (9.18)$$

or $$A = \pi r^2 \frac{\theta}{360} - 1/2\, r^2 \sin\theta$$

$$= r^2\left(\frac{\pi\theta}{360} - 1/2 \sin\theta\right) \qquad (9.19)$$

(ii) If chord $AC = c$ and height of arc $= h$ are known, the approximation formulae are:

(a) $$A = \frac{4}{3}h\sqrt{\left(\frac{5}{8}h\right)^2 + \left(\frac{c}{2}\right)^2} \qquad (9.20)$$

(b) $$A \simeq \frac{h^3}{2c} + \frac{2}{3}ch \qquad (9.21)$$

or $$A \simeq \frac{3h^3 + 4c^2h}{6c} \qquad (9.22)$$

Annulus (flat ring) (Fig. 9.9)

$$A = \pi R^2 - \pi r^2$$

$$= \pi(R^2 - r^2)$$

$$= \pi(R-r)(R+r) \qquad (9.23)$$

or $$A = 1/4\,\pi(D-d)(D+d) \qquad (9.24)$$

where $D = 2R$, $d = 2r$.

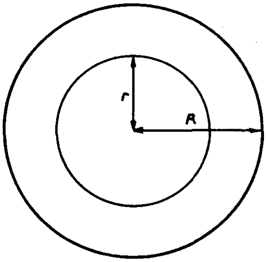


Fig. 9.9 Annulus

9.13 Areas involving non-circular curves

Ellipse (Fig. 9.10)

$$A = \frac{\pi}{4}ab \qquad (9.25)$$

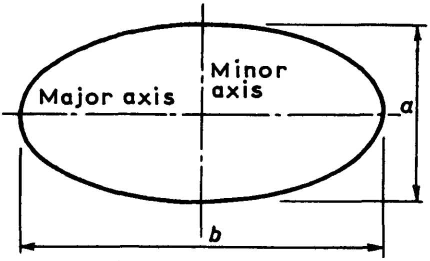


Fig. 9.10 Ellipse

where a and b are major and minor axes.

Parabola (Fig. 9.11)

$$A = \frac{2}{3}bh \qquad (9.26)$$

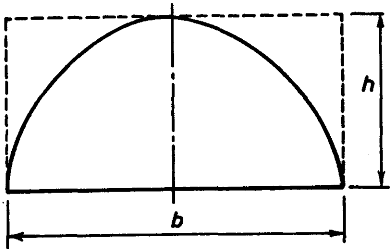


Fig. 9.11 Parabola

9.14 Surface areas

Curved surface of a cylinder (Fig. 9.12)

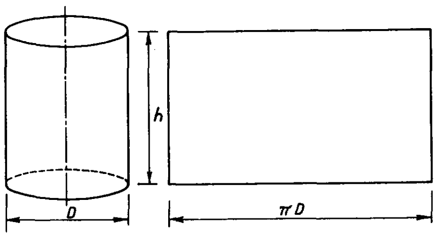


Fig. 9.12 Curved surface area of a cylinder

If the curved surface is laid out flat it will form a rectangle of length $2\pi r = \pi D$ and of height $h =$ height of cylinder.

$$\therefore \quad \text{Surface area (S.A.)} = 2\pi rh \qquad (9.27)$$

$$\text{S.A.} = \pi Dh \qquad (9.28)$$

Curved surface of a cone (Fig. 9.13)

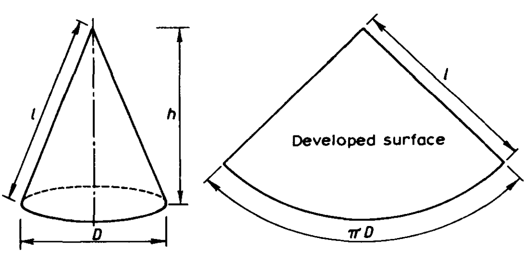


Fig. 9.13 Curved surface area of a cone

If the curved surface is laid out flat it will form the sector of a circle of radius l, i.e. the slant height of the cone.

By Eq. (9.17),

$$\text{Area of sector} = 1/2\, r^2\theta_{\text{rad}} = 1/2\, r \times r\theta$$
$$= 1/2 r \times \text{arc}$$
$$\text{which here} = 1/2 l \times \pi D$$
$$\text{S.A.} = 1/2\, \pi l D \tag{9.29}$$

i.e. $1/2 \times$ circumference of the base of the cone $\times$ slant height.

Surface area of a sphere (Fig. 9.14)

This is equal to the surface area of a cylinder of diameter D = diameter of the sphere and also of height D.

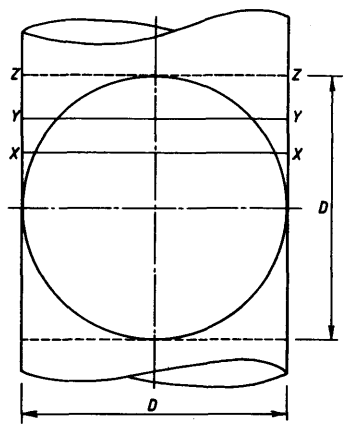


Fig. 9.14 Surface areas of a sphere

$$A = \pi D \times D$$
$$= \pi D^2 \tag{9.30}$$

or
$$A = 2\pi r \times 2r = 4\pi r^2 \tag{9.31}$$

Surface area of a segment of a sphere

In Fig. 9.14, if parallel planes are drawn perpendicular to the axis of the cylinder, the surface area of the segment bounded by these planes will be equal to the surface area of the cylinder bounded by these planes.

i.e.
$$\text{S.A.} = \pi D h \tag{9.32}$$
$$\text{S.A.} = 2\pi r h \tag{9.33}$$

where h = the height of the segment.

N.B. *The areas of similar figures are proportional to the squares of the corresponding sides* (9.34)

In Fig. 9.15,

$$\frac{\text{Area } \Delta ABC}{\text{Area } \Delta AB_1C_1} = \frac{AB^2}{AB_1^2}$$

$$= \frac{BC^2}{B_1C_1^2} = \frac{AC}{AC_1^2} = \frac{h_1^2}{h_2^2}$$

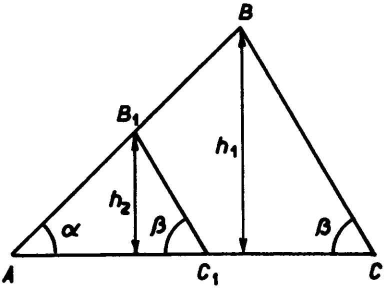


Fig. 9. 15 Areas of similar figures

Similarly, in Fig. 9.16,

Area of circle 1 $= \pi r_1^2 = 1/4\, \pi d_1^2$

Area of circle 2 $= \pi r_2^2 = 1/4\, \pi d_2^2$

$$\therefore \text{ as before } \frac{\text{Area 1}}{\text{Area 2}} = \frac{r_1^2}{r_2^2} \quad (9.35)$$

$$= \frac{d_1^2}{d_2^2} \quad (9.36)$$

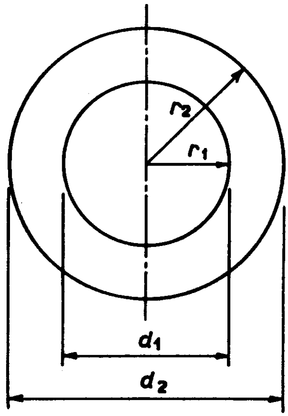


Fig. 9.16 Comparison of areas of circles

Example 9.1. The three sides of a triangular field are 663·75 links, 632·2 links and 654·05 links. Calculate its area in acres.

$a =$	663·75	$s - a =$	311·25
$b =$	632·20	$s - b =$	342·80
$c =$	654·05	$s - c =$	320·95
	2)1950·00	$s =$	975·00
$s =$	975·00		

By Eq. (11.3),

$$\begin{aligned}
\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{975{\cdot}00 \times 311{\cdot}25 \times 342{\cdot}80 \times 320{\cdot}95} \\
&= 182\,724 \text{ link}^2 \\
&= \underline{1{\cdot}827 \text{ acres}}
\end{aligned}$$

Example 9.2 A parallelogram has sides 147·2 and 135·7 ft. If the acute angle between the sides is $34°32'$ calculate its area in square yards.

By Eq. (11.7),

$$\begin{aligned}
\text{Area} &= ab \sin \alpha \\
&= 147{\cdot}2 \times 135{\cdot}7 \sin 34°32' \\
&= 11\,323 \text{ ft}^2 \\
&= \underline{1\,258 \text{ yd}^2}
\end{aligned}$$

Example 9.3 The area of a trapezium is 900 ft². If the perpendicular distance between the two parallel sides is 38 ft find the length of the parallel sides if their difference is 5 ft.

By Eq. (9.8),

$$A = \{x + (x-5)\}\frac{38}{2} = 900$$

$$\therefore \quad 2x - 5 = \frac{1800}{38}$$

$$x = \frac{47{\cdot}368 + 5}{2} = \frac{52{\cdot}368}{2}$$

$$= 26{\cdot}184$$

$\therefore$ Lengths of the parallel sides are 26·184 and 21·184.

Example 9.4 ABC is a triangular plot of ground in which AB measures 600 ft, (182·88 m). If angle $C = 73°$ and angle $B = 68°$ find the area in acres.

In triangle ABC,

$$\begin{aligned}
BC &= \frac{AB \sin A}{\sin C} = \frac{600 \sin\{180 - (73+68)\}}{\sin 73} \\
&= \frac{600 \sin 39°}{\sin 73°} \\
&= 394{\cdot}85 \text{ ft} \qquad (120{\cdot}345 \text{ m})
\end{aligned}$$

By Eq. (9.2),

$$\text{Area of triangle} = \tfrac{1}{2} BC.AB \sin B$$
$$= \tfrac{1}{2} \times 394{\cdot}85 \times 600 \sin 68$$
$$= 109\,829 \text{ ft}^2 \quad (10\,203 \text{ m}^2)$$
$$= \frac{109\,829}{9 \times 4840} \text{ acres}$$
$$= \underline{2{\cdot}521 \text{ acres.}}$$

Example 9.5 Calculate the area of the underground roadway from the measurements given. Assume the arch to be circular.

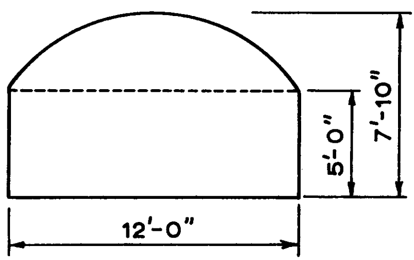


Fig. 9.17

In the segment (Fig. 9.18)

$$h = 7'\text{-}10'' - 5' = 2'\text{-}10''$$
$$c = 12'\text{-}0''$$
$$r^2 = (r-h)^2 + \left(\frac{c}{2}\right)^2$$
$$= r^2 - 2rh + h^2 + \tfrac{1}{4}c^2$$
$$\therefore \quad r = \frac{h^2 + \frac{1}{4}c^2}{2h} = \frac{2{\cdot}833^2 + \frac{144}{4}}{2 \times 2{\cdot}833}$$
$$= 7{\cdot}769 \text{ ft}$$
$$\sin\frac{\theta}{2} = \frac{c}{2r} = \frac{12}{2 \times 7{\cdot}769}$$
$$= 0{\cdot}77\,230$$
$$\frac{\theta}{2} = 50°33'40''$$
$$\therefore \quad \theta = 101°07'20''$$

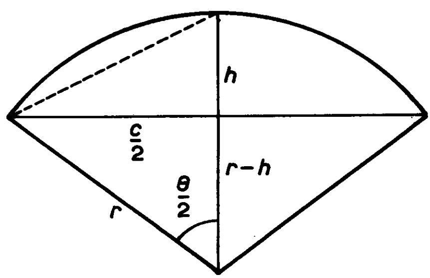


Fig. 9.18

$$\text{Area of segment} = \tfrac{1}{2} r^2(\theta - \sin\theta)$$
$$= \frac{7{\cdot}769^2}{2}(1{\cdot}764\,92 - 0{\cdot}981\,22)$$
$$= 30{\cdot}179 \times 0{\cdot}783\,70$$
$$= \underline{23{\cdot}651 \text{ ft}^2}$$
$$\text{Area of rectangle} = 5 \times 12 = 60 \text{ ft}^2$$
$$\therefore \quad \underline{\text{Total area} = 83{\cdot}651 \text{ ft}^2}$$

Check

By Eq. (9.20),

$$\text{Area of segment} \simeq \tfrac{4}{3}\times 2{\cdot}833\sqrt{\left(\frac{5\times 2{\cdot}833}{8}\right)^2+6^2}$$
$$\simeq 3{\cdot}776\sqrt{3{\cdot}135+36}$$
$$\simeq 3{\cdot}776\times 6{\cdot}294$$
$$\simeq \underline{23{\cdot}77\ \text{ft}^2}$$

By Eq. (9.22),

$$\text{Area of segment} \simeq \frac{3\times 2{\cdot}833^3+4\times 144\times 2{\cdot}833}{6\times 12}$$
$$\simeq \underline{23{\cdot}61\ \text{ft}^2}$$

Example 9.6 A square of 6 ft sides is to be subdivided into three equal parts by two straight lines parallel to the diagonal. Calculate the perpendicular distance between the parallel lines.

Triangle $ABC = \frac{1}{2}$ area of square

Triangle $BEF = \frac{1}{3}$ area of square

Length of diagonal $AC = 6\sqrt{2}$

$\therefore$ Length $BJ = 3\sqrt{2}$
$= 4{\cdot}242$

$$\frac{\text{Area of }\Delta\ BFE}{\text{Area of }\Delta\ ABC} = \frac{2}{3}$$

$$\therefore\quad \frac{BK^2}{BJ^2} = \frac{2}{3}$$

$$BK^2 = \frac{4{\cdot}242^2}{1{\cdot}5}$$

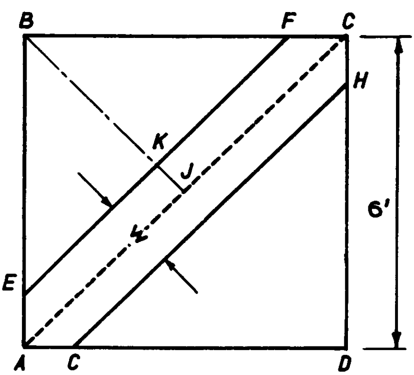


Fig. 9.19

$$BK = \frac{4{\cdot}242\times\sqrt{1{\cdot}5}}{1{\cdot}5}$$
$$= 3{\cdot}464\ \text{ft}$$

$$\therefore\quad KJ = BJ - BK$$
$$= 4{\cdot}242 - 3{\cdot}464 = 0{\cdot}778$$

$\therefore$ <u>Width apart of parallel lines = 1·556 ft</u>

Example 9.7 The side of a square paddock is 65 yd (59·44 m). At a point in one side 19½ yd (17·83 m) from one corner a horse is tethered by a rope 39 yd (35·66 m) long.

What area of grazing does the horse occupy?

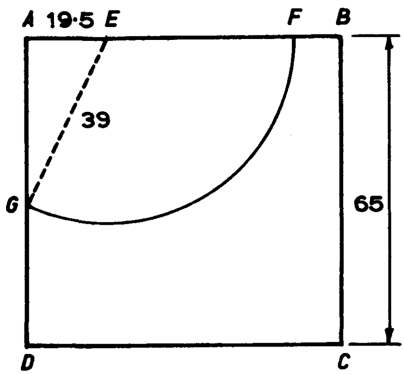


Fig. 9.20

The area occupied consists of a right-angled triangle AEG + a sector EFG.

$$\text{In triangle } AEG \quad \cos \hat{E} = \frac{19\tfrac{1}{2}}{39} = 0{\cdot}5$$

$$\hat{E} = 60°$$

By Eq. (9.2),

$$\text{Area of triangle } AEG = \tfrac{1}{2} \times 19\tfrac{1}{2} \times 39 \sin 60$$

$$= 329{\cdot}31\,\text{yd}^2 \quad (275{\cdot}34\,\text{m}^2)$$

By Eq. (9.17),

$$\text{Area of sector} = \tfrac{1}{2} \times 39^2 \times 120^\circ_{\text{rad}}$$

$$= 760{\cdot}5 \times 2{\cdot}094$$

$$= 1592{\cdot}49\,\text{yd}^2 \quad (1331{\cdot}52\,\text{m}^2)$$

$$\text{Total area} = \underline{1921{\cdot}80\,\text{yd}^2 \quad (1606{\cdot}86\,\text{m}^2) \text{ of grazing}}$$

Example 9.8 In order to reduce the amount of subsidence from the workings of a seam the amount of extraction is limited to 25% by driving 12 ft wide roadways. What must be the size of the square pillars left to fulfil this condition?

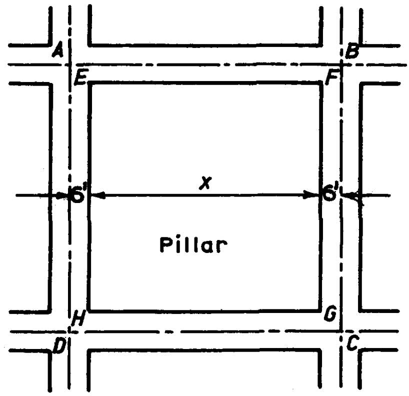


Fig. 9.21

N.B. The areas of similar figures are proportional to the squares of the relative dimensions,

$$\text{i.e.} \quad \frac{100}{75} = \frac{(x+12)^2}{x^2}$$

$$\begin{aligned} \text{i.e. } 100x^2 &= 75(x^2 + 24x + 144) \\ &= 75x^2 + 1800x + 10\,800 \\ \therefore \quad 25x^2 - 1800x - 10\,800 &= 0 \\ x^2 - 72x - 432 &= 0 \end{aligned}$$

By the formula for solving quadratic equations.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $b = -72$, $c = -432$, $a = 1$,

$$\begin{aligned} x &= \frac{72 \pm \sqrt{5184 + 1728}}{2} \\ &= \frac{72 \pm \sqrt{6912}}{2} \\ &= 36 \pm 41{\cdot}57 = \underline{77{\cdot}57 \text{ ft}} \end{aligned}$$

Alternative Solution

Size of square representing 100%	$= \sqrt{100}$	$= 10$	(AB)
Size of square representing 75%	$= \sqrt{75}$	$= 8{\cdot}66$	(EF)
Difference in size		$= 1{\cdot}34$	
Actual difference in size		$= 12$ ft	

If x is the size of the pillar, $1{\cdot}34 : 12 :: 8{\cdot}66 : x$

$$\therefore \qquad x = \frac{12 \times 8{\cdot}66}{1{\cdot}34}$$

$$= 77{\cdot}57 \text{ ft}$$

Example 9.9 A circular shaft is found to be 4 ft out of vertical at the bottom. If the diameter of the shaft is 20 ft, find the area of the crescent-shaped portion at the bottom of the shaft which is outside the circumference from the true centre at the surface.

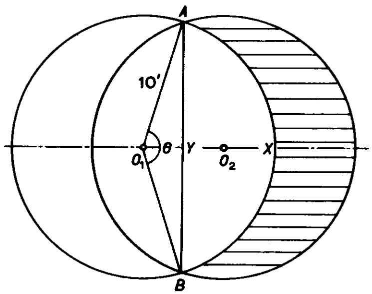


Fig. 9.22

Area of crescent-shaped portion = Area of circle − 2 (area of segment AXB).

Area of circle $= \pi r^2 = 3{\cdot}1416 \times 100 = 314{\cdot}16 \text{ ft}^2$

Area of sector $AXBO_1 = \frac{1}{2}r^2\theta$

$$\theta/2 = \cos^{-1}\frac{OY}{OA} = \cos^{-1}\frac{2}{10}$$

$$\frac{\theta}{2} = 78°32'$$

$$\theta = 157°04'$$

Area of sector $AXBO_1 = \frac{1}{2}100 \times 2{\cdot}74\,133 = 137{\cdot}07$

Area of triangle $OAB = \frac{1}{2}100 \sin 157°04' = \underline{19{\cdot}48}$

Area of segment $AXB = 117{\cdot}59$

Area of crescent $= 314{\cdot}16 - 2 \times 117{\cdot}59 = \underline{78{\cdot}98 \text{ ft}^2}$

Example 9.10 In a quadrilateral $ABCD, \hat{A} = 55°10'$, $\hat{B} = 78°30'$ $\hat{C} = 136°20'$. ($\hat{A}$ and $\hat{C}$ are diagonally opposite each other) AB = 620 links, DC = 284 links.

(a) Plot the figure to scale and from scaled values obtain the area in acres.

(b) Calculate the area in acres.

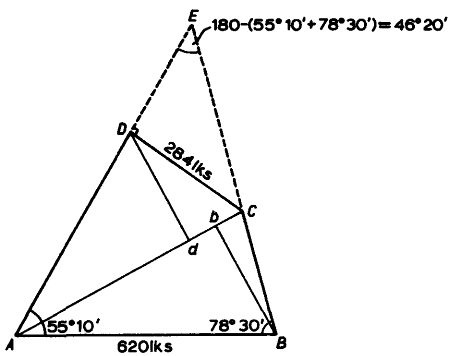


Fig. 9.23

Construction (N.B. $\hat{D} = 360 - (55°10' + 78°30' + 136°20') = 90°$)
Draw a line parallel to AD 284 links away. This will cut AE at C.

From scaling,

$$AC = 637 \text{ links}$$
$$Dd = 256 \text{ links}$$
$$Bb = \underline{296} \text{ links}$$
$$552$$

$$\text{Area} = \tfrac{1}{2}(637 \times 552) = 175\,810 \text{ sq links}$$
$$= 1{\cdot}758\,1 \text{ acres}$$

Calculation

$$\text{In triangle } AEB, \quad AE = \frac{AB \sin B}{\sin E} = \frac{620 \sin 78°30'}{\sin 46°20'} = 839{\cdot}89 \text{ lks}$$

$$\text{Area of triangle } AEB = \tfrac{1}{2} AE.AB \sin A$$
$$= \tfrac{1}{2} 839{\cdot}89 \times 620 \sin 55°10' = 213\,713 \text{ sq lks}$$

$$\text{In triangle } DEC, \quad DE = DC \cot 46°20' =$$
$$= 284 \cot 46°20' = 271{\cdot}08 \text{ lks}$$

Area of triangle DEC $= \frac{1}{2}DE.DC$

$= \frac{1}{2}271{\cdot}08 \times 284 \qquad = 38\,493$ sq lks

Area of triangle $ABCD$ $= 213\,713 - 38\,493 \qquad = 175\,220$ sq lks

$= \underline{1{\cdot}7522 \text{ acres}}$

9.2 Areas of Irregular Figures

In many cases an irregular figure can conveniently be divided into a series of regular geometrical figures, the total area being the sum of the separate parts.

If the boundary of the figure is very irregular the following methods may be employed.

9.21 Equalisation of the boundary to give straight lines (Fig. 9.24)

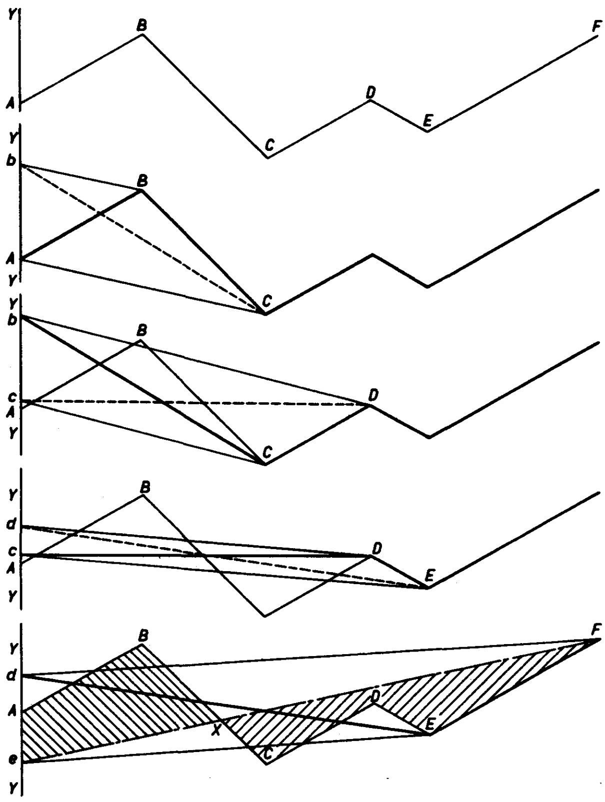


Fig. 9.24 Equalisation of an irregular boundary

The irregular boundary $ABCDEF$ is to be equalised by a line from a point on YY to F.

Construction

(1) Join A to C; draw a line Bb parallel to AC cutting YY at b. Triangle AbC is then equal to triangle ABC.
(triangles standing on the same base and between the same parallels are equal in area).

(2) Repeat this procedure.
Join bD, the parallel through C to give c on YY.

(3) This process is now repeated as shown until the final line eF equalises the boundary so that area $eABX$ = area $XFEDC$.

N.B. (a) This process may be used to reduce a polygon to a triangle of equal area.

(b) With practice there is little need to draw the construction lines but merely to record the position of the points b, c, d, e, etc. on line YY.

Reduction of a polygon (Fig. 9.25)

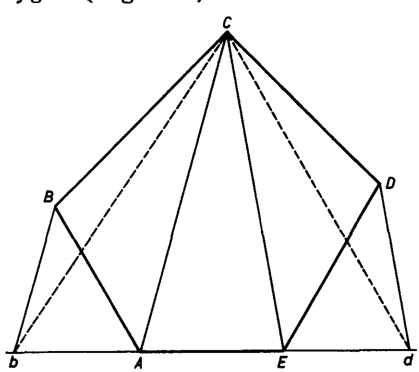


Fig. 9.25 Reduction of a polygon to a triangle of equal area

Area of triangle bCd = Area of polygon $ABCDE$.

(4) Where the boundary strips are more tortuous the following methods may be adopted.

9.22 The mean ordinate rule (Fig. 9.26)

The figure is divided into a number of strips of equal width and the lengths of the ordinates o_1, o_2, o_3 etc. measured.

(N.B. If the beginning or end of the figure is a point the ordinate is included as o_n = zero.)

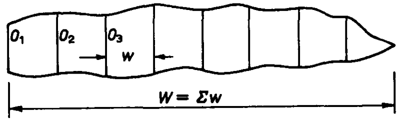


Fig. 9.26 The mean ordinate rule

The area is then calculated as

$$A = \frac{(o_1 + o_2 + o_3 + o_4 + \dots o_n)}{n} \times (n-1)w \qquad (9.37)$$

where n = number of ordinates

or $$A = \frac{\Sigma \text{ordinates} \times W}{n} \qquad (9.38)$$

where $W = \Sigma w$.

This method is not very accurate as it implies that the average ordinate is multiplied by the total width W.

9.23 The mid-ordinate rule (Fig. 9.27)

Here the figure is similarly divided into equal strips but these are then sub-divided, each strip having a mid-ordinate, *aa*, *bb*, *cc* etc.

The average value of these mid-ordinates is then multiplied by the total width W.

i.e. Area = mW

where m = the mean of the mean ordinates. (9.39)

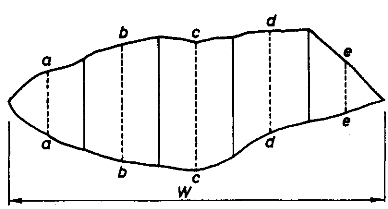


Fig. 9.27 The mid-ordinate rule

The only advantage of this method is that the number of scaled values is reduced.

9.24 The trapezoidal rule (Fig. 9.28)

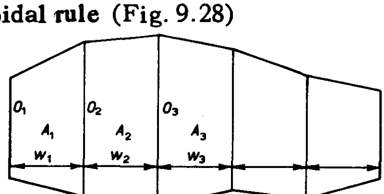


Fig. 9.28 The trapezoidal rule

This is a more accurate method which assumes that the boundary between the extremities of the ordinates are straight lines.

The area of the first trapezium $A_1 = w_1\left(\dfrac{o_1 + o_2}{2}\right)$

$$= \frac{w_1}{2}(o_1 + o_2)$$

The area of the second trapezium $A_2 = \dfrac{w_2}{2}(o_2 + o_3)$

The area of the last trapezium $A_{n-1} = \dfrac{w_{n-1}}{2}(o_{n-1} + o_n)$

If $w_1 = w_2 = w_3 = w_n = w$;

then the total area $= \dfrac{w}{2}[o_1 + 2o_2 + 2o_3 + \dots 2o_{n-1} + o_n]$ (9.40)

9.25 Simpson's rule (Fig. 9.29)

This assumes that the boundaries are curved lines and are considered as portions of parabolic arcs of the form $y = ax^2 + bx + c$.

The area of the figure Aab_1cCB is made up of two parts, the trapezium $AabcC$ + the curved portion above the line abc.

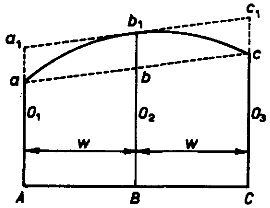


Fig. 9.29 Simpson's rule

The area of $ab_1cb = \frac{2}{3}$ of the parallelogram $aa_1\,b_1\,c_1\,cb$.

$$= \frac{4w}{3}\left[o_2 - \tfrac{1}{2}(o_1 + o_3)\right]$$

$$\therefore \quad \text{Total area} = 2w\left[\tfrac{1}{2}(o_1 + o_3)\right] + \frac{4w}{3}\left[o_2 - \tfrac{1}{2}(o_1 + o_3)\right]$$

$$= \frac{w}{3}[o_1 + 4o_2 + o_3] \qquad (9.41)$$

N.B. This is of the same form as the prismoidal formula with the linear values of the ordinates replacing the cross-sectional areas.

If the figure is divided into an even number of parts giving an odd number of ordinates, the total area of the figure is given as

$$A = \frac{w}{3}[(o_1 + 4o_2 + o_3) + (o_3 + 4o_4 + o_5) + \dots o_{n-2} + 4o_{n-1} + o_n]$$

$$\therefore \quad A = \frac{w}{3}[o_1 + 4o_2 + 2o_3 + 4o_4 + 2o_5 + \dots 2o_{n-2} + 4o_{n-1} + o_n] \qquad (9.42)$$

The rule therefore states that:

"if the figure is divided into an even number of divisions, the total area is equal to one third of the width between the ordinates multiplied by the sum of the first and last ordinate + twice the sum of the remaining odd ordinates + four times the sum of the even ordinates" This rule is more accurate than the others for most irregular areas and volumes met with in surveying.

Example 9.11 A plot of land has two straight boundaries *AB* and *BC* and the third boundary is irregular. The dimensions in feet are *AB* = 720, *BC* = 650 and the straight line *CA* = 828. Offsets from *CA* on the side away from *B* are 0, 16, 25, 9, 0 feet at chainages 0, 186, 402, 652, and 828 respectively from *A*.

(a) Describe briefly three methods of obtaining the area of such a plot.

(b) Obtain, by any method, the area of the above plot in acres, expressing the result to two places of decimals.

(R.I.C.S.)

The area of the figure can be found by

(1) the use of a planimeter;

(2) the equalisation of the irregular boundary to form a straight line and thus a triangle of equal area

(3) the solution of the triangle *ABC* + the area of the irregular boundary above the line by one of the ordinate solutions.

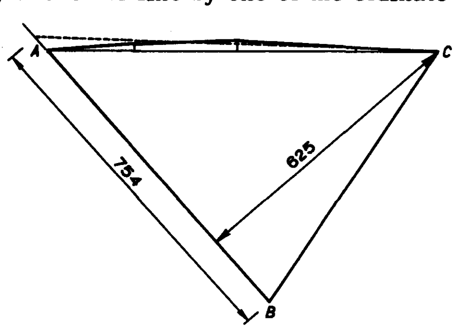


Fig. 9.30

By (2), Area $= \frac{1}{2}(754 \times 625) = \underline{235\,625 \text{ ft}^2} = \underline{5{\cdot}41 \text{ acres}}$

By (3), Area of triangle $ABC = \sqrt{s(s-a)(s-b)(s-c)}$ (Eq. 9.3)

where $s = \frac{1}{2}(a+b+c)$

$$\begin{array}{lrclrcll}
\text{i.e.} & a & = & 650 & s-a & = & 449 & \\
& b & = & 828 & s-b & = & 271 & \\
& c & = & 720 & s-c & = & \underline{379} & \\
& & & 2)\,\underline{2198} & s & = & \underline{1099} & \textit{Check} \\
& s & = & 1099 & & & &
\end{array}$$

$$\text{Area} = \sqrt{1099 \times 449 \times 271 \times 379}$$
$$= \underline{225\,126 \text{ ft}^2}$$

Area of the irregular boundary

(a) By the mean ordinate rule, Eq. (9.38),

$$\text{Area} = \frac{0 + 16 + 25 + 9 + 0}{5} \times 828$$
$$= \underline{8280 \text{ ft}^2}$$

$$\underline{\text{Total Area} \quad 233\,406 \text{ ft}^2}$$
$$= \underline{5{\cdot}35 \text{ acres}}$$

(b) By the trapezoidal rule, Eq. (9.40),

$$\text{Area} = \tfrac{1}{2}[186(0+16) + (402-186)(16+25) + (652-402)(25+9) + (828-652)(9+0)]$$
$$= \tfrac{1}{2}[186 \times 16 + 216 \times 41 + 250 \times 34 + 176 \times 9]$$
$$= \tfrac{1}{2}[2976 + 8856 + 8500 + 1584]$$
$$= \underline{10\,958 \text{ ft}^2}$$

$$\underline{\text{Total Area} \quad 236\,084 \text{ ft}^2}$$
$$= \underline{5{\cdot}43 \text{ acres}}$$

N.B. As the distance apart of the offsets is irregular neither the full Trapezoidal Rule nor Simpson's Rule are applicable.

9.26 The planimeter

This is a mechanical integrator used for measuring the area of irregular figures.

It consists essentially of two bars OA and AB, with O fixed as a fulcrum and A forming a freely moving joint between the bars. Thus A is allowed to rotate along the circumference of a circle of radius OA whilst B can move in any direction with a limiting circle OB.

Theory of the Planimeter (Fig. 9.31)

Let the joint at A move to A_1 and the tracing point B move first to B_1 and then through a small angle $\delta\alpha$ to B_2.

If the whole motion is very small, the area traced out by the tracing bar AB is $ABB_1B_2A_1$

i.e. $AB \times \delta h + \frac{1}{2}AB^2\delta\alpha$

or $\delta A = l\delta h + \frac{1}{2}l^2\delta\alpha$ (9.43)

where δA = the small increment of area,

l = the length of the tracing bar AB,

δh = the perpendicular height between the parallel lines AB and A_1B_1,

$\delta\alpha$ = the small angle of rotation.

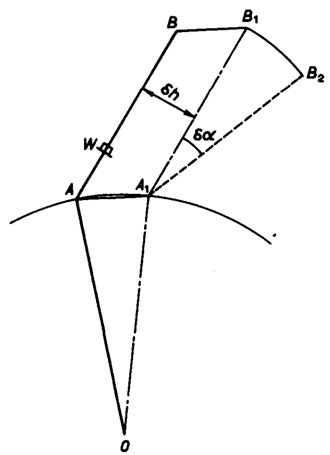


Fig. 9.31 Theory of the planimeter

A small wheel is now introduced at W on the tracing bar which will rotate, when moved at right-angles to the bar AB, and slide when moved in a direction parallel to its axis, i.e. the bar AB.

Let the length $AW = k.AB$.

Then the recorded value on the wheel will be

$$\delta W = \delta h + kAB\,\delta\alpha$$

i.e.
$$\delta h = \delta w - kAB\,\delta\alpha = \delta w - kl\,\delta\alpha$$

which when substituted in Eq. (9.43) gives

$$\delta A = l(\delta w - kl\,\delta\alpha) + \frac{1}{2}l^2\delta\alpha$$

$$= l\delta w + l^2\left(\tfrac{1}{2} - k\right)\delta\alpha \quad (9.44)$$

To obtain the total area with respect to the recorded value on the wheel and the total rotation of the arm, by integrating,

$$A = lw + l^2\left(\tfrac{1}{2} - k\right)\alpha \tag{9.45}$$

where A = the area traced by the bar,
w = the total displacement recorded on the wheel,
α = the total angle of rotation of the bar.

Two cases are now considered:

(1) When the fulcrum (O) is outside the figure being traced.
(2) When the fulcrum (O) is inside the figure being traced.

(1) *When the fulcrum O is outside the figure* (Fig. 9.32)

Commencing at a the joint is at A.

Moving to the right, the line *abcd* is traced by the pointer whilst the bar traces out the positive area (A_1) *abcdDCBA*.

Moving to the left, the line *defa* is traced out by the pointer whilst the negative area (A_2) *defaAFED* is traced out by the bar.

The difference between these two areas is the area of the figure *abcdefa*, i.e.

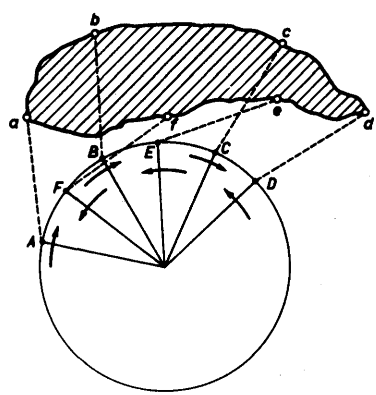


Fig. 9.32 Planimeter fulcrum outside the figure

$$A = A_1 - A_2 = lw + l^2\left(\tfrac{1}{2} - k\right)\alpha \qquad \text{(Eq. 9.45)}$$

but $\alpha = 0$

$\therefore$ $$A = lw \tag{9.46}$$

N.B. The joint has moved along the arc $ABCD$ to the right, then along $DEFA$ to the left.

In measuring such an area the following procedure should be followed:

(1) With the pole and tracing arms approximately at right-angles, place the tracing point in the centre of the area to be measured.
(2) Approximately circumscribe the area, to judge the size of the area compared with the capacity of the instrument. If not possible the pole should be placed elsewhere, or if the area is too

large it can be divided into sections, each being measured separately.

(3) Note the position on the figure where the drum does not record – this is a good starting point (A).

(4) Record the reading of the vernier whilst the pointer is at A.

(5) Circumscribe the area carefully in a *clockwise* direction and again read the vernier on returning to A.

(6) The difference between the first and second reading will be the required area. (This process should be repeated for accurate results).

(7) Some instruments have a variable scale on the tracing arm to give conversion for scale factors.

(2) *When the fulcrum O is inside the figure* (Fig. 9.33)

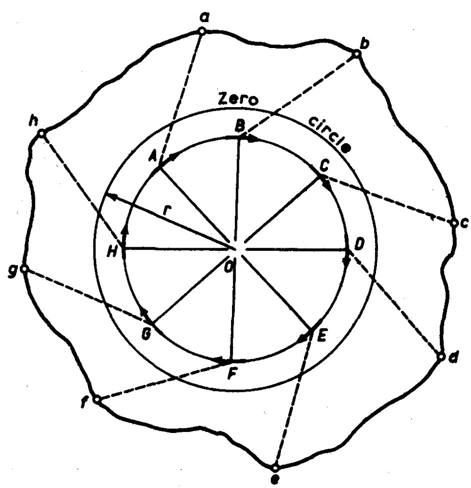


Fig. 9.33 Planimeter fulcrum inside the figure

In this case the bar traces out the figure (A_T) *abcdefgha* – the area of the circle (A_c) $ABCDEFGHA$; it has rotated through a full circle, i.e. $\alpha = 2\pi$.

$$A_T - A_c = lw + l^2\left(\tfrac{1}{2} - k\right)2\pi$$

$$\therefore \quad A_T = lw + l^2\left(\tfrac{1}{2} - k\right)2\pi + A_c$$

$$= lw + l^2\left(\tfrac{1}{2} - k\right)2\pi + \pi b^2 \quad \text{(where } b = OA\text{)}$$

$$= lw + \pi\{b^2 + l^2(1 - 2k)\} \tag{9.47}$$

This is explained as follows Fig. 9.34:

If the pointer P were to rotate without the wheel W moving, the angle OWP would be 90°

The figure thus described is known as the *zero circle* of radius r;

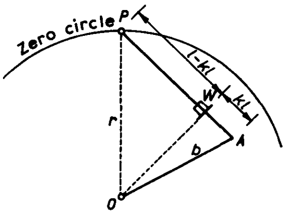


Fig. 9.34 Theory of the zero circle

i.e.

$$OW^2 = b^2 - (kl)^2$$

$$r^2 = OW^2 + (l - kl)^2$$

$$= b^2 - (kl)^2 + l^2 - 2kl^2 + (kl)^2$$

$$= b^2 + l^2(1 - 2k) \qquad (9.48)$$

∴ In Eq. (9.47),

$$A_T = lw + \pi r^2$$

$$= lw + \text{the area of the zero circle} \qquad (9.49)$$

The value of the zero circle is quoted by the manufacturer.

N.B. (1) $A_T - A_C = lw$. If $A_C > A_T$ then lw will be negative, i.e. the second reading will be less than the first, the wheel having a resultant negative recording.

(2) The area of the zero circle is converted by the manufacturer into revolutions on the measuring wheel and this *constant* is normally added to the recorded number of revolutions.

Example 9.12

$A_T > A_C$	1st reading	3·597
	2nd reading	12·642
	Difference	9·045
	Constant	23·515
	Total value	32·560
$A_T < A_C$	1st reading	6·424
	2nd reading	3·165
	Difference	−3·259
	Constant	23·515
	Total value	20·256

9.3 Plan Areas

9.31 Units of area

1 sq foot (ft^2)	=	144 sq inches (in^2)		
1 sq yard (yd^2)	=	9 ft^2		
1 acre	=	4 roods		
	=	10 sq chains	=	100 000 sq links
	=	4840 yd^2	=	43 560 ft^2
1 sq mile	=	640 acres		

Conversion factors

1 in^2	=	6·4516 cm^2	1 cm^2	=	0·155 000 in^2
1 ft^2	=	0·092 903 m^2			
1 yd^2	=	0·836 127 m^2	1 m^2	=	1·195 99 yd^2
1 sq chain	=	404·686 m^2			
1 rood	=	1011·71 m^2			
1 acre	=	4046·86 m^2	1 km^2	=	247·105 acres
	=	0·404 686 hectare (ha)	1 ha	=	2·471 05 acres
1 sq mile	=	2·589 99 km^2			
	=	258·999 ha			

N.B. The hectare is not an S.I. unit.

The British units of land measurement are the Imperial Acre and the Rood (the pole or perch is no longer valid).

The fractional part of an acre is generally expressed as a decimal although the rood is still valid.

Thus 56·342 acres becomes

$$\begin{array}{rr} & 56{\cdot}342 \text{ acres} \\ & \underline{4} \\ \underline{56 \text{ acres}} & \underline{1{\cdot}368 \text{ roods}} \end{array}$$

The use of the Gunter chain has been perpetuated largely because of the relationship between the acre and the square chain.

Thus 240 362 sq links = 24·036 2 sq chains

= 2·403 62 acres

The basic unit of area in the proposed International System is the square metre (m^2).

9.32 Conversion of planimetric area in square inches into acres

Let the scale of the plan be $\dfrac{1}{x}$.

i.e. $$1\text{ in.} = x\text{ in.}$$

$$\therefore \quad 1\text{ sq in.} = x^2\text{ sq in.}$$

$$= \frac{x^2}{12 \times 12 \times 9 \times 4840}\text{ acres}$$

Example 9.13

Find the conversion factors for the following scales. (a) 1/2500 (b) 6 in. to 1 mile. (c) 2 chains to 1 inch.

(a) $$1\text{ in}^2 = \frac{2500^2}{12 \times 12 \times 9 \times 4840} = \underline{0{\cdot}995\text{ acres}}$$

$$(4026{\cdot}6\text{ m}^2 = 0{\cdot}4026\text{ ha})$$

(b) 6 in. to 1 mile $\left(\dfrac{1}{10\,560}\right)$.

i.e. $$1\text{ in.} = \frac{1760 \times 36}{6} = 10\,560\text{ in.}$$

$$\therefore \quad 1\text{ in}^2 = \frac{10\,560^2}{144 \times 43\,560} = \underline{17{\cdot}778\text{ acres}}$$

$$(71\,945\text{ m}^2 = 7{\cdot}1945\text{ ha})$$

Alternatively, 6 in. = 1 mile

$$\therefore \quad 36\text{ in}^2 = 1\text{ mile}^2 = 640\text{ acres}$$

$$1\text{ in}^2 = \frac{640}{36} = \underline{17{\cdot}778\text{ acres}}$$

(c) 2 chains to 1 inch.

$$1\text{ in.} = 200\text{ links}$$

$$1\text{ in}^2 = 40\,000\text{ sq links}$$

$$= \underline{0{\cdot}4\text{ acres}} \qquad (1618{\cdot}7\text{ m}^2 = 0{\cdot}16187\text{ ha})$$

Alternatively, 2 chains to 1 inch (1/1584)

$$1\text{ in.} = 2 \times 66\text{ ft} = 132 \times 12 = 1584\text{ ft}$$

$$1\text{ in}^2 = \frac{1584^2}{144 \times 43\,560} = \underline{0{\cdot}4\text{ acre}}$$

9.33 Calculation of area from co-ordinates

Method 1. By the use of an enclosing rectangle (Fig. 9.35)

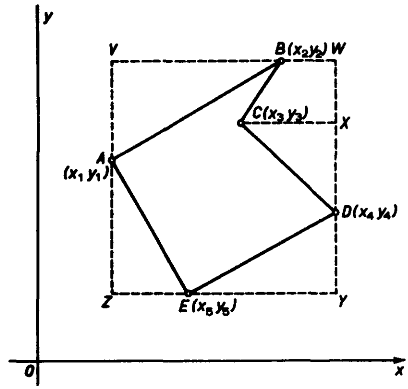


Fig. 9.35 Calculation of area by enclosing rectangle

The area of the figure $ABCDE$ = the area of the rectangle $VWXYZ$ $- \{\Delta AVB + BWXC + \Delta CXD + \Delta DYE + \Delta AEZ\}$.

This is the easiest method to understand and remember but is laborious in its application.

Method 2. By formulae using the total co-ordinates

Applying the co-ordinates to the above system we have:

$$\text{Area of rectangle } VWXYZ = (x_4 - x_1)(y_2 - y_5)$$

$$\text{of triangle } AVB = \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$$

$$\text{of trapezium } BWXC = \frac{1}{2}(y_2 - y_3)\{(x_4 - x_2) + (x_4 - x_3)\}$$

$$\text{of triangle } CXD = \frac{1}{2}(y_3 - y_4)(x_4 - x_3)$$

$$DYE = \frac{1}{2}(y_4 - y_5)(x_4 - x_5)$$

$$AEZ = \frac{1}{2}(y_1 - y_5)(x_5 - x_1)$$

i.e.
$$\begin{aligned} A = (x_4y_2 - x_4y_5 - x_1y_2 + x_1y_5) - \tfrac{1}{2}[\, & x_2y_2 - x_2y_1 - x_1y_2 + x_1y_1 + 2x_4y_2 \\ & - 2x_4y_3 - x_2y_2 + x_2y_3 - x_3y_2 + x_3y_3 \\ & + x_4y_3 - x_4y_4 - x_3y_3 + x_3y_4 + x_4y_4 \\ & - x_4y_5 - x_5y_4 + x_5y_5 + x_5y_1 - x_5y_5 \\ & - x_1y_1 + x_1y_5\,] \end{aligned}$$

$$\therefore \quad A = \frac{1}{2}[y_1(x_2 - x_5) + y_2(x_3 - x_1) + y_3(x_4 - x_2) + y_4(x_5 - x_3) + y_5(x_1 - x_4)]$$

This may be summarised as

$$A = \tfrac{1}{2}\Sigma y_n(x_{n+1} - x_{n-1}) \qquad (9.50)$$

i.e. Area = half the sum of the product of the total latitude of each station × the difference between the total departures of the preceding and following stations.

This calculation is best carried out by a tabular system.

Example 9.14

The co-ordinates of the corners of a polygonal area of ground are taken in order, as follows, in feet:

A (0, 0); *B* (200, −160); *C* (630, −205); *D* (1000, 70);
E (720, 400); *F* (310, 540); *G* (−95, 135), returning to *A*.

Calculate the area in acres.

Calculate also the co-ordinates of the far end of a straight fence from *A* which cuts the area in half.

To calculate the area of the figure *ABCDEFG* the co-ordinates are tabulated as follows:

Station	(a) T.Lat.	(b) T.Dep.	(c) Preceding ↑ Dep.	(d) Following ↓ Dep.	(c)−(d)	(a)×{(c) − (d)} +	(a)×{(c) − (d)} −
A	0	0	200	−95	295		
B	−160	200	630	0	630		100 800
C	−205	630	1000	200	800		164 000
D	70	1000	720	630	90	6300	
E	400	720	310	1000	−690		276 000
F	540	310	−95	720	−815		440 100
G	135	−95	0	310	−310		41 850
						6300	1022 750
							6 300
							2) 1016 450
							508 225 ft²

∴ Total Area = 508 225 ft² (47 215·63 m²)
= 11·667 acres (4·721 56 ha)

From a visual inspection it is apparent that the bisector of the area *AX* will cut the line *ED*.

The area of the figure *ABCD* can be found by using the above figures.

A	0	0	200	1000	−800	
B	−160	200	630	0	630	100 800
C	−205	630	1000	200	800	164 000
D	70	1000	0	630	−630	44 100
						2) 308 900
						154 450 ft²

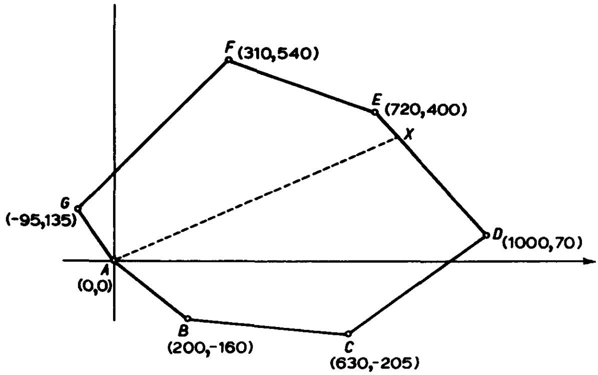


Fig. 9.36

∴ The area of the triangle AXD must equal

$$\frac{1}{2}(508\,225) - 154\,450 = 99\,662{\cdot}5\ \text{ft}^2$$

Also area $= \frac{1}{2}AD.DX \sin D$

$$\therefore \quad DX = \frac{2 \times 99\,662{\cdot}5}{AD \sin D}$$

To find length AD and angle D.

Bearing $DA = \tan^{-1} -1000/-70 = \text{S } 85°59'50''\text{ W} = 265°59'50''$

Length $DA = 1000/\sin 85°59'50'' = 1002{\cdot}45$ ft

Bearing $DE = \tan^{-1} -280/330 = \text{N } 40°18'50''\text{ W} = \underline{319°41'10''}$

Angle D $= 53°41'20''$

$$\therefore \quad DX = \frac{2 \times 99\,662{\cdot}5}{1002{\cdot}45 \sin 53°41'20''} = 246{\cdot}76\ \text{ft}$$

To *find the co-ordinates of X.* (N 40°18′50″ W 246·76 ft).

	$E_D = 1000{\cdot}00$	
sin bearing 0·646 97	$\Delta E = -159{\cdot}65$	$E_X = 840{\cdot}35$ ft
cos bearing 0·762 51	$\Delta N = +188{\cdot}16$	
	$N_D = 70{\cdot}00$	$N_X = 258{\cdot}16$ ft

Check on Area

A	0	0	200	840·35	−640·35		
B	−160	200	630	0	630		100 800
C	−205	630	1000	200	800		164 000
D	70	1000	840·35	630	210·35	14 724·5	
X	258·16	840·35	0	1000	−1000		258 160
							522 960
							14 724·5
							508 235·5 ft²

Method 3. *By areas related to one of the co-ordinate axes* (Fig. 9.37)

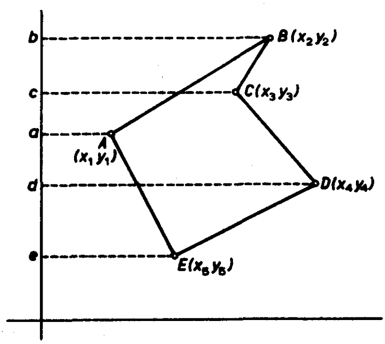


Fig. 9.37 Areas related to one axis

Area *ABCDE* = (*bBCc* + *cCDd* + *dDEe*) − (*bBAa* + *aAEe*)

Using the co-ordinates designated, trapeziums

$$\left.\begin{aligned} bBCc &= \tfrac{1}{2}(x_2+x_3)(y_2-y_3) \\ cCDd &= \tfrac{1}{2}(x_3+x_4)(y_3-y_4) \\ dDEe &= \tfrac{1}{2}(x_4+x_5)(y_4-y_5) \end{aligned}\right\} - \left\{\begin{aligned} bBAa &= \tfrac{1}{2}(x_2+x_1)(y_2-y_1) \\ aAEe &= \tfrac{1}{2}(x_1+x_5)(y_1-y_5) \\ & \end{aligned}\right. \qquad (9.51)$$

i.e. $$\tfrac{1}{2}[(x_2y_2-x_2y_3+x_3y_2-x_3y_3)+(x_3y_3-x_3y_4+x_4y_3-x_4y_4)$$
$$+(x_4y_4-x_4y_5+x_5y_4-x_5y_3)-(x_2y_2-x_2y_1+x_1y_2-x_1y_1)$$
$$-(x_1y_1-x_1y_5+x_5y_1-x_5y_5)]$$

$$= \tfrac{1}{2}[y_1(x_2-x_5)+y_2(x_3-x_1)+y_3(x_4-x_2)+y_4(x_5-x_3)+y_5(x_1-x_4)]$$

as before

$$\text{Area} = \frac{1}{2}\Sigma y_n(x_{n+1} - x_{n-1}) \qquad \text{(Eq. 9.50)}$$

Method 4. Area by 'Latitudes' and 'Longitudes' (Fig. 9.38)

Here *Latitude* is defined as the partial latitude of a line
Longitude is defined as the distance from the y axis to the centre of the line.

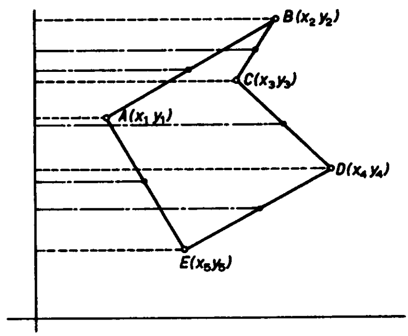


Fig. 9.38 Calculation of area by latitudes and longitudes

From Eq. (9.51),

$$A = \frac{1}{2}[\{(x_2+x_3)(y_2-y_3)+(x_3+x_4)(y_3-y_4)+(x_4+x_5)(y_4-y_5)\}$$

$$-\{(x_2+x_1)(y_2-y_1)+(x_1+x_5)(y_1-y_5)\}]$$

$$= \frac{1}{2}[(x_1+x_2)(y_1-y_2)+(x_2+x_3)(y_2-y_3)+(x_3+x_4)(y_3-y_4)$$

$$+(x_4+x_5)(y_4-y_5)+(x_5+x_1)(y_5-y_1)]$$

$$= \frac{1}{2}\Sigma(x_n + x_{n+1})(y_n - y_{n+1}) \qquad (9.52)$$

where $\frac{1}{2}(x_n + x_{n+1})$ = the longitude of a line.

$(y_n - y_{n+1})$ = the partial latitude of a line, i.e. the latitude.

N.B. (1) It is preferable to use double longitudes and thus produce double areas, the total sum being finally divided by 2

(2) This method is adaptable for tabulation using total departures and partial latitudes or vice versa.

Example 9.15

From the previous Example 9.14 the following table is compiled.

Stn.	T. Dep.	P. Lat.	Sum of Adj. T. Dep.	Double Area +	Double Area −
A	0				
		−160	200		32 000
B	200				
		−45	830		37 350
C	630				
		+275	1630	448 250	
D	1000				
		+330	1720	567 600	
E	720				
		+140	1030	144 200	
F	310				
		−405	215		87 075
G	−95				
		−135	−95	12 825	
A	0				
				1 172 875	156 425
				156 425	
				2)1 016 450	
				508 225 ft^2	

9.34 Machine calculations with checks

Using Eq. (9.52),

	(1) y	(2) x	(3) dy	(4) Σx	(5) = (3)×(4)	(6) dx	(7) Σy	(8) = (6)×(7)
A	0	0	−160	200	−32 000	200	−160	−32 000
B	−160	200	− 45	830	−37 350	430	−365	−156 950
C	−205	630	+275	1630	448 250	370	−135	−49 950
D	70	1000	+330	1720	567 600	−280	470	−131 600
E	400	720	+140	1030	144 200	−410	940	−385 400
F	540	310	−405	215	−87 075	−405	675	−273 375
G	135	−95	−135	−95	+12 825	95	135	+12 825
	+1145	+2860	+745	+5625	+1 172 875	+1095	+2220	+12 825
	−365	−95	−745	−95	−156 425	−1095	−660	−1 029 275
	+780	+2765		+5530	+1 016 450		1560	1 016 450
	(×2)	(×2)						
	1560	5530			508 225 ft^2			508 225ft^2

N.B. (1) From the total co-ordinates in columns 1 and 2, the sum and difference between adjacent stations are derived.

e.g. $dy_{AB} = (-160-0) \quad = -160 \qquad \Sigma y_{AB} = -160+0 \quad = -160$

$dy_{BC} = (-205+160) = -\ 45 \qquad \Sigma y_{BC} = -160-205 = -365$

(2) All the arithmetical checks should be carried out to prove the insertion of the correct values.

(a) 2 $\times$ the sum of the total latitudes $= \Sigma y$ (col. 7).
(b) 2 $\times$ the sum of the total departures $= \Sigma x$ (col. 4).
(c) The Algebraic sum of columns 3 and 6 should equal zero.

(3) The product of columns 3 and 4 gives column 5, i.e. the double area:
Also the product of columns 6 and 7 gives column 8, i.e. the double area:
Columns 5 and 8 when totalled should check.

Alternative method

From the previous work it is seen that the area is equal to one-half of the sum of the products obtained by multiplying the ordinate (latitude) of each point by the difference between the abscissae (departure) of the following and preceding points.

i.e.

$$A = \tfrac{1}{2}(x_2y_1 - x_5y_1 + x_3y_2 - x_1y_2 + x_4y_3 - x_2y_3 + x_5y_4 - x_3y_4 + x_1y_5 - x_4y_5) \tag{9.53}$$

This may be written as,

$$A = \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{bmatrix} \tag{9.54}$$

This is interpreted as 'the area equals one half of the sum of the products of the co-ordinates joined by solid lines minus one half of the sum of the products of the co-ordinates joined by dotted lines,

This method has more multiplications but only *one subtraction.*

Using the previous example;

$$A = \frac{1}{2}\begin{bmatrix} 0 & 200 & 630 & 1000 & 720 & 310 & -95 \\ 0 & -160 & -205 & 70 & 400 & 540 & 135 \end{bmatrix}$$

$$= \tfrac{1}{2}[-182700 - 833750]$$

$$= \tfrac{1}{2}[1\,016\,450] \qquad \text{(negative sign neglected)}$$

$$= \underline{508\,225 \text{ ft}^2}$$

9.4 Subdivision of Areas*

9.41 The subdivision of an area into specified parts from a point on the boundary (Fig. 9.39)

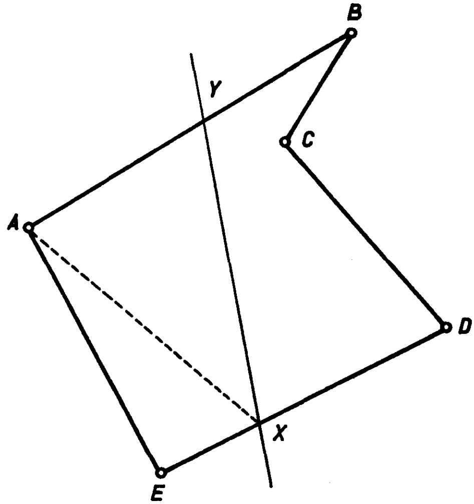


Fig. 9.39 Subdivision of an area from a point on the boundary

Consider the area $ABCDE$ to be equally divided by a line starting from X on the line ED.

(1) Plot the co-ordinates to scale.

(2) By inspection or trial and error decide on the approximate line of subdivision XY.

(3) Select a station nearest to the line XY, i.e. A or B.

(4) Calculate the total area $ABCDE$.

(5) Calculate the area AXE.

(6) Calculate the area $AXY = \frac{1}{2}ABCDE - AXE$.

(7) Calculate the length and bearing of AX.

(8) Calculate the bearing of line ED.

(9) Calculate the length AY in triangle AYX.

N.B. Area of triangle $AYX = \frac{1}{2}AX.XY \sin \hat{X}$.
As the area, AX and $\hat{X}$ are known, AY is calculated:

$$AY = \frac{\text{Area of triangle } AYX}{\frac{1}{2}AX \sin \hat{X}} \tag{9.55}$$

(10) Calculate the co-ordinates of Y.

* For a complete analysis of this the reader is advised to consult *The Basis of Mine Surveying* by M.H. Haddock.

9.42 The subdivision of an area by a line of known bearing (Fig. 9.40)

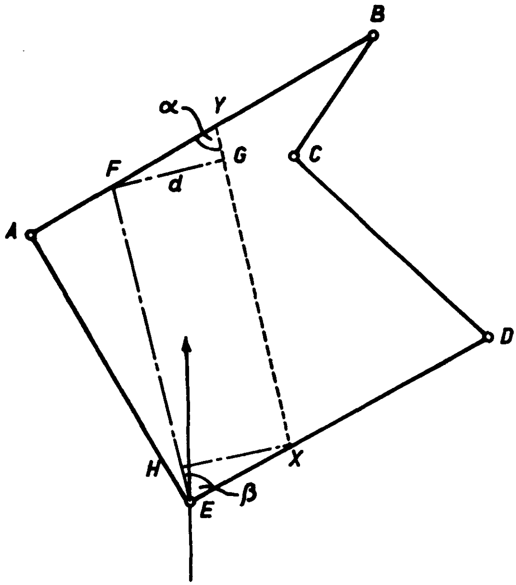


Fig. 9.40 Subdivision of an area by a line of known bearing

Construction

Given the area $ABCDE$ and the bearing of the line of sub-division XY, then EF on the given bearing and XY will be parallel to this, a perpendicular distance d away.

Draw FG perpendicular to XY.
HX perpendicular to EF.

$$\text{The area } AYXE = \tfrac{1}{2}\text{area } ABCDE$$

$$= \Delta AFE + \Delta FYG + FGXH + \Delta HXE.$$

(1) From the co-ordinates the length and bearing of AE can be calculated.

(2) In the triangle AFE the area can be obtained by first solving for the length EF.

(3) The area of the figure $FYGXEH$ can thus be obtained in terms of d, i.e.

$$\text{triangle } FYG = \tfrac{1}{2}d^2\cot a$$

$$\text{rectangle } FGXH = d(EF - HE)$$

$$= d\,(EF - d\cot\beta)$$

$$\text{triangle } HXE = \tfrac{1}{2}d^2\cot\beta$$

$$\therefore \quad dEF + \tfrac{1}{2}d^2(\cot\alpha - \cot\beta) = \text{Area of } AYXE - \text{Area of } \Delta AFE. \tag{9.56}$$

This is a quadratic in d as the angles α and β are found from the bearings.

From the value of d the co-ordinates of F and E can be obtained.

9.43 The subdivision of an area by a line through a known point inside the figure (Fig. 9.41)

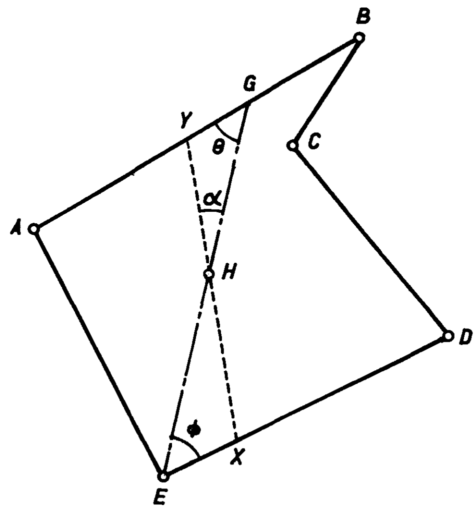


Fig. 9.41 Subdivision of an area by a line through a known point inside the figure

Construction

Given the area $ABCDE$ and the co-ordinates of the fixed point H, join EH and produce to cut AB in G. Assume the dividing line XY is rotated α^0 about H.

From the co-ordinates:

(1) Calculate the length and bearing EH.

(2) In the triangle AGE calculate the length EG (this gives the length AG) and the area.

(3) The required area $AYXE = \Delta AGE - \Delta YGH + \Delta EHX$
$= \Delta AGE - \text{area } (A)$

To find the missing area,

$$A = \tfrac{1}{2}EH.HX \sin\alpha + \tfrac{1}{2}GH.HY \sin\alpha$$

$$= \tfrac{1}{2}EH \times \frac{EH \sin\phi \sin\alpha}{\sin(\alpha+\phi)} + \tfrac{1}{2}GH \times \frac{GH \sin\theta \sin\alpha}{\sin(\alpha+\theta)}$$

$$= \frac{1}{2}\left[\frac{EH^2}{\cot\alpha + \cot\phi} + \frac{GH^2}{\cot\alpha + \cot\theta}\right]$$

$$\therefore \quad 2A = \frac{EH^2}{\cot\alpha + \cot\phi} + \frac{GH^2}{\cot\alpha + \cot\theta} \tag{9.57}$$

As the lengths EH and GH are known, and ϕ and θ are obtainable from the bearings, this is a quadratic equation in $\cot\alpha$, from which α may be found, and thus the co-ordinates of X and Y.

These problems are best treated from first principles based on the foregoing basic ideas.

Example 9.16. In a quadrilateral $ABCD$, the co-ordinates of the points, in metres, are as follows:

Point	E	N
A	0	0
B	0	−893·8
C	+634·8	−728·8
D	+1068·4	+699·3

Find the area of the figure by calculation.

If E is the mid-point of AB, find, graphically or by calculation, the co-ordinates of a point F, on the line CD, such that the area $AEFD$ equals the area $EBCF$.

N.B. Co-ordinates of $E = \frac{1}{2}(A + B)$

i.e. $0, \frac{1}{2} \times -893{\cdot}8 = 0, -446{\cdot}9$

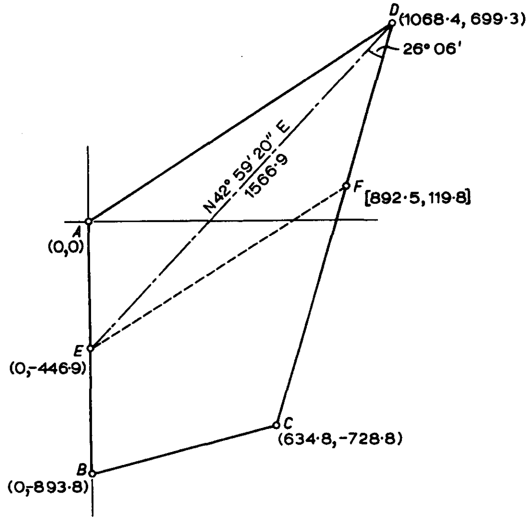


Fig. 9.42

	x	y	dx	Σy		dy	Σx	
A	0	0	0	−893·8	-	−893·8	0	104 742·00
B	0	−893·8	634·8	−1622·6	−1 030 026·48	165·0	634·8	2 432 339·92
C	634·8	−728·8	433·6	−29·5	−12 791·20	1428·1	1703·2	
D	1068·4	+699·3	−1068·4	699·3	−747 132·12	−699·3	1068·4	−747 132·12
	1703·2	699·3	1068·4	−699·3		1593·1	3406·4	2 537 081·92
		−1622·6	−1068·4	−2545·9	−1 789 949·80	−1593·1		−747 132·12
	1703·2	−923·3		−1846·6	−1 789 949·80		3406·4	+1 789 949·80
	3406·4	−1846·6						

Area = 894 974·9 m²

Referring to Fig. 9.42,

$$\text{Bearing } ED = \tan^{-1}\frac{1068{\cdot}4 - 0}{699{\cdot}3 + 446{\cdot}9} = \tan^{-1} 0{\cdot}932\,12 = \underline{\text{N } 42°59'20'' \text{ E}}$$

$$\text{Length } ED = 1068{\cdot}4 \sin 42°59'20'' = \underline{1566{\cdot}9}$$

$$\text{In triangle } ADE, \text{ Area} = \tfrac{1}{2}AE.ED \sin 42°59'20''$$

$$= \tfrac{1}{2} \times 446{\cdot}9 \times 1566{\cdot}9 \sin 42°59'20''$$

$$= \underline{238\,735{\cdot}4 \text{ m}^2}$$

$$\therefore \text{ Area triangle } EDF = \frac{894\,974{\cdot}9}{2} - 238\,735{\cdot}4 \text{ sq ft} = \underline{208\,752 \text{ m}^2}$$

$$\text{Bearing } DF = \text{Bearing } DC = \tan^{-1}\frac{634{\cdot}8 - 1068{\cdot}4}{-728{\cdot}8 - 699{\cdot}3}$$

$$= \tan^{-1} 0{\cdot}30362 = \underline{\text{S } 16°53'20'' \text{ W}}$$

$$\therefore \quad \text{Angle } ADF = 42°59'20'' - 16°53'20'' = \underline{26°06'}$$

Using Eq. (9.55),

$$DF = \frac{\text{Area } \Delta EDF}{\tfrac{1}{2}ED \sin EDF} = \frac{2 \times 208\,752}{1566{\cdot}9 \sin 26°06'} = \underline{605{\cdot}66 \text{ m}}$$

To obtain the co-ordinates of F,

Line DF S 16°53′20″ W Length 605·66 m

P. Dep. $605{\cdot}66 \sin 16°53'20'' = -175{\cdot}9$

P. Lat. $605{\cdot}66 \cos 16°53'20'' = -579{\cdot}5$

∴ *T. Dep. of F* $1068{\cdot}4 - 175{\cdot}9 = \underline{\underline{892{\cdot}5 \text{ m}}}$

T. Lat. of F $699{\cdot}3 - 579{\cdot}5 = \underline{\underline{119{\cdot}8 \text{ m}}}$

Example 9.17

With the previous co-ordinate values let the bearing of the dividing line be N 57°35′10″ E.

Construction

Draw line AG on this bearing and EF parallel to this a perpendicular distance d away.

$$\text{Bearing } AD = \tan^{-1}\frac{1068{\cdot}4}{699{\cdot}3}$$

$$= \tan^{-1} 1{\cdot}527\,81$$

$$= \underline{\text{N } 56°47'40'' \text{ E}}$$

$$\text{Length } AD = 1068{\cdot}4 \operatorname{cosec} 56°47'40''$$

$$= \underline{1276{\cdot}9 \text{ m}}$$

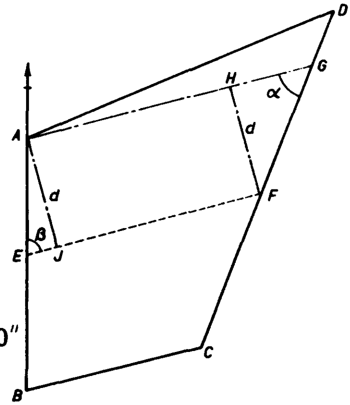


Fig. 9.43

In triangle ADG,

$$\hat{A} = 57°35'10'' - 56°47'40'' = 0°47'30''$$

$$\hat{D} = 56°47'40'' - 16°53'20'' = 39°54'20''$$

$$\hat{G} = 180 - (57°35'10'' - 16°53'20'') = \underline{139°18'10''}$$

$$\underline{180°00'00''}$$

$$\therefore \quad AG = \frac{AD \sin D}{\sin G} = \frac{1276{\cdot}9 \sin 39°54'20''}{\sin 139°18'10''} = \underline{1256{\cdot}3 \text{ m}}$$

$$\text{Area triangle } ADG = \tfrac{1}{2} AD.AG \sin \hat{A}$$

$$= \tfrac{1}{2} \times 1276{\cdot}9 \times 1256{\cdot}3 \sin 0°47'30''$$

$$= \underline{11\,084{\cdot}8 \text{ m}^2}$$

$$\text{Area} \quad AGFE = \tfrac{1}{2} \text{Area } ABCD - \Delta ADG$$

$$= 447\,487{\cdot}5 - 11\,084{\cdot}8 = \underline{436\,402{\cdot}7 \text{ m}^2}$$

$$\text{In figure } AGFE, \text{ Area} = \Delta AJE + AHFJ + \Delta FHG$$

$$= \tfrac{1}{2} d^2 \cot\beta + d(AG - d\cot\alpha) + \tfrac{1}{2} d^2 \cot\alpha$$

$$= 1256{\cdot}3\, d + \tfrac{1}{2} d^2(\cot\beta - \cot\alpha)$$

where $\alpha = 57°35'10'' - 16°53'20'' = 40°41'50''$

$\beta = 57°35'10''$

$$\therefore \quad 436\,402{\cdot}7 = 1256{\cdot}3\, d - 0{\cdot}263\,88\, d^2$$

This is a quadratic equation in d

$$0{\cdot}26388\,d^2 - 1256{\cdot}3\,d + 436\,402{\cdot}7 = 0$$

$$\therefore \quad d = \frac{1256{\cdot}3 \pm\sqrt{(1256{\cdot}3^2 - 4\times 0{\cdot}26388\times 436\,402{\cdot}7)}}{2\times 0{\cdot}26388}$$

$$= \frac{1256{\cdot}3 \pm\sqrt{(1\,578\,289{\cdot}7 - 460\,631{\cdot}8)}}{2\times 0{\cdot}26388}$$

$$= \frac{1256{\cdot}3 \pm\sqrt{1\,117\,657{\cdot}9}}{2\times 0{\cdot}26388}$$

$$= \frac{1256{\cdot}3 \pm 1057{\cdot}2}{2\times 0{\cdot}26388} = \frac{2313{\cdot}5}{2\times 0{\cdot}26388} \text{ or } \frac{199{\cdot}1}{2\times 0{\cdot}26388}$$

The first answer is not in accordance with the data given.

$$\therefore \quad d = 377{\cdot}26 \text{ m}$$

$\therefore$ Co-ordinates of E = 0, and $-377{\cdot}26$ cosec $57°35'10''$

= Total Dep. 0

Total Lat. $-446{\cdot}9$ m

$$\text{Length } EF = AG - HG + EJ$$
$$= 1256{\cdot}3 - 377{\cdot}26\cot\alpha + 377{\cdot}26\cot\beta$$
$$= 1256{\cdot}3 + 377{\cdot}26(\cot\beta - \cot\alpha)$$
$$= 1256{\cdot}3 - 199{\cdot}1 = 1057{\cdot}2$$

$\therefore$ Co-ordinates of F:

P. Dep. $1057{\cdot}2 \sin 57°35'10''$ = $+892{\cdot}5$ m

P. Lat. $1057{\cdot}2 \cos 57°35'10''$ = $+566{\cdot}7$ m

Total Dep. of F = $0 + 892{\cdot}5$ = $+892{\cdot}5$ m

Total Lat. of F = $-446{\cdot}9 + 566{\cdot}7$ = $+119{\cdot}8$ m

Example 9.18

Given the previous co-ordinate values let the dividing line pass through a point whose co-ordinates are (703·8, 0).

From previous information,

AD is N 56°47′40″ E, 1276·9 m

DC is S 16°53′20″ W

In triangle ADG, $$AG = \frac{AD\sin\hat{D}}{\sin\hat{G}} = \frac{1276{\cdot}9\sin 39°54'20''}{\sin 106°53'20''} = 855{\cdot}9 \text{ m}$$

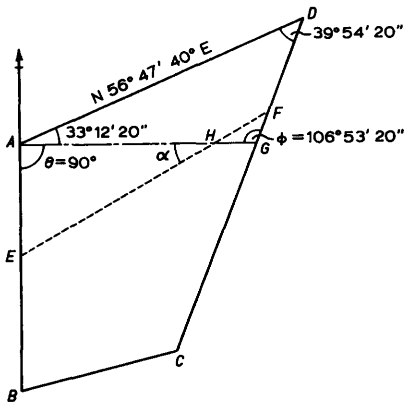


Fig. 9.44

$$\therefore \quad AH = 703{\cdot}8 \quad \text{(due E)}$$

$$HG = 855{\cdot}9 - 703{\cdot}8 = 152{\cdot}1 \text{ m}$$

$$\text{Area} = \tfrac{1}{2} AD.AG \sin \hat{A}$$

$$= \tfrac{1}{2} \times 1276{\cdot}9 \times 855{\cdot}9 \sin 33°12'20'' = 299\,268 \text{ m}^2$$

$$\text{Now Area } ADFE = \tfrac{1}{2} \text{ Area } ABCD = 447\,488 \text{ m}^2$$

$$= \Delta ADF + \Delta AHE - \Delta HFG$$

$$\therefore \Delta AHE - \Delta HFG = 447\,488 - 299\,268 = 148\,220 \text{ m}^2$$

i.e. by Eq. 9.57, $$148\,220 = \frac{AH^2}{2(\cot\alpha + \cot\theta)} - \frac{HG^2}{2(\cot\alpha + \cot\phi)}$$

As $\theta = 90°$,

$$296\,440 \cot\alpha(\cot\alpha + \cot\phi) = AH^2(\cot\alpha + \cot\phi) - HG^2 \cot\alpha$$

i.e. $296\,440 \cot^2\alpha + \cot\alpha[296\,440 \cot\phi - AH^2 + HG^2] - AH^2 \cot\phi = 0$

thus $\quad 296\,440 \cot^2\alpha - 562\,200 \cot\alpha + 150\,388 = 0$

Solving the quadratic gives $\alpha = 32°25'$

The co-ordinates of E are thus 0, and $703{\cdot}8 \tan 32°25'$

i.e. $\underline{(0, -446{\cdot}9 \text{ m})}$

Exercises 9

1. In the course of a chain survey, three survey lines forming the sides of a triangle were measured as follows:

Line	Length (links)	Inclination
AB	570	level
BC	310	1 in 10
CA	495	7°

On checking the chain after the survey, it was found that its length was 101 links.

Calculate the correct plan area of the triangle.

(E.M.E.U. Ans. 0·77249 acres)

2. A piece of ground has been surveyed with a Gunter's chain with the following results in chains: *AB* 11·50, *CA* 8·26, *DB* 10·30, *CE* 12·47, *BC* 12·20, *CD* 9·38, *DE* 6·63. Calculate the area in acres. Subsequently it was found that the chain was 0·01 chain too long. Find the discrepancy in the previous calculation and indicate its sign.

(L.U. Ans. 12·320 acres; −0·248 acres)

3. Undernoted are data relating to three sides of an enclosure, *AB*, *BC* and *CD* respectively, and a line joining the points *D* and *A*.

Line	Azimuth	Length (ft)
AB	010°00′	541·6
BC	088°55′	346·9
CD	159°19′	601·8
DA	272°01′	654·0

The fourth side of the enclosure is an arc of a circle to the south of *DA*, and the perpendicular distance from the point of bisection of the chord *DA* to the curve is 132·6 ft. Calculate the area of the enclosure in acres.

(Ans. 7·66 acres)

4. Plot to a scale of 40 inches to 1 mile, a square representing 2½ acres. By construction, draw an equilateral triangle of the same area and check the plotting by calculation.

(Ans. Side of square 2·5 in.
Side of triangle 3·8 in.)

5. The following offsets 15 ft apart were measured from a chain line to an irregular boundary:

23·8, 18·6, 14·2, 16·0, 21·4, 30·4, 29·6, 24·2 ft.

Calculate the area in acres.

(Ans. 0·0531 acres)

6. Find the area in square yards enclosed by the straight line boundaries joining the points *ABCDEFA* whose co-ordinates are:

	Eastings (ft)	Northings (ft)
A	250	75
B	550	175
C	700	425
D	675	675
E	450	675
F	150	425

(R.I.C.S. Ans. 24791·6 yd^2)

7. The following table gives the co-ordinates in feet of points on the perimeter of an enclosed area *ABCDEF*. Calculate the area of the land enclosed therein. Give your answer in statute acres and roods.

Point	Departure		Latitude	
	+	−	+	−
A		74·7	105·2	
B	63·7		261·4	
C	305·0		74·5	
D	132·4			140·4
E		54·5		192·4
F		571·9		108·3

(R.I.C.S. Ans. 4 acres 0·5 roods)

8. Using the data given in the traverse table below, compute the area in acres contained in the figure *ABCDEA*.

Side	Latitude (ft)	Departure (ft)
AB	+1327	−758
BC	+766	+805
CD	−952	+987
DE	−1949	+537
EA	+808	−1572

(I.C.E. Ans. 73·3 acres)

9. State in square inches and decimals thereof what an area of 10 acres would be represented by, on each of three plans drawn to scale of (a) 1 inch = 2 chains (b) 1/2500 and (c) 6 in. = 1 mile.

(Ans. (a) 25 in^2; (b) 10·04 in^2; (c) 0·562 in^2)

10. State what is meant by the term 'zero circle' when used in connection with the planimeter.

A planimeter reading tens of square inches is handed to you to enable you to measure certain areas on plans drawn to scales of (a) 1/360 (b) 2 chains to 1 inch (c) 1/2500 (d) 6 in. to 1 mile and (e) 40 in. to 1 mile. State the multiplying factor you would use in each

instance to convert the instrumental readings into acres.

(Ans. (a) 0·2066 (b) 4·0 (c) 9·96 (d) 177·78 (e) 4·0)

11. The following data relate to a closed traverse:

Line	Azimuth	Length (m)
AB	241°30′00″	301·5
BC	149°27′00″	145·2
CD	034°20′30″	415·7
DE	079°18′00″	800·9

Calculate (a) the length and bearing of the line *EA* to the nearest 30″,

(b) the area of the figure *ABCDEA*,

(c) the length and bearing of the line *BX* which will divide the area into two equal parts,

(d) the length of a line *XY* of bearing 068°50′ which will divide the area into two equal parts.

(Ans. (a) 307°54′ ; (b) 89 290 m^2; (c) 526·4 m 091°13′48″; (d) 407·7 m)

Bibliography

HADDOCK, M.H., *The Basis of Mine Surveying* (Chapman & Hall).
CLENDINNING, J., *The Principles of Surveying* (Blackie).
McADAM, R., *Colliery Surveying* (Oliver and Boyd).
MIDDLETON, R.E. and CHADWICK, O., *A Treatise on Surveying* (Spon).
THOMAS, W.N., *Surveying* (Edward Arnold).
SALMON, V.G., *Practical Surveying and Fieldwork* (Griffin).
PARRY, R. and JENKINS, W.R., *Land Surveying* (Estates Gazette).
DAVIES, R.E., FOOTE, F.S. and KELLY, J.W., *Surveying: Theory and Practice* (McGraw-Hill).
MINISTRY OF TECHNOLOGY, *Changing to the Metric System* (H.M.S.O.).
SMIRNOFF, M.V., *Measurements for Engineering and Other Surveys* (Prentice-Hall).

10 VOLUMES

10.1 Volumes of Regular Solids

The following is a summary of the most important formulae.

Prism (Fig. 10.1)

$$V = \text{cross-sectional area} \times \text{perpendicular height}$$

i.e. $$V = Ah = A_1h_1 \qquad (10.1)$$

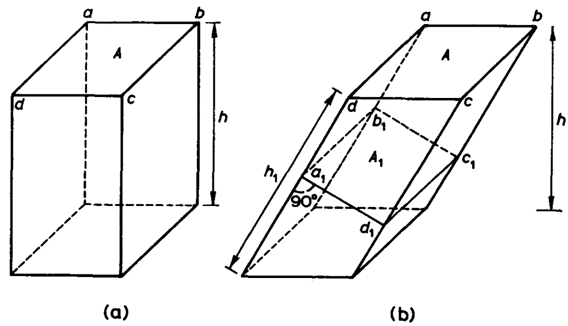


Fig. 10.1

Cylinder (Fig. 10.2)

This is a special case of the prism.

$$V = Ah = \pi r^2 h \qquad (10.2)$$

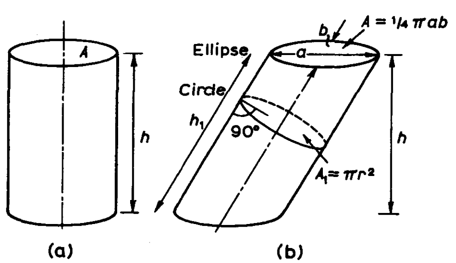


Fig. 10.2

In Fig. 10.2 (b) the cylinder is cut obliquely and thus the end area becomes an ellipse, i.e. $V = Ah$.

$$\therefore \quad V = \frac{1}{4}\pi abh \qquad (10.3)$$

$$= \frac{1}{2}\pi arh \quad (\text{as } b = 2r) \qquad (10.4)$$

$$= A_1h_1 = \pi r^2 h_1 \qquad (10.5)$$

Pyramid (Fig. 10.3)

$$V = \frac{1}{3}\text{ base area} \times \text{perpendicular height}$$

$$= \frac{1}{3}Ah \qquad (10.6)$$

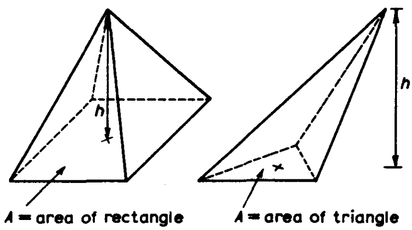


Fig. 10.3

Cone (Fig. 10.4)

This is a special case of the pyramid.

$$V = \frac{1}{3}Ah = \frac{1}{3}\pi r^2 h \qquad (10.7)$$

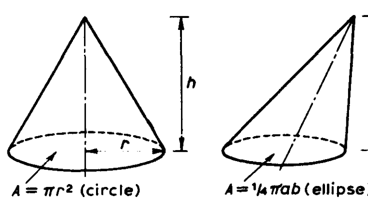


Fig. 10.4

In Fig. 10.4 (b) the base is in the form of an ellipse.

$$\therefore \quad V = \frac{1}{12}\pi abh \qquad (10.8)$$

$$= \frac{1}{6}\pi arh \quad (\text{as } b = 2r) \qquad (10.9)$$

Frustum of Pyramid (Fig. 10.5)

$$V = \frac{h}{3}(A + B + \sqrt{AB}) \qquad (10.10)$$

where h = perpendicular height

A and B = areas of larger and smaller ends respectively.

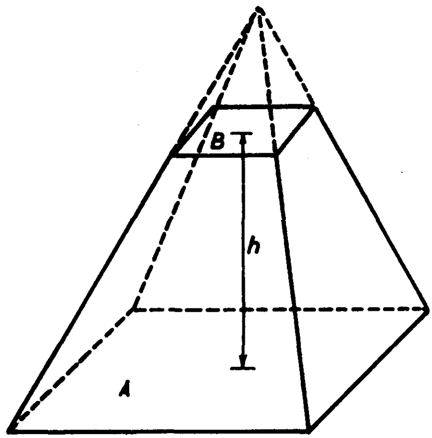


Fig. 10.5

Frustum of Cone (Fig. 10.6)

This is a special case of the frustum of the pyramid in which $A = \pi R^2$, $B = \pi r^2$.

$$\therefore \quad V = \frac{h}{3}[\pi R^2 + \pi r^2 + \sqrt{\pi R^2 \pi r^2}]$$

$$= \frac{\pi h}{3}[R^2 + r^2 + Rr] \qquad (10.11)$$

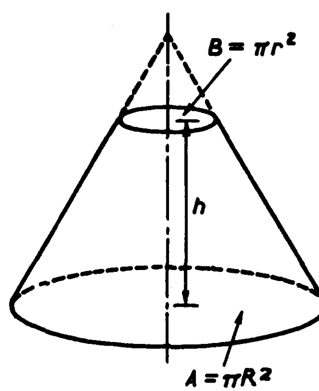


Fig. 10.6

Wedge (Fig. 10.7)

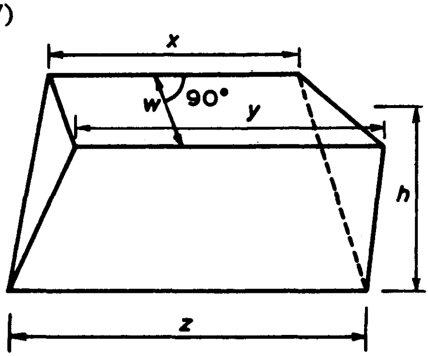


Fig. 10.7

V = Sum of parallel edges × width of base × $\frac{1}{6}$ perpendicular height.

$$\text{i.e.} \quad = \frac{wh}{6}(x + y + z) \tag{10.12}$$

The above formulae relating to the pyramid are proved as follows (Fig. 10.8).

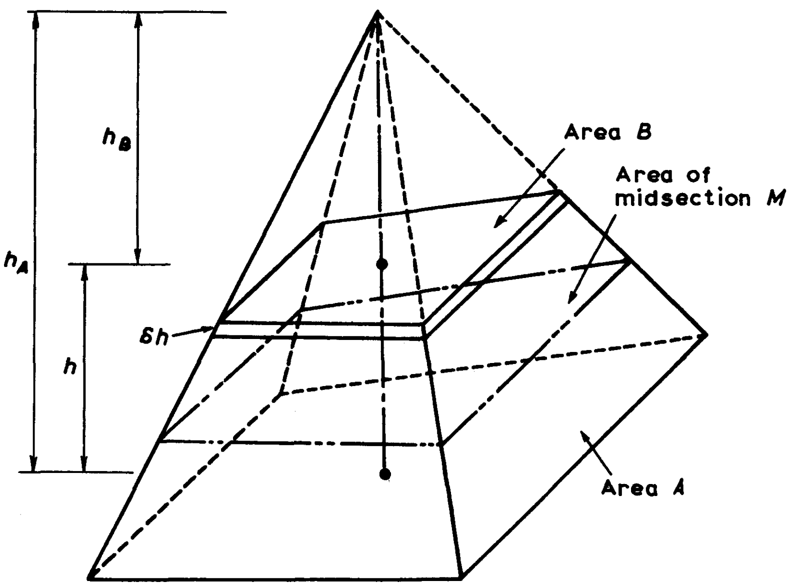


Fig. 10.8

Let A = the base area
h_A = the perpendicular height
B = the area of any section parallel to the base and at a perpendicular distance h_B from the vertex.

Then
$$\delta V = B\delta h$$

but
$$\frac{A}{B} = \frac{h_A^2}{h_B^2} \qquad \therefore \quad B = \frac{Ah_B^2}{h_A^2}$$

$$\therefore \quad V = \frac{A}{h_A^2}\int_0^{h_A} h_B^2\, dh_B$$

$$= \frac{A}{h_A^2} \times \frac{h_A^3}{3} = \underline{\frac{1}{3}Ah_A} \tag{Eq. 10.6}$$

In the case of the frustum $(h_A - h_B = h)$,

$$V = \frac{A}{h_A^2}\int_{h_B}^{h_A} h_B^2\, dh_B$$

$$= \frac{A}{3h_A^2}[h_A^3 - h_B^3]$$

$$= \frac{A}{3h_A^2}(h_A - h_B)(h_A^2 + h_A h_B + h_B^2)$$

$$= \frac{h}{3}\left[A + \frac{Ah_B}{h_A} + \frac{Ah_B^2}{h_A^2}\right] \qquad (10.13)$$

But $$\frac{A}{B} = \frac{h_A^2}{h_B^2}$$

$$\therefore \quad B = \frac{Ah_B^2}{h_A^2} \quad \text{and} \quad \frac{\sqrt{B}}{\sqrt{A}} = \frac{h_B}{h_A}$$

$$\therefore \quad \underline{V = \frac{h}{3}[A + \sqrt{AB} + B]} \qquad \text{(Eq.10.10)}$$

If C is the area of the mid-section of the pyramid, then

$$\frac{C}{A} = \frac{\left(\frac{h_A}{2}\right)^2}{h_A^2} = \frac{h_A^2}{4h_A^2}$$

$$\therefore \quad A = 4C$$

$$\therefore \quad V = \frac{Ah_A}{3} = \frac{(A + 4C)h_A}{6} \qquad (10.14)$$

Similarly, if M is the area of the mid-section between A and B, then

$$\frac{M}{A} = \frac{\{\frac{1}{2}(h_A + h_B)\}^2}{h_A^2} = \frac{(h_A + h_B)^2}{4h_A^2}$$

$$\therefore \quad 4M = \frac{A(h_A^2 + 2h_A h_B + h_B^2)}{h_A^2}$$

$$= A\left(1 + 2\frac{h_B}{h_A} + \frac{h_B^2}{h_A^2}\right)$$

$$= A + 2A\frac{h_B}{h_A} + B$$

$$\therefore \quad \frac{Ah_B}{h_A} = \tfrac{1}{2}(4M - A - B)$$

Substituting this value in Eq. (10.13),

$$V = \frac{h}{6}[A + 4M + B] \tag{10.15}$$

The Prismoidal Formula

From Eq. 10.15 it is seen that the volumes of regular solids can be expressed by the same formula, viz. the volume is equal to the sum of the two parallel end areas + four times the area of the mid-section × 1/6 the perpendicular height, i.e.

$$V = \frac{h}{6}[A + 4M + B] \qquad \text{(Eq. 10.15)}$$

This formula is normally associated with the *prismoid* which is defined as 'a solid having two parallel end areas A and B, which may be of any shape, provided that the surfaces joining their perimeters are capable of being generated by straight lines.'

N.B. The mean area is derived from the average of the corresponding dimensions of the two end areas but not by taking the average of A and B.

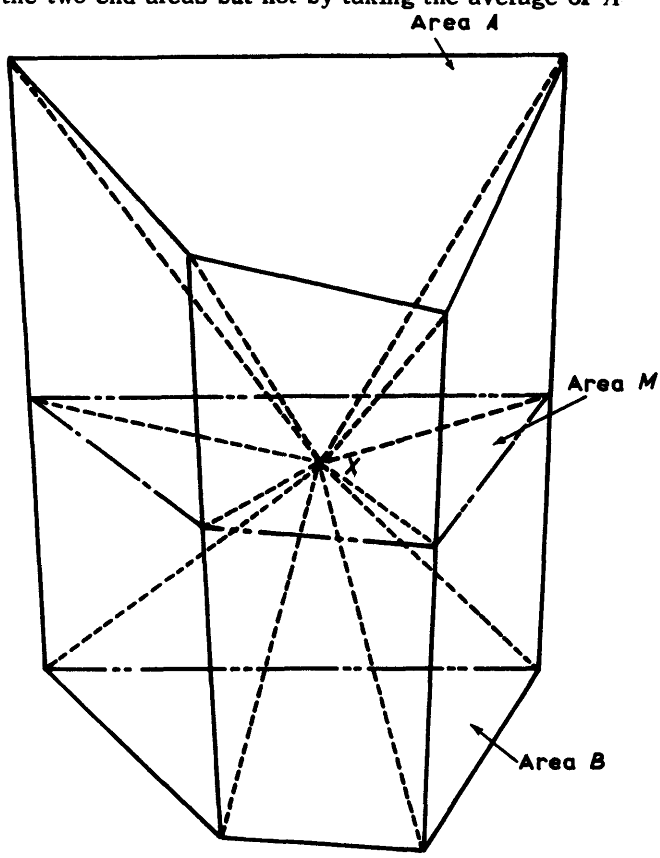


Fig. 10.9

Newton's proof of this formula is to take any point X on the mid-section and join it to all twelve vertices of the three sections, Fig. 10.9. The total volume then becomes the sum of the ten pyramids so formed.

This formula is similarly applicable to the cone and sphere.

The cone (Fig. 10.10)

$$V = \frac{h}{6}\left[\pi r^2 + 4\pi\left(\frac{r}{2}\right)^2 + 0\right]$$
$$= \frac{2\pi r^2 h}{6}$$
$$= \underline{\frac{1}{3}\pi r^2 h} \quad \text{(Eq. 10.7)}$$

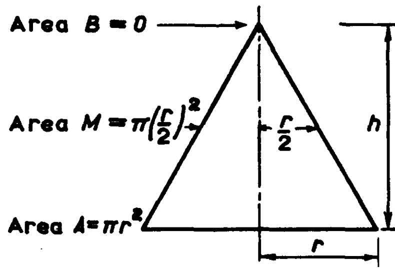


Fig. 10.10

The sphere (Fig. 10.11)

$$V = \frac{2r}{6}[0 + 4\pi r^2 + 0]$$
$$= \underline{\frac{4}{3}\pi r^3} \qquad (10.16)$$

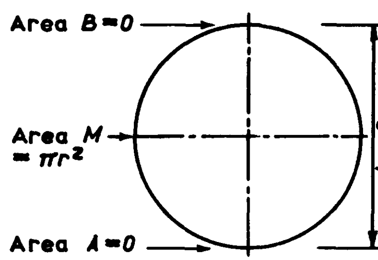


Fig. 10.11

N.B. The relative volumes of a cone, sphere and cylinder, all of the same diameter and height, are respectively 1, 2 and 3, Fig. 10.12.

$$\text{Cone} = \frac{2r}{3} \times \pi r^2$$
$$= \underline{\frac{2}{3}\pi r^3} \qquad (10.17)$$
$$\text{Sphere} = \underline{\frac{4}{3}\pi r^3} \quad \text{(Eq. 10.16)}$$
$$\text{Cylinder} = 2r \times \pi r^2$$
$$= \underline{2\pi r^3} \qquad (10.18)$$

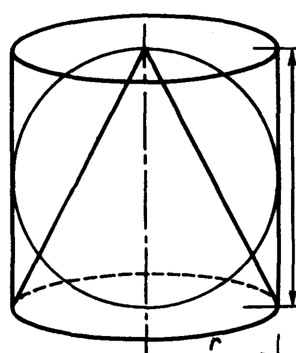


Fig. 10.12

Applying the prismoidal formula to the frustum of a cone,

$$V = \frac{h}{6}\left[\pi R^2 + \pi r^2 + 4\pi\left(\frac{R+r}{2}\right)^2\right]$$

$$= \frac{\pi h}{6}[R^2 + r^2 + R^2 + 2Rr + r^2]$$

$$= \frac{\pi h}{3}[R^2 + r^2 + Rr] \qquad \text{(Eq. 10.11)}$$

Applying the prismoidal formula to the wedge,

$$V = \frac{h}{6}\left[\frac{w}{2}(x+y) + 4\left\{\frac{x+z}{2} + \frac{z+y}{2}\right\}\frac{w}{4} + 0\right]$$

$$= \frac{wh}{6}\left[\frac{x}{2} + \frac{y}{2} + \frac{x}{2} + \frac{z}{2} + \frac{z}{2} + \frac{y}{2}\right]$$

$$= \frac{wh}{6}[x + y + z] \qquad \text{(Eq. 10.12)}$$

It thus becomes very apparent that of all the mensuration formulae the PRISMOIDAL *is the most important.*

If in any solid having an x axis the areas (A) normal to this axis can be expressed in the form

$$A = bx^2 + cx + d$$

then the prismoidal formula applies precisely.

The *sphere* may be regarded as made up of an infinite number of small pyramids whose apexes meet at the centre of the sphere. The heights of these pyramids are then equal to the radius of the sphere.

$\therefore$ Volume of each pyramid $=$ area of base $\times \dfrac{r}{3}$

Volume of sphere $=$ surface area of sphere $\times \dfrac{r}{3}$

Surface area of sphere $= \dfrac{\text{volume of sphere}}{1/3r} = 4\pi r^2$ (10.19)

Sector of a sphere (Fig. 10.13)

This is a cone OAC with a spherical cap. ABC.

The volume can be derived from the above arguments:

Volume of sector $=$ (curved surface area of segment) $\times\ r/3$

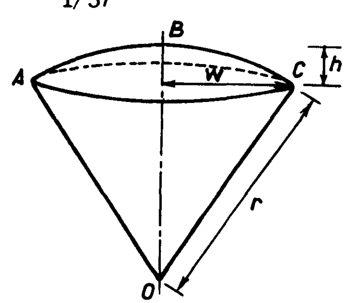


Fig. 10.13

(by Eq. 9.33) $$= 2\pi r \times h \times \frac{r}{3}$$

$$= \underline{\frac{2}{3}\pi r^2 h} \qquad (10.20)$$

Segment of sphere

This is the sector less the cone *OAC*

$$\therefore \quad V = \frac{2}{3}\pi r^2 h - \frac{1}{3}\pi w^2(r-h)$$

$$\text{But} \quad w^2 = r^2 - (r-h)^2 = r^2 - r^2 + 2rh - h^2$$

$$\therefore \quad V = \frac{2}{3}\pi r^2 h - \frac{1}{3}\pi(2rh - h^2)(r-h)$$

$$= \frac{2}{3}\pi r^2 h - \frac{2}{3}\pi r^2 h + \pi r h^2 - \frac{1}{3}\pi h^3$$

$$= \underline{\frac{1}{3}\pi h^2 (3r - h)} \qquad (10.21)$$

10.2 Mineral Quantities

Flat seams

The general formula for calculating tonnage is:

$$\text{Tonnage} = \frac{\text{plan area (ft}^2) \times \text{thickness (ft)} \times 62{\cdot}5 \times \text{S.G.}}{2240} \text{ tons} \qquad (10.22)$$

Here $62{\cdot}5 \simeq$ the weight of 1 ft^3 of water in pounds

S.G. = the specific gravity of the mineral.

This gives the tonnage in a seam before working and takes no account of losses.

Taking coal as a typical mineral, alternative calculations may be made.

$$\text{Tonnage} = \text{plan area (acres)} \times \text{thickness (in.)} \times 101 \text{ S.G.} \qquad (10.23)$$

Here 1 acre of water 1 inch thick weighs approximately 101 tons.

When the specific gravity of coal is not known, either

(a) Assume S.G. of 1·25 – 1·3.
(b) Assume 125 tons per inch/acre
1250 – 1500 tons per foot/acre.
(c) Assume 1 yd^3 of coal weighs 0·9 – 1·0 ton, or
(d) Assume 1 ft^3 of coal weighs 80 lb.

For loss of tonnage compared with the 'solid' estimate assume 15–20%.

Based on the International System (S.I.) units, the following conversion factors are required:

1 ft	=	0·3048 m	1 ton	=	1016·05 kg
1 ft^2	=	0·092 903 m^2	1 cwt	=	50·802 3 kg
1 acre	=	4046·86 m^2	1 lb	=	0·453 592 37 kg
1 ft^3	=	0·028 316 m^3			
1 yd^3	=	0·764 555 m^3			

The weight of water is 1 g/cm^3 at 4°C (i.e. 1000 kg/m^3)

$\therefore$ Coal weighs $\simeq$ 1250 – 1300 kg/m^3 ($\simeq$ 1000 kg/yd^3)

1 gallon = 4·546 09 litres = 0·004 546 m^3

Inclined seams

The tonnage may be obtained by using either (a) the inclined area or (b) the vertical thickness, i.e.

$$V = A \,.\, t \sec\alpha \tag{10.24}$$

where V = plan area

t = thickness

α = angle of inclination of full dip of seam

Example 10.1 In a pillar and stall, or stoop and room workings, the stalls or rooms are 12 ft in width and the pillars are 40 yd square. Calculate the approximate tonnage of coal extracted from the stalls or rooms, in a seam 7 ft 9 in. in thickness, dipping 19° from the horizontal, under a surface area 1½ acres in extent. Assume a yield of 125 tons per inch-acre, and deduct 3¾% for loss in working.

(M.Q.B./M)

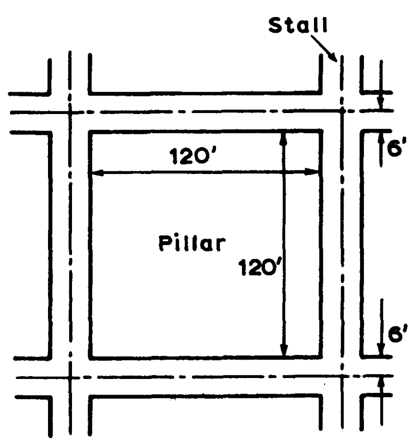


Fig. 10.14

In Fig. 10.14, Total Area $= (120 + 12)^2$

Pillar Area $= 120^2$

$$\therefore \quad \%\ \text{extraction} = 100 - \left(\frac{120}{132}\right)^2 \times 100$$

$$= 100\left\{1 - \left(\frac{120}{132}\right)^2\right\}$$

$$= 17{\cdot}36\%$$

Plan area of extraction $= 1{\cdot}5 \times 17{\cdot}36/100$ acres

Inclined area of extraction $= 1{\cdot}5 \times (17{\cdot}36/100) \times \sec 19°$

Volume of coal extracted $= 1{\cdot}5 \times (17{\cdot}36/100) \sec 19 \times 7{\cdot}75 \times 12 \times 125$

$= 3201{\cdot}5$ tons

$$\text{Loss of volume} = \frac{3201{\cdot}5 \times 15}{400} = 120 \text{ tons}$$

$\therefore$ Approximate tonnage extracted $= 3200 - 120$

$= \underline{3080 \text{ tons}}$

Exercises 10(a) (Regular solids)

1. A circular shaft is being lined with concrete, of average thickness 18 in. The finished inside diameter is 22 ft and a length of 60 ft is being walled. In addition 23 yd^3 of concrete will be required for a walling curb.

If each yd^3 of finished concrete requires (a) 700 lb cement (b) 1600 lb sand and (c) 2500 lb aggregate, find, to the nearest ton, the quantity of each material required to carry out the operation.

(Ans. 84 tons cement; 192 tons sand; 300 tons aggregate)

2. A colliery reservoir, circular in shape, with sides sloping at a uniform gradient and lined with concrete is to be constructed on level ground to the undernoted inside dimensions:

Top diameter 40 m
Bottom diameter 36 m
Depth 9 m

The excavation is to be circular, 42 m in diameter, with vertical sides 10·5 m deep.

Calculate the volume of concrete required.

(Ans. 122·63 m^3)

3. Two shafts – one circular of 20 ft diameter, and the other rectangular 20 ft by 10 ft – are to be sunk to a depth of 710 yd. The material excavated is to be deposited in the form of a truncated cone within an area of level ground 100 yd square. If the top of the heap is to

be level and the angle of repose of the material 35°, what will be the ultimate height of the heap with the diameter at its maximum? Assume the proportion of broken to unbroken strata to be 5 to 3 by volume (take $\pi = (22/7)$).

(Ans. 38·9 ft)

4. An auxiliary water tank in the form of a cylinder with hemispherical ends is placed with its long axis horizontal. The internal dimensions of the tank are (i) length of cylindrical portion 24 m (ii) diameter 5 m (iii) overall length 29 m.

Calculate (a) the volume of the tank and

(b) the amount of water it contains to the nearest 100 litres when filled to a depth of 1·07 m.

(Ans. (a) 536·69 m^3; (b) 11 400 litres)

5. Two horizontal drifts of circular cross-section and 16 ft excavated diameter cross each other at right-angles and on the same level. Calculate the volume of excavation in ft^3 which is common to both drifts.

(M.Q.B./S Ans. 2731 ft^3)

6. The plan of a certain building on level ground is a square with sides 200 ft in length for which mineral support is about to be acquired. The south side of the building is parallel to the line of strike of the seam, the full dip of which is due South at the rate of 12 in. to the yard. The floor of the seam is 360 yd under the surface at the centre of the building.

Draw a plan of the building and protecting block to a scale of 1 in = 200 ft, allowing a lateral margin equal to one third of the depth of the seam at the edge of the protecting block opposite the nearest point of the protected area. Thereafter calculate the tonnage of coal contained in the protecting block, the seam thickness being 70 in. and the sp. gr. 1·26.

(M.Q.B./S Ans. 182 860 tons)

7. A solid pier is to have a level top surface 20 ft wide. The sides are to have a batter of 2 vertical to 1 horizontal and the seaward end is to be vertical and perpendicular to the pier axis. It is to be built on a rock stratum with a uniform slope of 1 in 24, the direction of this maximum slope making an angle whose tangent is 0·75 with the direction of the pier. If the maximum height of the pier is to be 20 ft above the rock, diminishing to zero at the landward end, calculate the volume of material required.

(L.U. Ans. 160 000 ft^3)

8. A piece of ground has a uniform slope North and South of 1 vertical to 20 horizontal. A flat area 200 ft by 80 ft is to be made by cutting and filling, the two volumes being equal. Compare the volumes of

excavation if the 200 ft runs (a) North and South (b) East and West.
The side slopes are to be 1 vertical to 2 horizontal.

(L.U. Ans. (a) 24 300 ft^3; (b) 4 453 ft^3)

10.3 Earthwork Calculations

There are three general methods of calculating volumes, which use (1) cross-sectional areas, (2) contours, (3) spot heights.

10.31 Calculation of volumes from cross-sectional areas

In this method cross-sections are taken at right-angles to some convenient base line which generally runs longitudinally through the earthworks. The method is specifically applicable to transport constructions such as roads, railways and canals but may be applied to any irregular volume.

The cross-sectional areas may be irregular and thus demand the use of one of the previously discussed methods, but in many transport constructions the areas conform to various typical shapes, viz. sections (a) without crossfall, (b) with crossfall, (c) with part cut and part fill, (d) with variable crossfall.

(a) *Sections without crossfall, i.e. surface level* (Fig. 10.15)

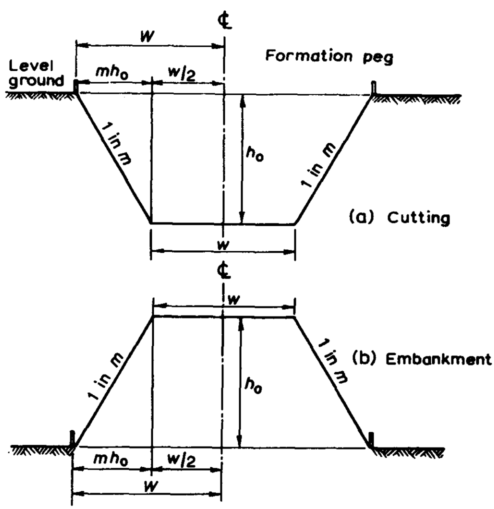


Fig. 10.15 Sections without crossfall

The sections may be cuttings or embankments but in either case the following terms are used:

Formation width (w)

Formation height (h_o), measured on centre line (℄)

Side width (W), for the fixing of formation pegs, measured from centre line.

Side slopes or batter 1 in m, i.e. 1 vertical to m horizontal

$$\text{Thus} \quad \underline{W = \frac{w}{2} + mh_0} \tag{10.25}$$

$$\text{Cross-sectional area} = \frac{h_0}{2}(w + 2W)$$

$$= \frac{h_0}{2}(w + w + 2mh_0)$$

$$\underline{A = h_0(w + mh_0)} \tag{10.26}$$

Example 10.2 A cutting formed in level ground is to have a formation width of 40 ft (12·19 m) with the sides battering at 1 in 3. If the formation height is 10 ft (3·05 m) find (a) the side width, (b) the cross-sectional area.

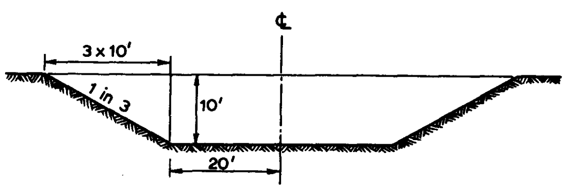


Fig. 10.16

Here W = 40 ft (12·19 m)

h_o = 10 ft (3·05 m)

m = 3

$$\therefore \quad W = \frac{w}{2} + mh_o$$

$$= 20 + 3 \times 10 = \underline{50 \text{ ft}} \ (15{\cdot}24 \text{ m})$$

$$\text{Area } A = h_0(w + mh_0)$$

$$= 10(40 + 30) = \underline{700 \text{ ft}^2} \ (65{\cdot}03 \text{ m}^2)$$

The metric values are shown in brackets.

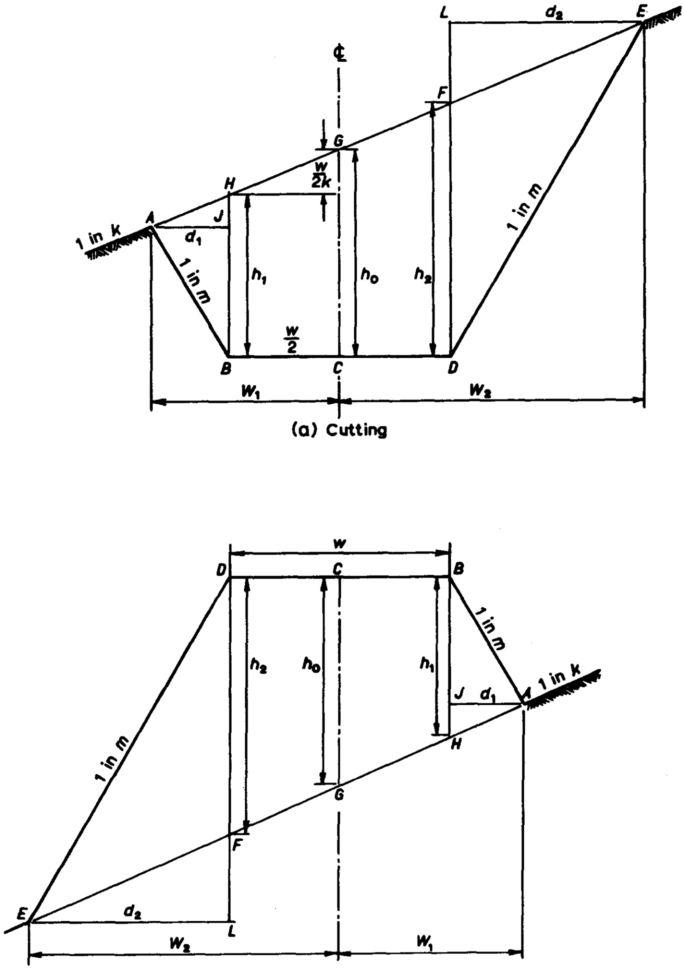
(b) *Sections with crossfall of* 1 *in* k (often referred to as a two-level section)

In both the cutting and embankment the total area is made up of three parts, Fig. 10.17.

(1) Triangle AHB Area = $\frac{1}{2} h_1 d_1$

(2) Trapezium $BHFD$ Area = $h_0 w$

(3) Triangle DFE Area = $\frac{1}{2} h_2 d_2$.



(a) Cutting

(b) Embankment

Fig. 10.17 Sections with crossfall

In both figures,
$$h_1 = h_0 - \frac{w}{2k} \tag{10.27}$$

$$h_2 = h_0 + \frac{w}{2k} \tag{10.28}$$

By the rate of approach method(see p. 432)

$$d_1 = \frac{h_1}{\dfrac{1}{m} + \dfrac{1}{k}} = \frac{h_1 mk}{k + m} \tag{10.29}$$

and
$$d_2 = \frac{h_2}{\dfrac{1}{m} - \dfrac{1}{k}} = \frac{h_2 mk}{k - m} \tag{10.30}$$

$$\therefore \quad W_1 = \frac{w}{2} + d_1 = \frac{w}{2} + \frac{\left(h_0 - \dfrac{w}{2k}\right) mk}{k + m} \tag{10.31}$$

$$W_2 = \frac{w}{2} + d_2 = \frac{w}{2} + \frac{\left(h_0 + \dfrac{w}{2k}\right) mk}{k - m} \tag{10.32}$$

$$\underline{\text{Total area} = \frac{1}{2}h_1 d_1 + h_0 w + \frac{1}{2}h_2 d_2} \tag{10.33}$$

The area of the cross-section is best solved by working from first principles, but if the work is extensive a complete formula may be required.

Given the initial information as w, h_0, m and k, substitution of these values into the various steps gives, from Eq. (10.33),

$$A = \frac{\left(h_0 - \dfrac{w}{2k}\right)^2 mk}{2(k+m)} + \frac{\left(h_0 + \dfrac{w}{2k}\right)^2 mk}{2(k-m)} + wh_0$$

$$= \frac{mk\left[\left\{h_0^2 - \dfrac{h_0 w}{k} + \left(\dfrac{w}{2k}\right)^2\right\}(k-m) + \left\{h_0^2 + \dfrac{h_0 w}{k} + \left(\dfrac{w}{2k}\right)^2\right\}(k+m)\right]}{2(k^2 - m^2)} + wh_0$$

$$= \frac{mk\left[2h_0^2 k + 2\left(\dfrac{w}{2k}\right)^2 + \dfrac{2wh_0 m}{k}\right]}{2(k^2 - m^2)} + wh_0$$

$$= \frac{m\left[h_0^2 k^2 + \left(\dfrac{w}{2}\right)^2 + wh_0 m\right]}{k^2 - m^2} + wh_0 \tag{10.34}$$

Example 10.3 The ground slopes at 1 in 20 at right-angles to the centre line of a proposed embankment which is to be 40 ft (12·19 m) wide at a formation level of 10 ft (3·05 m) above the ground. If the batter of the sides is 1 in 2, calculate (a) the side width, (b) the area of the cross-section.

In Fig. 10.18,

$$w = 40 \text{ ft } (12{\cdot}19 \text{ m})$$
$$h_0 = 10 \text{ ft } (3{\cdot}05 \text{ m})$$

$$m = 2$$
$$k = 20$$

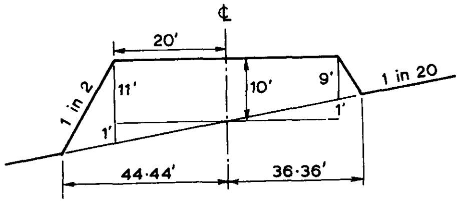


Fig. 10.18

Then $h_1 = 10 - \dfrac{20}{20} = 9$ ft

$h_2 = 10 + 1 = 11$ ft

$$d_1 = \frac{9 \times 2 \times 20}{20 + 2} = \frac{360}{22} = 16{\cdot}36 \text{ ft}$$

$$d_2 = \frac{11 \times 2 \times 20}{20 - 2} = \frac{440}{18} = 24{\cdot}4\dot{4} \text{ ft}$$

$\therefore$ $\underline{W_1 = 20 + 16{\cdot}36 = 36{\cdot}36 \text{ ft}}$ (11·08 m)

$\underline{W_2 = 20 + 24{\cdot}44 = 44{\cdot}44 \text{ ft}}$ (13·55 m)

$$\begin{aligned}
\text{Area} &= \tfrac{1}{2}[h_1 d_1 + h_2 d_2] + w h_0 \\
&= \tfrac{1}{2}[9 \times 16{\cdot}36 + 11 \times 24{\cdot}4\dot{4}] + 40 \times 10 \\
&= \tfrac{1}{2}[147{\cdot}28 + 268{\cdot}88] + 400 \\
&= \underline{608{\cdot}08 \text{ ft}^2} \ (56{\cdot}49 \text{ m}^2)
\end{aligned}$$

By Eq. (10.34),

$$\begin{aligned}
A &= \frac{2[100 \times 400 + 400 + 40 \times 10 \times 2]}{400 - 4} + 40 \times 10 \\
&= \frac{40\,000 + 400 + 800}{198} + 400 \\
&= \underline{608{\cdot}08 \text{ ft}^2}
\end{aligned}$$

or converted into S.I. units,

$$\begin{aligned}
A &= \frac{2[3{\cdot}05^2 \times 400 + 6{\cdot}095^2 + 12{\cdot}19 \times 3{\cdot}05 \times 2]}{396} + 12{\cdot}19 \times 3{\cdot}05 \\
&= \frac{[3721 + 37{\cdot}15 + 74{\cdot}36]}{198} + 37{\cdot}18 \\
&= 19{\cdot}36 + 37{\cdot}18 = \underline{56{\cdot}54 \text{ m}^2}
\end{aligned}$$

(c) *Sections with part cut and part fill* (Fig. 10.19)

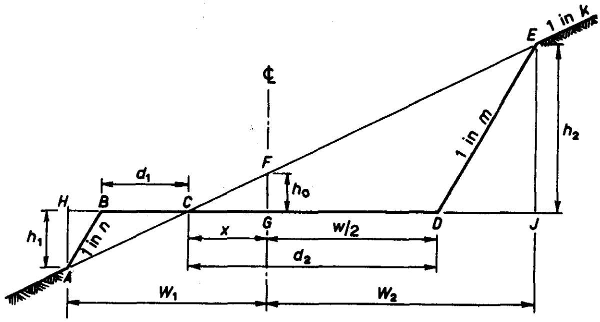


Fig. 10.19 Section part cut/part fill

As before, the formation width $BD = w$
the formation height $FG = h_0$
the ground slope $= 1$ in k

but here the batter on the cut and the fill may differ, so

batter of fill is 1 in n
batter of cut is 1 in m.

The total area is made up of only 2 parts:

(1) Triangle ABC Area $= \frac{1}{2} h_1 d_1$
(2) Triangle CED Area $= \frac{1}{2} h_2 d_2$

$$d_1 = \frac{w}{2} - x = \frac{w}{2} - kh_0 \tag{10.35}$$

$$d_2 = \frac{w}{2} + x = \frac{w}{2} + kh_0 \tag{10.36}$$

By the rate of approach method and noting that h_1 and h_2 are now required, it will be seen that to conform to the basic figure of the method the gradients must be transformed into n in 1, m in 1 and k in 1.

$$\therefore \qquad h_1 = \frac{d_1}{k - n} \tag{10.37}$$

and

$$h_2 = \frac{d_2}{k - m} \tag{10.38}$$

Side width

$$W_1 = \frac{w}{2} + HB$$

$$= \frac{w}{2} + nh_1 \tag{10.39}$$

$$W_2 = \frac{w}{2} + DJ$$

$$= \frac{w}{2} + mh_2 \qquad (10.40)$$

$$\text{Area of fill} = \tfrac{1}{2} h_1 d_1$$

$$= \frac{d_1^2}{2(k-n)}$$

$$= \frac{\left(\frac{w}{2} - kh_0\right)^2}{2(k-n)} \qquad (10.41)$$

$$\text{Area of cut} = \tfrac{1}{2} h_2 d_2$$

$$= \frac{d_2^2}{2(k-m)}$$

$$= \frac{\left(\frac{w}{2} + kh_0\right)^2}{2(k-m)} \qquad (10.42)$$

In the above h_0 has been treated as $-$ve, occurring in the cut. If it is $+$ve and the centre line is in fill, then

$$\text{Area of fill} = \frac{\left(\frac{w}{2} + kh_0\right)^2}{2(k-n)} \qquad (10.43)$$

$$\text{Area of cut} = \frac{\left(\frac{w}{2} - kh_0\right)^2}{2(k-m)} \qquad (10.44)$$

N.B. If $h_0 = 0$ and $m = n$,

$$\text{Area of cut} = \text{Area of fill} = \frac{w^2}{8(k-m)} \qquad (10.45)$$

Example 10.4 A proposed road is to have a formation width of 40 feet with side slopes of 1 in 1 in cut and 1 in 2 in fill. The ground falls at 1 in 3 at right-angles to the centre line which has a reduced level of 260·3 ft. If the reduced level of the road is to be 262·8 ft, calculate (a) the side width, (b) the area of cut, (c) the area of fill.

$$d_1 = \frac{w}{2} - (-h_0 k) = \frac{w}{2} + h_0 k$$

$$= 20 + 2{\cdot}5 \times 3 = 27{\cdot}5 \text{ ft}$$

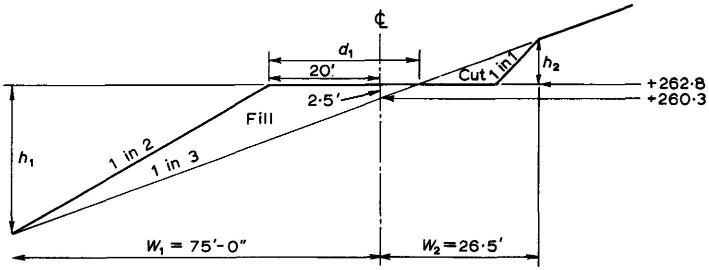


Fig. 10.20

$$d_2 = 20 - 7{\cdot}5 = 12{\cdot}5 \text{ ft}$$

$$h_1 = \frac{d_1}{k-n} = \frac{27{\cdot}5}{3-2} = 27{\cdot}5 \text{ ft}$$

$$h_2 = \frac{d_2}{k-m} = \frac{12{\cdot}5}{3-1} = 6{\cdot}25 \text{ ft}$$

$$W_1 = \frac{w}{2} + nh_1 = 20 + 2 \times 27{\cdot}5 = 75{\cdot}0 \text{ ft}$$

$$W_2 = \frac{w}{2} + mh_2 = 20 + 1 \times 6{\cdot}5 = 26{\cdot}5 \text{ ft}$$

Area of cut $= \frac{1}{2} d_2 h_2 = \frac{1}{2} \times 12{\cdot}5 \times 6{\cdot}25 = 39{\cdot}06 \text{ ft}^2$

Area of fill $= \frac{1}{2} d_1 h_1 = \frac{1}{2} \times 27{\cdot}5 \times 27{\cdot}5 = 378{\cdot}13 \text{ ft}^2$

By Eqs. 10.43/10.44,

$$\text{Area of cut} = \frac{\left(\frac{w}{2} - kh_0\right)^2}{2(k-m)} = \frac{(20 - 7{\cdot}5)^2}{2(3-1)} = 39{\cdot}06 \text{ ft}^2$$

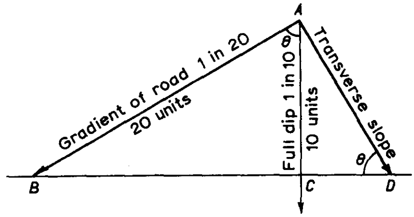
$$\text{Area of fill} = \frac{\left(\frac{w}{2} + kh_0\right)^2}{2(k-n)} = \frac{(20 + 7{\cdot}5)^2}{2(3-2)} = 378{\cdot}13 \text{ ft}^2$$

Example 10.5 An access road to a small mine is to be constructed to rise at 1 in 20 across a hillside having a maximum slope of 1 in 10. The road is to have a formation width of 15 ft, and the volumes of cut and fill are to be equalised. Find the width of cutting, and the volume of excavation in 100 ft of road. Side slopes are to batter at 1 in 1 in cut and 1 in 2 in fill.

(N.R.C.T.)

To find the transverse slope (see page 413)

Fig. 10.21



Let AB be the proposed road dipping at 1 in 20 (20 units)
AC the full dip 1 in 10 (10 units)
AD the transverse slope 1 in t (t units).

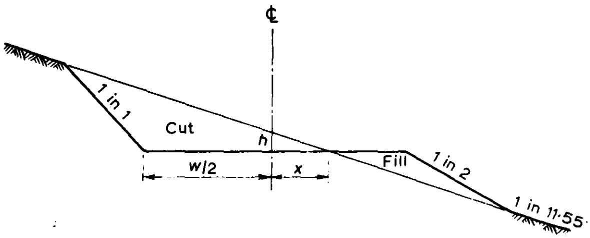
In triangle ABC,

$$\cos\theta = \frac{10}{20}$$

$$\therefore \theta = 60^\circ$$

In triangle ADC, $$AD = t = \frac{10}{\sin 60} = 11{\cdot}55 \text{ (gradient value)}$$

Fig. 10.22



If area of cut = area of fill, from Eqs. (10.43) and (10.44) for h +ve,

$$\frac{\left(\frac{w}{2} - kh\right)^2}{2(k-m)} = \frac{\left(\frac{w}{2} + kh\right)^2}{2(k-n)}$$

i.e. $$\frac{(7{\cdot}5 - 11{\cdot}55h)^2}{11{\cdot}55 - 1} = \frac{(7{\cdot}5 + 11{\cdot}55h)^2}{11{\cdot}55 - 2}$$

$$7{\cdot}5 - 11{\cdot}55h = \sqrt{\frac{10{\cdot}55}{9{\cdot}55}}(7{\cdot}5 + 11{\cdot}55h)$$

$$= 1{\cdot}051(7{\cdot}5 + 11{\cdot}55h)$$

$$\therefore \quad h = \frac{-0{\cdot}383}{23{\cdot}689} = -0{\cdot}016\,17 \quad \text{(i.e. in cut)}$$

$$\therefore \quad x = kh = 11{\cdot}55 \times 0{\cdot}01617$$

$$= -0{\cdot}187 \text{ ft}$$

$\therefore$ Width of cutting $= 7{\cdot}5 + 0{\cdot}187 = 7{\cdot}687$ say $7{\cdot}69$ ft

$$\text{Area of cutting} = \frac{(7{\cdot}5 + 0{\cdot}187)^2}{2(11{\cdot}55 - 1)}$$

$$= \frac{7{\cdot}69^2}{21{\cdot}10}$$

$$= \underline{2{\cdot}80 \text{ ft}^2}$$

$$\text{Volume of cutting} = 2{\cdot}80 \times 100 \text{ ft}^3$$

$$= \underline{10{\cdot}37 \text{ yd}^3}$$

(d) *Sections with variable crossfall (three-level section)*

If the cross-section is very variable, it may be necessary to determine the area either (a) by an ordinate method or (b) by plotting the section and obtaining the area by scaling or by planimeter.

If the section changes ground slope at the centre line the following analysis can be applied, Fig. 10.23.

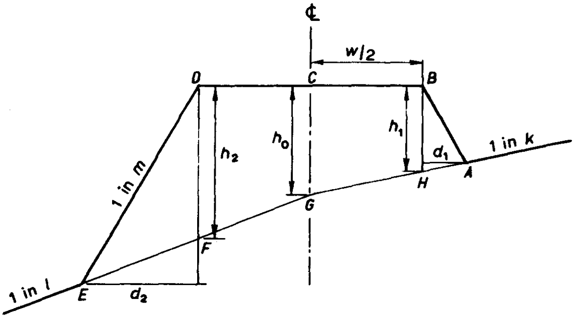


Fig. 10.23 Section with variable crossfall

The total area is made up of four parts:

(1) Triangle AHB Area $= \frac{1}{2}h_1 d_1$

(2) Trapezium $BHGC$ Area $= \frac{w}{4}(h_0 + h_1)$

(3) Trapezium $CGFD$ Area $= \frac{w}{4}(h_0 + h_2)$

(4) Triangle DFE Area $= \frac{1}{2}h_2 d_2$

$$\therefore \text{ Total Area} = \tfrac{1}{2}[h_1 d_1 + h_2 d_2 + \tfrac{w}{2}(2h_0 + h_1 + h_2)] \qquad (10.46)$$

$$\text{Here} \quad h_1 = h_0 - \frac{w}{2k}$$

$$h_2 = h_0 + \frac{w}{2l}$$

$$d_1 = \frac{h_1 mk}{k+m}$$

$$d_2 = \frac{h_2 ml}{l - m}$$

Side width $$W_1 = \frac{w}{2} + d_1$$

$$W_2 = \frac{w}{2} + d_2$$

N.B. k and l have both been assumed +ve, and the appropriate change in sign will be required if GA and GE are different from that shown.

If the level of the surface is known, relative to the formation level, at the edges of the cutting or embankment (Fig. 10.24)

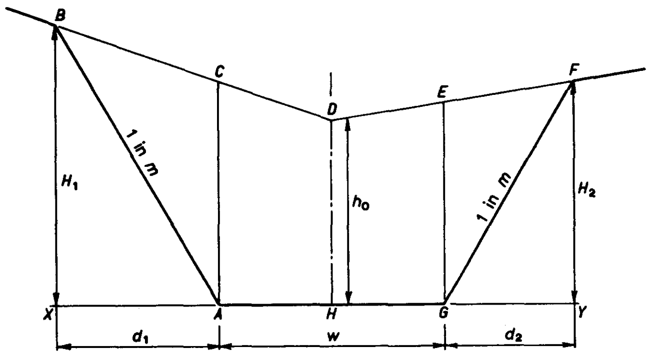


Fig. 10.24 Section with levels at formation pegs

$$\text{Area } ABCDHA = \text{Area } XBDH - \text{Area } XBA$$
$$= \frac{1}{2}\left[\left(\frac{w}{2} + d_1\right)(H_1 + h_0) - H_1 d_1\right]$$
$$= \frac{1}{2}\left[\left(\frac{w}{2} + mH_1\right)(H_1 + h_0) - mH_1^2\right]$$

$$\text{Area } DEFGH = \text{Area } DFYH - \text{Area } FYG$$
$$= \frac{1}{2}\left[\left(\frac{w}{2} + mH_2\right)(H_2 + h_0) - mH_2^2\right]$$

$$\therefore \quad \text{Total Area} = \frac{1}{2}\left[\left(\frac{w}{2} + mH_1\right)(H_1 + h_0) + \left(\frac{w}{2} + mH_2\right)(H_2 + h_0) - m(H_1^2 + H_2^2)\right]. \qquad (10.47)$$

Exercises 10(b) (Cross-sectional areas)

9. At a point A on the surface of ground dipping uniformly due South 1 in 3, excavation is about to commence to form a short cutting for a branch railway bearing N 30° E and rising at 1 in 60 from A. The

width at formation level is 20 ft and the sides batter at 1 vertical to 1 horizontal.

Plot two cross-sections at points B and C 100 ft and 150 ft respectively from A and calculate the cross-sectional area at B.

(N.R.C.T. Ans. 1323·3 ft^2)

10. Calculate the side widths and cross-sectional area of an embankment to a road with a formation width of 40 ft. The sides slope 1 in 2 when the centre height is 10 ft and the existing ground has a crossfall of 1 in 12 at right-angles to the centre line of the embankment.

(N.R.C.T. Ans. 34·28 ft; 48·01 ft; 622·8 ft^2)

11. A road is to be constructed on the side of a hill having a crossfall of 1 vertically to 8 horizontally at right-angles to the centre line of the road; the side slopes are to be similarly 1 to 2 in cut and 1 to 3 in fill; the formation is 50 ft wide and level. Find the distance of the centre line of the road from the point of intersection of the formation with the natural ground to give equality of cut and fill, ignoring any consideration of 'bulking'.

(L.U. Ans. 1·14 ft on the fill side)

12. A road is to be constructed on the side of a hill having a crossfall of 1 vertically to 10 horizontally at right-angles to the centre line of the road; the side slopes are to be similarly 1 to 2 in cut and 1 to 3 in fill; the formation is 80 ft wide and level. Find the position of the centre line of the road with respect to the point of intersection of the formation and the natural ground, (a) to give equality of cut and fill, (b) so that the area of cut shall be 0·8 of the area of fill in order to allow for bulking.

(L.U. Ans. (a) 1·34 ft on the fill side; (b) 0·90 ft on the cut side)

13. The earth embankment for a new road is to have a top width of 40 ft and side slopes of 1 vertically to 2 horizontally, the reduced level of the top surface being 100·0 O.D.

At a certain cross-section, the chainages and reduced levels of the natural ground are as follows, the chainage of the centre line being zero, those on the left and right being treated as negative and positive respectively:

Chainage (ft)	−50	−30	−15	−0	+10	+44
Reduced level (ft)	86·6	88·6	89·2	90·0	90·7	92·4

Find the area of the cross-section of the filling to the nearest square foot, by calculation.

(L.U. Ans. 567 ft^2)

14. A 100 ft length of earthwork volume for a proposed road has a constant cross-section of cut and fill, in which the cut area equals the

fill area. The level formation is 30 ft wide, the transverse ground slope is 20° and the side slopes in cut and fill are respectively ½(horizontal) to 1(vertical) and 1(horizontal) to 1(vertical).

Calculate the volume of excavation in 100 ft length.

(L.U. Ans. 209·2 yd³)

10.32 Alternative formulae for the calculation of volumes from the derived cross-sectional areas

Having computed the areas of the cross-sections, the volumes involved in the construction can be computed by using one of the ordinate formulae but substituting the area of the cross-section for the ordinate.

(1) *Mean Area Rule*

$$V = \frac{W}{n}(A_1 + A_2 + A_3 + \dots + A_n)$$

i.e. $$V = \frac{W}{n}\Sigma A \qquad (10.48)$$

where W = total length between end sections measured along centre line.

n = no. of sectional areas.

ΣA = sum of the sectional areas.

N.B. This is not a very accurate method.

(2) *Trapezoidal (or End Area) Rule* (Fig. 10.25)

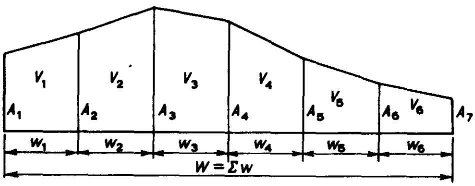


Fig. 10.25 Trapezoidal rule

$$V_1 = \frac{w_1}{2}(A_1 + A_2)$$

$$V_2 = \frac{w_2}{2}(A_2 + A_3)$$

$$V_3 = \frac{w_3}{2}(A_3 + A_4)$$

$$V_{n-1} = \frac{w_{n-1}}{2}(A_{n-1} + A_n)$$

If $W_1 = W_2 = W_n$, then

$$V = (V_1 + V_2 + V_3 + \dots + V_{n-1})$$

$$= \frac{w}{2}[A_1 + 2A_2 + 2A_3 + 2A_4 + \dots + 2A_{n-1} + A_n] \quad (10.49)$$

(3) *Prismoidal Rule* (Fig. 10.26)

As the cross-sections are all parallel and the distance apart can be made equal, the alternate sections can be considered as the mid-section.

The formula assumes that the mid-section is derived from the mean of all the linear dimensions of the end areas. This is difficult to apply in practice but the above application is considered justified particularly if the distance apart of the sections is kept small.

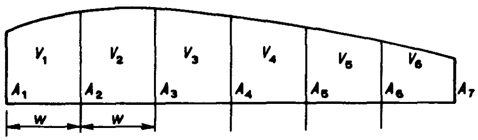


Fig. 10.26 Prismoidal rule

Thus
$$(V_1 + V_2) = \frac{2w}{6}(A_1 + 4A_2 + A_3)$$

$$(V_3 + V_4) = \frac{w}{3}(A_3 + 4A_4 + A_5)$$

$$(V_5 + V_6) = \frac{w}{3}(A_5 + 4A_6 + A_7)$$

$$\therefore \quad \text{Total volume} = \frac{w}{3}[A_1 + 4A_2 + 2A_3 + 4A_4 + 2A_5 + 4A_6 + A_7]$$

If the number of sections is odd, then

$$V = \frac{w}{3}[A_1 + 4\Sigma \text{ even areas} + 2\Sigma \text{ odd areas} + A_n] \quad (10.50)$$

which is *Simpson's rule* applied to volumes.

Prismoidal Corrections

If having applied the end areas rule it is then required to find a closer approximation, a correction can be applied to change the derived value into the amount that would have been derived had the prismoidal rule been applied. For areas A_1 and A_2 s units apart,

By the end areas formula, $V_E = \frac{s}{2}(A_1 + A_2)$

By the prismoidal formula, $V_P = \frac{s}{6}(A_1 + 4A_m + A_2)$

The difference will be the value of the correction, i.e.

$$c = V_E - V_P = \frac{s}{6}[3A_1 + 3A_2 - A_1 - 4A_m - A_2]$$

$$\underline{c = \frac{s}{6}[2(A_1 + A_2) - 4A_m]} \qquad (10.51)$$

For sections without crossfall

Let the two end sections A_1 and A_2 be s ft apart with formation width of w ft and formation heights h_1 and h_2.

Then, by Eq. (10.26),

$$A_1 = h_1(w + mh_1)$$

$$A_2 = h_2(w + mh_2)$$

$$A_m = \frac{1}{2}(h_1 + h_2)\left\{w + \frac{1}{2}m(h_1 + h_2)\right\}$$

Putting these values into Eq. (10.51),

$$c = \frac{s}{6}\Big[2\{w(h_1 + h_2) + m(h_1^2 + h_2^2)\} - 2\{w(h_1 + h_2) + \frac{m}{2}(h_1^2 + h_2^2 + 2h_1h_2)\}\Big]$$

$$= \frac{sm}{6}[2h_1^2 + 2h_2^2 - h_1^2 - h_2^2 - 2h_1h_2]$$

$$\underline{c = \frac{sm}{6}(h_1 - h_2)^2} \qquad (10.52)$$

For sections with crossfall

From Eq. (10.34),

$$A_1 = \frac{m\left[h_1^2 k^2 + \frac{1}{4}w^2 + wh_1 m\right]}{(k^2 - m^2)} + wh_1$$

$$A_2 = \frac{m\left[h_2^2 k^2 + \frac{1}{4}w^2 + wh_2 m\right]}{(k^2 - m^2)} + wh_2$$

$$A_m = \frac{m\left[\frac{1}{4}(h_1 + h_2)^2 k^2 + \frac{1}{4}w^2 + \frac{1}{2}wm(h_1 + h_2)\right]}{(k^2 - m^2)} + \frac{1}{2}w(h_1 + h_2)$$

Substituting these values in Eq. (10.51),

Prismoidal Correction $c = \frac{s}{6}[2(A_1 + A_2) - 4A_m]$

S

$$c = \frac{sm}{6(k^2 - m^2)}[2\{k^2(h_1^2 + h_2^2) + \tfrac{1}{2}w^2 + wm(h_1 + h_2) + wm(k^2 - m^2)(h_1 + h_2)\} - 4\{\tfrac{1}{4}k^2(h_1 + h_2)^2 + \tfrac{1}{4}w^2 + \tfrac{1}{2}wm(k^2 - m^2)(h_1 + h_2)\}]$$

$$c = \frac{sm}{6(k^2 - m^2)}[2k^2(h_1^2 + h_2^2) - k^2(h_1 + h_2)^2]$$

$$c = \frac{smk^2(h_1 - h_2)^2}{6(k^2 - m^2)} \qquad (10.53)$$

For sections with cut and fill

From Eq. (10.42),

$$A_1 = \frac{\left(\frac{w}{2} + kh_1\right)^2}{2(k - m)}$$

$$A_2 = \frac{\left(\frac{w}{2} + kh_2\right)^2}{2(k - m)}$$

$$A_m = \frac{\left\{\frac{w}{2} + \frac{k}{2}(h_1 + h_2)\right\}^2}{2(k - m)}$$

Substituting these values in Eq. (10.51),

$$\text{Prismoidal correction for cut} = \frac{s}{12(k - m)}\left[2\left\{\left(\frac{w}{2} + kh_1\right)^2 + \left(\frac{w}{2} + kh_2\right)^2\right\} - 4\left\{\frac{w}{2} + \frac{k}{2}(h_1 + h_2)\right\}^2\right]$$

$$= \frac{sk^2(h_1 - h_2)^2}{12(k - m)} \qquad (10.54)$$

$$\text{Prismoidal correction for fill} = \frac{s}{12(k - n)}\left[2\left\{\left(\frac{w}{2} - kh_1\right)^2 + \left(\frac{w}{2} - kh_2\right)^2\right\} - 4\left\{\frac{w}{2} - \frac{k}{2}(h_1 + h_2)\right\}^2\right]$$

$$= \frac{sk^2(h_1 - h_2)^2}{12(k - n)} \qquad (10.55)$$

Example 10.6 An embankment is to be formed with its centre line on the surface (in the form of a plane) on full dip of 1 in 20. If the formation width is 40 ft and formation height are 10, 15, and 20 ft at intervals of 100 feet, with the side slopes 1 in 2, calculate the volume between the end sections.

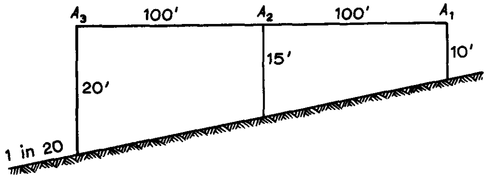
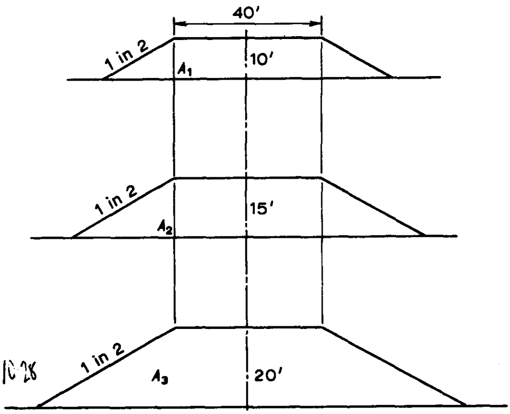


Fig. 10.27 Longitudinal section



10·28

Fig. 10.28 Cross-sections

$$\text{Area (1)} = h_1(w + mh_1)$$
$$= 10(40 + 2 \times 10)$$
$$= \underline{600 \text{ ft}^2}$$

$$\text{Area (2)} = 15(40 + 2 \times 15)$$
$$= \underline{1050 \text{ ft}^2}$$

$$\text{Area (3)} = 20(40 + 2 \times 20)$$
$$= \underline{1600 \text{ ft}^2}$$

Volume

(1) *By Mean Areas*

$$V = \frac{W}{n}(\Sigma A)$$

$$= \frac{200}{3}(600 + 1050 + 1600)$$
$$= \underline{216\,666{\cdot}7 \text{ ft}^3}$$

(2) *By End Areas (Trapezoidal)*

$$V = \frac{w}{2}[A_1 + 2A_2 + A_3]$$
$$= \frac{100}{2}[600 + 2100 + 1600]$$
$$= \underline{215\,000 \text{ ft}^3}$$

(3) *By the prismoidal rule (treating the whole as one prismoid)*

$$V = \frac{w}{3}[A_1 + 4A_2 + A_3]$$
$$= \frac{100}{3}[600 + 4200 + 1600]$$
$$= \underline{213\,333{\cdot}3 \text{ ft}^3}$$

(4) *By Prismoidal Correction to End Areas*

(By end Areas in each section) $V_1 = \frac{100}{2}(600 + 1050) = 82\,500 \text{ ft}^3$

$$V_2 = \frac{100}{2}(1050 + 1500) = \underline{132\,500 \text{ ft}^3}$$

$$V_T = \underline{215\,000 \text{ ft}^3}$$

By outer Areas $V = \frac{200}{2}(600 + 1600) = \underline{220\,000 \text{ ft}^3}$

Applying Prismoidal Correction to adjacent areas,

$$(V_E - V_P)_1 = \frac{sm}{6}(h_1 - h_2)^2$$
$$= \frac{100 \times 2}{6}(15 - 10)^2 = 833{\cdot}33 \text{ ft}^3$$
$$(V_E - V_P)_2 = \frac{100 \times 2}{6}(20 - 15)^2 = 833{\cdot}33 \text{ ft}^3$$

Total Correction $= +1666{\cdot}\dot{6}6 \text{ ft}^3$

$\therefore$ Volume $V_P = 215\,000 - 1666{\cdot}67 = \underline{213\,333{\cdot}33 \text{ ft}^3}$

Applying Prismoidal Correction to outer areas,

$$V_E - V_P = \frac{200 \times 2}{6}(20 - 10)^2 = 6\,666{\cdot}67$$
$$V_P = 220\,000 - 6\,666{\cdot}67 = \underline{213\,333{\cdot}33 \text{ ft}^3}$$

N.B. (1) The correct value is only obtained by applying the prismoidal correction to volumes obtained by adjacent areas unless, as here, the whole figure is symmetrical.

(2) The prismoidal correction has little to commend it in preference to the application of the prismoidal formula if all the information is readily available.

Example 10.7 Given the previous example but with the centre line turned through 90°

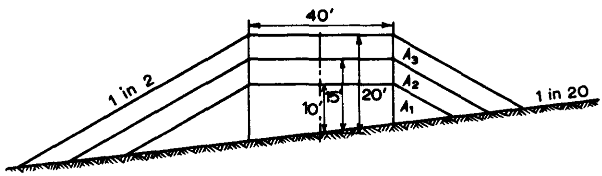


Fig. 10.29 Cross-sections

From Eq. (10.34),

$$A = \frac{m[h^2 k^2 + \frac{1}{4} w^2 + whm]}{k^2 - m^2} + wh$$

Cross-sectional Areas

$$A_1 = \frac{2}{20^2 - 2^2}[10^2 \times 20^2 + \tfrac{1}{4} \times 40^2 + 40 \times 10 \times 2] + 40 \times 10$$

$$= \frac{1}{198}[40\,000 + 400 + 800] + 400 = \underline{608{\cdot}08 \text{ ft}^2}$$

$$A_2 = \frac{1}{198}[90\,000 + 400 + 1200] + 600 = \underline{1062{\cdot}62 \text{ ft}^2}$$

$$A_3 = \frac{1}{198}[160\,000 + 400 + 1600] + 800 = \underline{1618{\cdot}18 \text{ ft}^2}$$

Volume

(1) *By Mean Areas* (Eq. 10.48)

$$V = \frac{200}{3}[608{\cdot}08 + 1062{\cdot}62 + 1618{\cdot}18] = \underline{219\,258{\cdot}7 \text{ ft}^3}$$

(2) *By End Areas* (Eq. 10.49)

(taking all sections) $V = \frac{100}{2}[608{\cdot}08 + 2 \times 1062{\cdot}62 + 1618{\cdot}18] = \underline{217\,575 \text{ ft}^3}$

(taking outer sections) $V = \frac{200}{2}[608{\cdot}08 + 1618{\cdot}18] = \underline{222\,626 \text{ ft}^3}$

(3) *By the prismoidal rule* (Eq. 10.50) (treating the whole as one prismoid)

$$V = \frac{100}{3}[608{\cdot}08 + 4 \times 1062{\cdot}62 + 1618{\cdot}18] = 215\,891{\cdot}5 \text{ ft}^3$$

(4) *By applying Prismoidal Correction to End Areas*

Applying prismoidal correction to each section (Eq. 10.53),

$$c = \frac{Smk^2}{6(k^2 - m^2)} \times (h_1 - h_2)^2$$

$$c_1 = \frac{100 \times 2 \times 20^2}{6 \times 396} \times (15 - 10)^2 = 841{\cdot}75 \text{ ft}^3$$

$$c_2 = \frac{100 \times 2 \times 20^2}{6 \times 396} \times (20 - 15)^2 = \underline{841{\cdot}75 \text{ ft}^3}$$

$$c_T = 1683{\cdot}50$$

$$\therefore\ V_P = 217\,575 - 1683{\cdot}5 = \underline{215\,891{\cdot}5 \text{ ft}^3}$$

Applying prismoidal correction to outer areas,

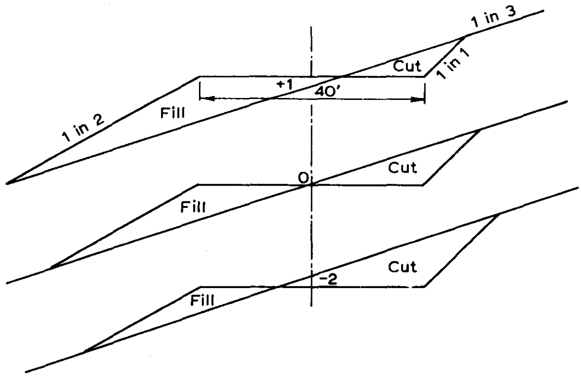
$$c = \frac{200 \times 2 \times 20^2}{6 \times 396} \times (20 - 10)^2 = 6734 \text{ ft}^3$$

$$\therefore\ V_P = 222\,626 - 6734 = \underline{215\,892 \text{ ft}^3}$$

N.B. Where the figure is symmetrical the prismoidal correction again gives the same value, but it would be unwise to apply the latter method where the prismoids are long and the cross-sectional areas very variable, unless it is applied to each section in turn, as shown above.

Example 10.8 A road has a formation width of 40 ft, and the side slopes are 1 in 1 in cut and 1 in 2 in fill. The ground slopes at 1 in 3 at right-angles to the centre line. Sections at 100 ft centres are found to have formation heights of +1 ft, 0, and −2 ft respectively. Calculate the volumes of cut and fill over this length.

Fig. 10.30



Areas of cut

From Eq. (10.44),

$$(h+\text{ve})\quad \text{Area of Cut} = \frac{\left(\frac{w}{2} - kh\right)^2}{2(k - m)}$$

$$\therefore \quad A_1 = \frac{\left(\frac{40}{2} - 3 \times 1\right)^2}{2(3 - 1)} = \underline{72{\cdot}25 \text{ ft}^2}$$

$$A_2 = \frac{(20 - 0)^2}{4} = \underline{100 \text{ ft}^2}$$

$$(h-\text{ve})\quad A_3 = \frac{(20 + 3 \times 2)^2}{4} = \underline{169 \text{ ft}^2}$$

Areas of fill

From Eq. (10.43),

$$(h+\text{ve})\quad \text{Area of fill} = \frac{\left(\frac{w}{2} + Kh\right)^2}{2(k - n)}$$

$$\therefore \quad A_1' = \frac{(20 + 3)^2}{2(3 - 2)} = \underline{264{\cdot}5 \text{ ft}^2}$$

$$A_2' = \frac{(20 + 0)^2}{2} = \underline{200 \text{ ft}^2}$$

$$(h-\text{ve})\quad A_3' = \frac{(20 - 6)^2}{2} = \underline{98 \text{ ft}^2}$$

Volume of Cut

(1) *By Mean Areas*

$$V = \frac{200}{3}[72{\cdot}25 + 100 + 169] = \underline{22\,750 \text{ ft}^3}$$

(2) *By End Areas*

(taking all sections)

$$V = \frac{100}{2}[72{\cdot}25 + 2 \times 100 + 169] = \underline{22\,062{\cdot}5 \text{ ft}^3}$$

(taking outer sections)

$$V = \frac{200}{2}[72{\cdot}25 + 169] = \underline{24\,125 \text{ ft}^3}$$

(3) *By the prismoidal rule* (treating the whole as a prismoid)

$$V = \frac{100}{3}[72{\cdot}25 + 4 \times 100 + 169] = \underline{21\,375 \text{ ft}^3}$$

(4) *By applying Prismoidal Correction to End Areas*

Applying prismoidal correction to each section of cut,

$$c = \frac{Sk^2(h_1 - h_2)^2}{12(k - m)}$$

$$\therefore \quad c_1 = \frac{100 \times 3^2(1 - 0)^2}{12(3 - 1)} = 37{\cdot}5 \text{ ft}^3$$

$$c_2 = \frac{900 \times (2 - 0)^2}{24} = 150{\cdot}0 \text{ ft}^3$$

$$c_T = 187{\cdot}5 \text{ ft}^3$$

$$\therefore \quad V_P = 22062{\cdot}5 - 187{\cdot}5 = \underline{21875{\cdot}0 \text{ ft}^3}$$

Applying prismoidal correction to outer areas,

$$c = \frac{900\,(1 + 2)^2}{24} = 337{\cdot}5 \text{ ft}^3$$

$$\therefore \quad V_P = 24125 - 337{\cdot}5 = \underline{23787{\cdot}5 \text{ ft}^3}$$

Volumes of Fill

(1) *By Mean Areas*

$$V = \frac{200}{3}[264{\cdot}5 + 200 + 98] = \underline{37500 \text{ ft}^3}$$

(2) *By End Areas*

(taking all sections)

$$V = \frac{100}{2}[264{\cdot}5 + 2 \times 200 + 98] = \underline{38125 \text{ ft}^3}$$

(taking outer sections)

$$V = \frac{200}{2}[264{\cdot}5 + 98] = \underline{36250 \text{ ft}^3}$$

(3) *By the prismoidal rule* (treating the whole as a prismoid)

$$V = \frac{100}{3}[264{\cdot}5 + 4 \times 200 + 98] = \underline{38750 \text{ ft}^3}$$

(4) *By applying Prismoidal Correction to End Areas*

Applying prismoidal correction to each section of fill,

$$c_1 = \frac{100 \times 3^2 \times 1^2}{12(3 - 2)} = 75 \text{ ft}^3$$

$$c_2 = \frac{900 \times 2^2}{12} = \underline{300 \text{ ft}^3}$$

$$c_T = 375 \text{ ft}^3$$

$$\therefore \quad V_P = 38125 - 375 = \underline{37750 \text{ ft}^3}$$

Applying prismoidal correction to outer areas,

$$c = \frac{900\,(1 + 2)^2}{12} = 675 \text{ ft}^3$$

$$\therefore \quad V_P = 36\,250 - 675 = \underline{35\,575 \text{ ft}^3}$$

N.B. The prismoidal correction applied to each section gives a much closer approximation to the value derived by the prismoidal formula, although the latter in this case is not strictly correct as the middle height is not the mean of the two end heights.

Example 10.9 Calculate the volume between three sections of a railway cutting. The formation width is 20 ft; the sections are 100 ft apart; the side slopes are 1 in 2 and the heights of the surface above the formation level are as follows:

Section	Left	Centre	Right
1	17·6	16·4	17·0
2	21·2	20·0	18·8
3	19·3	17·9	16·3

From Eq. (10.47),

$$\text{Area} = \frac{1}{2}\left[\left(\frac{w}{2} + mH_1\right)(H_1 + h_0) + \left(\frac{w}{2} + mH_2\right)(H_2 + h_0) - m(H_1^2 + H_2^2)\right]$$

$$\text{Section 1, } A_1 = \frac{1}{2}\left[\left(\frac{20}{2} + 2 \times 17{\cdot}6\right)(17{\cdot}6 + 16{\cdot}4) + \left(\frac{20}{2} + 2 \times 17{\cdot}0\right)(17{\cdot}0 + 16{\cdot}4) - 2(17{\cdot}6^2 + 17{\cdot}0^2)\right] = \underline{904{\cdot}4 \text{ ft}^2}$$

$$\text{Section 2, } A_2 = \frac{1}{2}[(10 + 42{\cdot}4)(41{\cdot}2) + (10 + 37{\cdot}6)(38{\cdot}8) - 2(21{\cdot}2^2 + 18{\cdot}8^2)] = \underline{1200{\cdot}0 \text{ ft}^2}$$

$$\text{Section 3, } A_3 = \frac{1}{2}[(10 + 38{\cdot}6)(37{\cdot}2) + (10 + 32{\cdot}6)(34{\cdot}2) + 2(19{\cdot}3^2 + 16{\cdot}3^2)] = \underline{994{\cdot}2 \text{ ft}^2}$$

Using the prismoidal formula,

$$V = \frac{100}{3 \times 27}[904{\cdot}4 + 4 \times 1200 + 994{\cdot}2] = \underline{8270 \text{ yd}^3}$$

$$\underline{\text{Total Volume} = 8\,270 \text{ yd}^3}$$

10.33 Curvature Correction (Fig. 10.31)

When the centre line of the construction is curved, the cross-sectional areas will be no longer parallel but radial to the curve.

Volume of such form is obtained by using the *Theorem of Pappus* which states that '*a volume swept out by a constant area revolving about a fixed axis is given by the product of the area and the length of the path of the centroid of the area*'.

The volume of earthworks involved in cuttings and embankments as part of transport systems following circular curves may thus be determined by considering cross-sectional areas revolving about the centre of such circular curves.

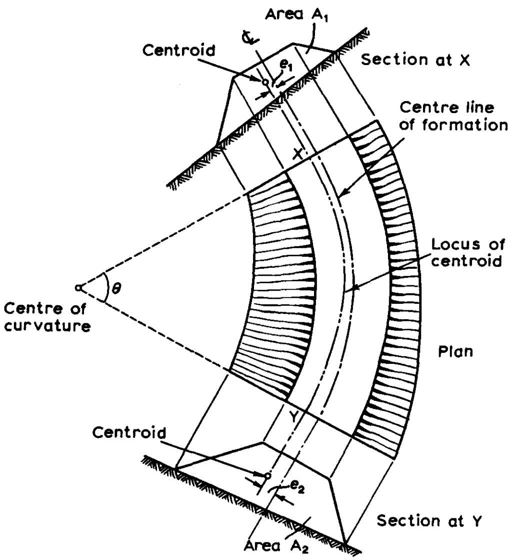


Fig. 10.31 Curvature correction

If the cross-sectional area is constant, then the volume will equal the product of this area and the length of the arc traced by the centroid.

If the sections are not uniform, an approximate volume can be derived by considering a mean eccentric distance $(e) = \frac{e_1 + e_2}{2}$ relative to the centre line of the formation.

This will give a mean radius for the path of the centroid $(R \pm e)$, the negative sign being taken as on the same side as the centre of curvature.

Length of path of centroid $XY = (R \pm e)\,\theta_{rad}$.

$$\text{but} \quad \theta_{rad} = \frac{S}{R} \quad \text{where } S = \text{length of arc on the centre line}$$

$$\therefore \quad XY = \frac{S}{R}(R \pm e) = S\left(1 \pm \frac{e}{R}\right)$$

$\therefore$ Volume is given approximately as

$$\underline{V = \frac{S}{2}(A_1 + A_2)\left(1 \pm \frac{e}{R}\right)} \tag{10.56}$$

Alternatively each area may be corrected for the eccentricity of its centroid.

If e_1 be the eccentricity of the centroid of an area A_1, then the volume swept out through a small arc $\delta\theta$ is $\delta V = A_1(R \pm e_1)\delta\theta$.

If the eccentricity had been neglected then

$$\delta V = A_1 R\delta\theta$$

$$\text{with a resulting error} = A_1 e_1 \delta\theta$$

$$= \frac{A_1 e_1}{R} \text{ per unit length} \tag{10.57}$$

Thus, if each area is corrected by an amount $\pm\frac{Ae}{R}$, these new equivalent areas can be used in the volume formula adopted.

10.34 Derivation of the eccentricity e of the centroid G

Centroids of simple shapes

Parallelogram (Fig. 10.32)

G lies on the intersection of the diagonals or the intersection of lines joining the midpoints of their opposite sides. (10.58)

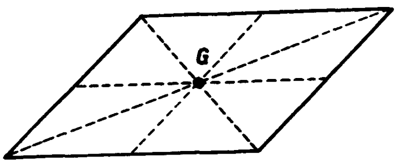


Fig. 10.32

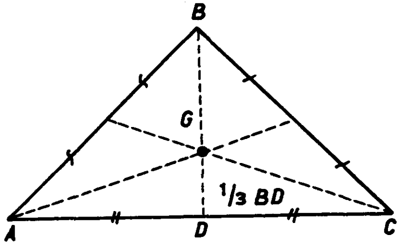


Fig. 10.33

Triangle (Fig. 10.33)

G lies at the intersection of the medians and is $\frac{2}{3}$ of their length from each apex. (10.59)

Trapezium (Fig. 10.34)

$$x = \frac{1}{3}h\left(\frac{a + 2b}{a + b}\right) \tag{10.60}$$

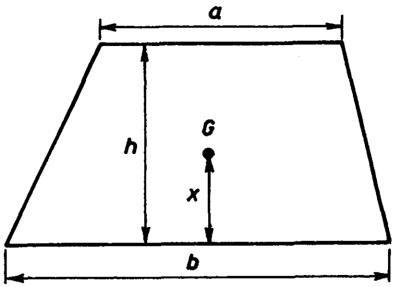


Fig. 10.34

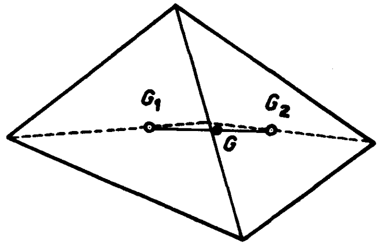


Fig. 10.35

A Compound Body (Fig. 10.35)

If the areas of the separate parts are A_1 and A_2 and their centroids G_1 and G_2, with the compounded centroid G,

$$G_1G = \frac{A_2 \times G_1G_2}{A_1 + A_2} \tag{10.61}$$

$$\text{or} \quad G_2G = \frac{A_1 \times G_1G_2}{A_1 + A_2} \tag{10.62}$$

Thus for typical cross-sectional areas met with in earthwork calculations, the figures can be divided into triangles and the centre of gravity derived from the compounding of the separate centroids of the triangles or trapezium, Fig. 10.36.

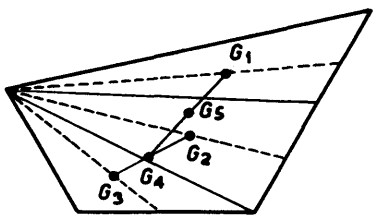


Fig. 10.36

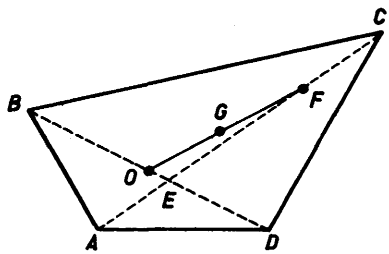


Fig. 10.37

Alternatively, Fig. 10.37,

Let the diagonals of $ABCD$ intersect at E.

$$BO = OD \quad \text{on line } BD$$

$$AE = FC \quad \text{on line } AC$$

$$\text{then} \quad \underline{2.OG = GF} \tag{10.63}$$

To find the eccentricity e *of the centroid* G

Case 1. *Where the surface has no crossfall,* the area is symmetrical and the centroid lies on the centre line, i.e. $e = 0$.

Case 2. *Where the surface has a crossfall* 1 *in* k (Fig. 10.38)

Let Total Area of $ABDE = A_T$

Area of triangle AEF

$$= \tfrac{1}{2}AF(H_2 - H_1)$$

$$= \frac{W_1(W_1 + W_2)}{k} \qquad (10.64)$$

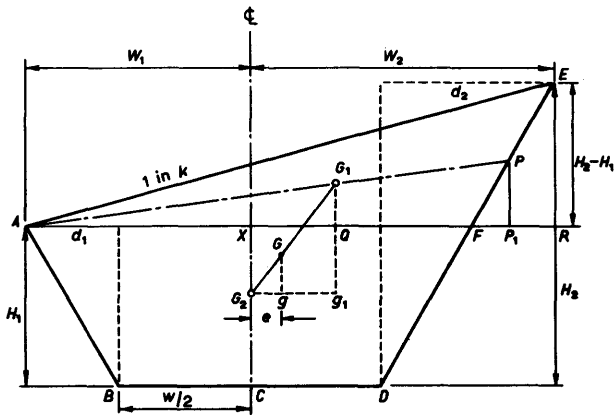


Fig. 10.38 Section with crossfall

Let G_1 and G_2 be the centroids of areas AEF and $AFDB$ respectively.

$$\text{Length } AQ = \text{horizontal projection of } AG_1$$

$$= \tfrac{2}{3}[AP_1]$$

$$= \tfrac{2}{3}\left[\frac{AR + AF}{2}\right] = \tfrac{1}{3}[W_1 + W_2 + 2W_1]$$

$$= \tfrac{1}{3}[3W_1 + W_2] = W_1 + \frac{W_2}{3}$$

Distance of Q from centre line,

$$\text{i.e.} \quad XQ, = G_2g_1 = W_1 + \frac{W_2}{3} - W_1$$

$$= \frac{W_2}{3} \qquad (10.65)$$

Distance of centroid G for the whole figure (from the centre line, i.e. e),

$$e = \frac{\text{Area } \Delta\ AEF \times XQ}{\text{Total Area } A_T} = \frac{W_1W_2(W_1 + W_2)}{3k \, . \, A_T} \qquad (10.66)$$

$$\text{Conversion Area } A_c = \pm\frac{A_T e}{R}$$

$$\text{i.e.} \quad A_c = \pm \frac{A_T[W_1 W_2 (W_1 + W_2)]}{3k A_T R}$$

$$= \frac{W_1 W_2 (W_1 + W_2)}{3kR}$$

$$\therefore \quad \text{Corrected Area} = A_T \pm \frac{W_1 W_2 (W_1 + W_2)}{3kR} \qquad (10.67)$$

Case 3. *Sections with part cut and part fill* (Fig. 10.39)

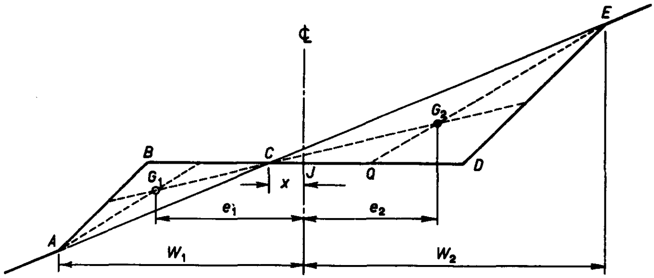


Fig. 10.39 Section part cut/part fill

For section in cut, i.e. triangle *CED*, *G* lies on the median *EQ*.

$$JQ = \tfrac{1}{2}\left(\frac{W}{2} + x\right) - x = \tfrac{1}{2}\left(\frac{W}{2} - x\right)$$

$$= \tfrac{1}{2}\left(\frac{W}{2} - kh_0\right)$$

$$e_2 = JQ + \tfrac{1}{3}(W_2 - JQ) = \tfrac{1}{3}(W_2 + 2JQ)$$

$$= \tfrac{1}{3}\left(W_2 + \frac{W}{2} - kh_0\right) \qquad (10.68)$$

$$\text{Similarly for fill} \quad e_1 = \tfrac{1}{3}\left(W_1 + \frac{W}{2} + kh_0\right) \qquad (10.69)$$

Example 10.10 Using the information in Example 10.6, viz. embankment with a surface crossfall of 1 in 20, side slopes 1 in 2, formation width 40 ft and formation heights of 10, 15 and 20 ft at 100 ft centres, if this formation lies with its centre line on the arc of a circle of radius 500 ft, calculate

(a) the side widths of each section,
(b) the eccentricity of their centroids,
(c) the volume of the embankment over this length for the centre of curvature (i) uphill (ii) downhill.

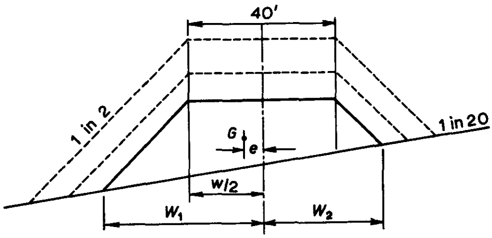


Fig. 10.40

(a) *Side widths*

Section 1

From Eq. (10.31),

$$W_1 = \frac{w}{2} + \frac{\left(h_0 - \frac{w}{2k}\right)mk}{k + m}$$

and from Eq. (10.32),

$$W_2 = \frac{w}{2} + \frac{\left(h_0 - \frac{w}{2k}\right)mk}{k - m}$$

i.e.
$$W_1 = \frac{40}{2} + \frac{\left(10 - \frac{40}{40}\right)2 \times 20}{20 + 2}$$

$$= 20 + \frac{9 \times 40}{22} = \underline{36{\cdot}36 \text{ ft}}$$

$$W_2 = 20 + \frac{11 \times 40}{18} = \underline{44{\cdot}44 \text{ ft}}$$

Section 2

$$W_1 = 20 + \frac{14 \times 40}{22} = \underline{45{\cdot}45 \text{ ft}}$$

$$W_2 = 20 + \frac{16 \times 40}{18} = \underline{55{\cdot}56 \text{ ft}}$$

Section 3

$$W_1 = 20 + \frac{19 \times 40}{22} = \underline{54{\cdot}55 \text{ ft}}$$

$$W_2 = 20 + \frac{21 \times 40}{18} = \underline{66{\cdot}67 \text{ ft}}$$

(b) *Eccentricity* (e)

From Eq. (10.66),
$$e = \frac{W_1 W_2 (W_1 + W_2)}{3kA}$$

$$e_1 = \frac{36{\cdot}36 \times 44{\cdot}44\,(36{\cdot}36 + 44{\cdot}44)}{3 \times 20 \times 608{\cdot}08}$$

$= \underline{3{\cdot}58 \text{ ft}}$ (Area 608·08 ft^2 from previous calculations)

$$e_2 = \frac{45{\cdot}45 \times 55{\cdot}56\,(45{\cdot}45 + 55{\cdot}56)}{3 \times 20 \times 1062{\cdot}62}$$

$$= \underline{4{\cdot}00 \text{ ft}}$$

$$e_3 = \frac{54{\cdot}55 \times 66{\cdot}67\,(54{\cdot}55 \times 66{\cdot}67)}{3 \times 20 \times 1618{\cdot}18}$$

$$= \underline{4{\cdot}54 \text{ ft}}$$

(c) *Volumes*

Using the above values of eccentricity in the prismoidal formula, the volume correction

$$V_c = \pm\frac{100}{3}\left[608{\cdot}08 \times \frac{3{\cdot}58}{500} + 4\left(1062{\cdot}62 \times \frac{4}{500}\right) + 1618{\cdot}18 \times \frac{4{\cdot}54}{500}\right]$$

$$= \pm\frac{1}{15}[608{\cdot}08 \times 3{\cdot}58 + 16 \times 1062{\cdot}62 + 1618{\cdot}18 \times 4{\cdot}54]$$

$$= \underline{\pm 1768{\cdot}4 \text{ ft}^3}$$

The correction is +ve if the centre of the curve lies on the uphill side.

$$\therefore \quad \text{Corrected volume} = 215\,891{\cdot}5 \pm 1768{\cdot}4 \text{ ft}^3$$

$$= \underline{217\,660 \text{ ft}^3}$$

$$\text{or} \quad \underline{214\,123 \text{ ft}^3}$$

A more convenient calculation of volume, without separately calculating the eccentricity, is to correct the areas using Eq. (10.67).

$$\text{Area Correction} \quad A_c = \pm\frac{W_1 W_2 (W_1 + W_2)}{3.k.R}$$

$$A_{c_1} = \pm\frac{36{\cdot}36 \times 44{\cdot}44\,(36{\cdot}36 + 44{\cdot}44)}{3 \times 20 \times 500}$$

$$= \underline{\pm 4{\cdot}35 \text{ ft}^2}$$

$$A_{c_2} = \pm\frac{45{\cdot}45 \times 55{\cdot}56\,(45{\cdot}45 + 55{\cdot}56)}{3 \times 20 \times 500}$$

$$= \underline{\pm 8{\cdot}50 \text{ ft}^2}$$

$$A_{c_3} = \pm\frac{54{\cdot}55 \times 66{\cdot}67\,(54{\cdot}55 + 66{\cdot}67)}{3 \times 20 \times 500}$$

$$= \underline{\pm 14{\cdot}70 \text{ ft}^2}$$

$\therefore$ Corrected Areas are:

$$A_1 = 608{\cdot}08 \pm 4{\cdot}35 = \underline{612{\cdot}43} \text{ or } \underline{603{\cdot}73 \text{ ft}^2}$$

$$A_2 = 1062{\cdot}62 \pm 8{\cdot}50 = \underline{1071{\cdot}12} \text{ or } \underline{1054{\cdot}12 \text{ ft}^2}$$

$$A_3 = 1618{\cdot}18 \pm 14{\cdot}70 = \underline{1632{\cdot}88} \text{ or } \underline{1603{\cdot}48 \text{ ft}^2}$$

Corrected volumes:

(i) With centre of curve on uphill side,

$$V = \frac{100}{3}[612{\cdot}43 + 4 \times 1071{\cdot}12 + 1632{\cdot}88]$$
$$= \underline{217\,657 \text{ ft}^3}$$

(ii) With centre of curve on downhill side,

$$V = \frac{100}{3}[603{\cdot}73 + 4 \times 1054{\cdot}12 + 1603{\cdot}48]$$
$$= \underline{214\,122 \text{ ft}^3}$$

10.4 Calculation of Volumes from Contour Maps

Here the volume is derived from the areas contained in the plane of the contour. For accurate determinations the contour interval must be kept to a minimum and this value will be the width (w) in the formulae previously discussed.

The areas will generally be obtained by means of a planimeter, the latter tracing out the enclosing line of the contour.

For most practical purposes the Prismoidal formula is satisfactory, with alternate areas as 'mid-areas' or, if the contour interval is large, on interpolated mid-contour giving the required 'mid-area' may be used.

10.5 Calculation of Volumes from Spot-heights

This method uses grid levels from which the depth of construction is derived.

The volume is computed from the mean depth of construction in each section forming a truncated prism, the end area of which may be rectangular but preferably triangular, Fig. 10.41.

$$V = \text{plan area} \times \text{mean height} \tag{10.70}$$

If a grid is used, the triangular prisms are formed by drawing diagonals, and then each prism is considered in turn.

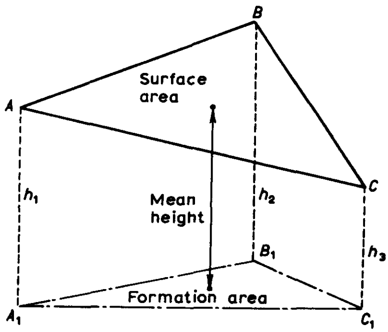


Fig. 10.41 Volume from spot-heights

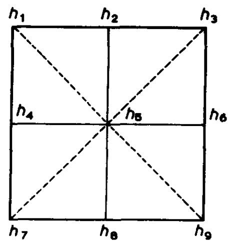


Fig. 10.42

The total volume is then derived (each triangle is of the same area) as one third of the area of the triangle multiplied by the sum of each height in turn multiplied by the number of applications of that height,

$$\text{i.e.} \quad V = \frac{\Delta}{3}[\Sigma nh] \tag{10.71}$$

$$\text{e.g.} \quad V = \frac{\Delta}{3}[2h_1 + 2h_2 + 2h_3 + 2h_4 + 8h_5 + 2h_6 + 2h_7 + 2h_8 + 2h_9]$$

(Fig. 10.42)

10.6 Mass-haul Diagrams

These are used in planning the haulage of large volumes of earthwork for construction works in railway and trunk road projects.

10.61 Definitions

Bulking An increase in volume of earthwork after excavation.

Shrinkage A decrease in volume of earthwork after deposition and compaction.

Haul Distance (d) The distance from the working face of the excavation to the tipping point.

Average Haul Distance (D) The distance from the centre of gravity of the cutting to that of the filling.

Free Haul Distance The distance, given in the Bill of Quantities, included in the price of excavation per cubic yard.

Overhaul Distance The extra distance of transport of earthwork volumes beyond the Free Haul Distance.

Haul The sum of the product of each load by its haul distance. This must equal the total volume of excavation multiplied by the average haul distance, i.e. $\Sigma . v . d = V.D.$

Overhaul The products of volumes by their respective overhaul distance. Excess payment will depend upon overhaul.

Station Yard A unit of overhaul, viz. 1 yd^3 × 100 ft.

Borrow The volume of material brought into a section due to a deficiency.

Waste The volume of material taken from a section due to excess.

In S.I. units the haul will be in m^3, the haul distances in metres and the new 'station' unit probably 1 m^3 moved 100 m.

10.62 Construction of the mass-haul diagram (Fig. 10.43)

(1) Calculate the cross-sectional areas at given intervals along the project.

(2) Calculate the volumes of cut and fill between the given areas relative to the proposed formation.

N.B. (a) Volumes of cut are considered positive.
(b) Volumes of fill are considered negative.

(3) Calculate the aggregated algebraic volume for each section.

(4) Plot the profile of the existing ground and the formation.

(5) Using the same scale for the horizontal base line, plot the mass haul curve with the aggregated volumes as ordinates.

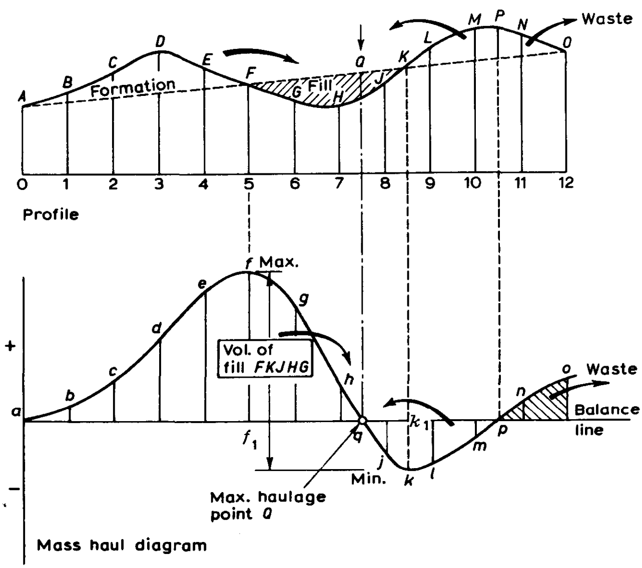


Fig. 10.43 Mass-haul curves

10.63 Characteristics of the mass-haul diagram

(1) A rising curve indicates cutting as the aggregate volume is increasing ($a–f$ is seen to agree with AF on the profile).

(2) A maximum point on the curve agrees with the end of the cut, i.e. $f–F$.

(3) A falling curve indicates filling as the aggregate volume is decreasing ($f–k$ is seen to agree with $F–K$ on the profile).

(4) The vertical difference between a maximum point and the next minimum point represents the volume of the embankment, i.e. $ff_1 + k_1k$ (the vertical difference between any two points not having a minimum or maximum between them represents the volume of earthwork between them.)

(5) If any horizontal line is drawn cutting the mass-haul curve (e.g. aqp), the volume of cut equals the volume of fill between these points. In each case the algebraic sum of the quantities must equal zero.

(6) When the horizontal balancing line cuts the curve, the area above the line indicates that the earthwork volume must be moved forward. When the area cut off lies below the balancing line, then the earthwork must be moved backwards.

(7) The length of the balancing line between intersection points, e.g. aq, qp, represents the maximum haul distance in that section (q is the maximum haulage point both forward, aq, and backwards, pq).

(8) The area cut off by the balancing line represents the haul in that section. N.B. As the vertical and horizontal scales are different, i.e. 1 in. = s ft horizontally and 1 in. = v yd^3, an area of a in^2 represents a haul of avs yd^3, ft = $\dfrac{avs}{100}$ station yards.

10.64 Free-haul and overhaul (Fig. 10.44)

The Mass-haul diagram is used for finding the overhaul charge as follows:

Free-haul distance is marked off parallel to the balance line on any haul area, e.g. bd. The ordinate cc_2 represents the volume dealt with as illustrated in the profile.

Any cut within the section ABB_1A_1 has to be transported through the free-haul length to be deposited in the section D_1E_1ED. This represents the 'overhaul' of volume (ordinate bb_1) which is moved from the centroid G_1 of the cut to the centroid G_2 of the fill.

The overhaul distance is given as the distance between the centroids less the free-haul distance.

$$\text{i.e.}\quad (G_1G_2) - bd$$

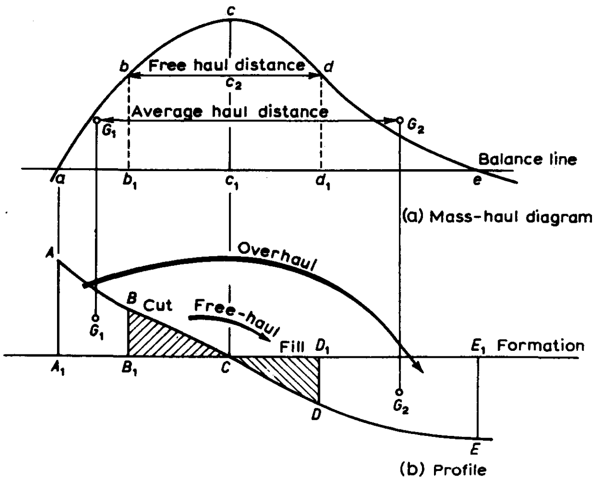


Fig. 10.44 Free-haul and overhaul

The amount of overhaul is given as the volume (ordinate $bb_1 = dd_1$) $\times$ the overhaul distance.

Where long haulage distances are involved, it may be more economical to waste material from the excavation and to borrow from a location within the free-haul limit.

If l is the overhaul distance, c the cost of overhaul and e the cost of excavation, then to move 1 yd^3 from cut to fill the cost is given as

$$e + lc$$

whereas the cost to cut, waste the material, borrow and tip without overhaul will equal $2e$.

$$\text{Economically} \quad e + lc = 2e$$

$$\therefore \quad l = \frac{e}{c} \quad \text{(assuming no cost for wasting)}$$

Thus if the cost of excavation is 2/6 per yd^3 and the cost of overhaul is 2d per station yard, then the total economic overhaul distance $= \dfrac{30}{2} = \underline{1500 \text{ ft}}$

If the free-haul is given as 500 ft the maximum economic haul $= 1500 + 500 = \underline{2000 \text{ ft}}$.

The overhaul distance is found from the mass-haul diagram by determining the distance from the centroid of the mass of the excavation to the centroid of the mass of the embankment.

The centroid of the excavation and of the embankment can be

determined (1) graphically, (2) by taking moments, (3) planimetrically. These methods are illustrated in the following example.

Example 10.11 Volumes of cut and fill along a length of proposed road are as follows:

Chainage	Volume (ft^3) Cut	Fill
0		
100	290	
200	760	
300	1680	
400	620	
480	120	
500		20
600		110
700		350
800		600
900		780
1000		690
1100		400
1200		120

Draw a mass diagram, and excluding the surplus excavated material along this length determine the overhaul if the free-haul distance is 300 ft. (I.C.E.)

Answer

Chainage	Volume (ft^3) Cut	Fill	Aggregate volume (ft^3)
0			
100	290		+ 290
200	760		+ 1050
300	1680		+ 2730
400	620		+ 3350
480	120		+ 3470
500		20	+ 3450
600		110	+ 3340
700		350	+ 2990
800		600	+ 2390
900		780	+ 1610
1000		690	+ 920
1100		400	+ 520
1200		120	+ 400
	3470	3070	
	3070		
Check	400		

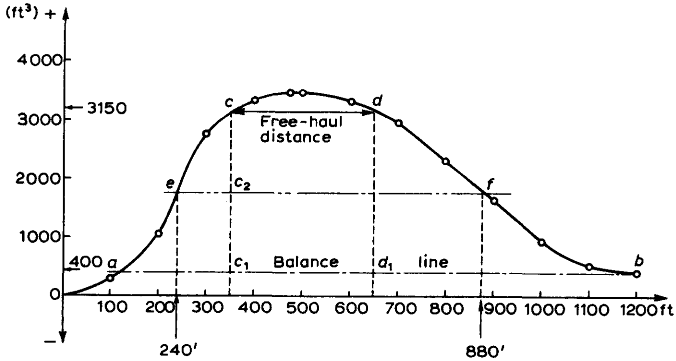


Fig. 10.45 Mass-haul diagram

(1) *Graphical Method*

(i) As the surplus of 400 ft^3 is to be neglected, the balancing line is drawn from the end of the mass-haul curve, parallel to the base line, to form a new balancing line *ab*.

(ii) As the free-haul distance is 300 ft, this is drawn as a balancing line *cd*.

(iii) From *c* and *d*, draw ordinates cutting the new base line at $c_1 d_1$.

(iv) To find the overhaul:

(a) Bisect cc_1 to give c_2 and draw a line through c_2 parallel to the base line and cutting the curve at *e* and *f*, which now represent the centroids of the masses acc_1 and dbd_1

(b) The average haul distance from acc_1 in excavation to make up the embankment dbd_1 = *ef*.

(c) The overhaul distance = the haul distance–the free-haul distance, i.e.

$$ef - cd$$

i.e. Scaled value = 640 – 300 = 340 ft.

(d) The overhaul of material at acc_1

= volume (cc_1) × overhaul distance ($ef - cd$)
= 2750 ft^3 × 340 ft
= 346·3 station yards

(2) *By taking moments*

With reference to the mass-haul curve and the tabulated volumes, moments are taken at *a* to find the centroid of the area acc_1.

At *a* the chainage is scaled as 120 ft.

Chainage	Volume (ft^3)	Distance (ft)	Product ($V \times D$)
(*a*) 120 – 200	1050 – 400 = 650	$\frac{1}{2}(200-120)$ = 40	26 000
200 – 300	2730 – 1050 = 1680	$\frac{1}{2}(300-200)+80 = 130$	218 400
300 – 350 (*c*)	3150 – 2730 = 420	$\frac{1}{2}(350-300)+180 = 205$	86 100
	Σv = 2750		ΣP 330 500

Thus the distance from *a* to the centroid

$$= \frac{330\,500}{2750} = 120{\cdot}2 \text{ ft}$$

$\therefore$ Chainage of the centroid $= 120 + 120{\cdot}2 = 240{\cdot}2$ ft.

Taking moments at *d*, chainage 650 ft:

Chainage	Volume (ft^3)	Distance (ft)	Product ($V \times D$)
(*d*) 650 – 700	3350 – 2990 = 160	$\frac{1}{2}(700-650)$ = 25	4000
700 – 800	2990 – 2390 = 600	$\frac{1}{2}(800-700)+50 = 100$	60 000
800 – 900	2390 – 1610 = 780	$\frac{1}{2}(900-800)+150 = 200$	156 000
900 – 1000	1610 – 920 = 690	$\frac{1}{2}(1000-900)+250 = 300$	207 000
1000 – 1100	920 – 520 = 400	$\frac{1}{2}(1100-1000)+350 = 400$	160 000
1100 – 1200	520 – 400 = 120	$\frac{1}{2}(1200-1100)+450 = 500$	60 000
	Σv = 2750		ΣP 647 000

Thus the distance from *d* to the centroid

$$= \frac{647\,000}{2750} = 235{\cdot}3 \text{ ft}$$

$\therefore$ Chainage of the centroid $= 650 + 235{\cdot}3 = 885{\cdot}3$ ft

Average haul distance $= 885{\cdot}3 - 240{\cdot}2 = 645{\cdot}1$

Length of overhaul $= 645{\cdot}1 - 300 = 345{\cdot}1$

$$\text{Overhaul} = \frac{2750 \times 345{\cdot}1}{2700} = \underline{351{\cdot}5} \text{ station yards.}$$

(3) *Planimetric Method*

Distance to centroid = Haul/volume

$$= \frac{\text{Area} \times \text{horizontal scale} \times \text{vertical scale}}{\text{volume ordinate}}$$

From area acc_1

Area scaled from mass-haul curve $= 0{\cdot}9375 \text{ in}^2$

Horizontal scale = 200 ft to 1 in.

$$\text{Vertical scale} = 1600 \text{ ft}^3 \text{ to 1 in.}$$

$$\therefore \quad \text{Haul} = 0{\cdot}9375 \times 200 \times 1600 = 300\,000$$

$$\text{Volume (ordinate } cc_1) = 2750$$

$$\text{Distance to centroid} = 300\,000/2750 = 109{\cdot}1 \text{ ft}$$

$$\text{Chainage of centroid} = 350 - 109{\cdot}1 = \underline{240{\cdot}9 \text{ ft}}$$

For area dbd_1

$$\text{Area scaled} = 1{\cdot}9688 \text{ in}^2$$

$$\therefore \quad \text{Haul} = 1{\cdot}9688 \times 320\,000 = 630\,016$$

$$\text{Volume} = (\text{ordinate } dd_1) = 2750$$

$$\text{Distance to centroid} = 229{\cdot}1 \text{ ft}$$

$$\text{Chainage of centroid} = 650 + 229{\cdot}1 = \underline{879{\cdot}1 \text{ ft}}$$

$$\text{Average haul distance} = 879{\cdot}1 - 240{\cdot}9 = 638{\cdot}2 \text{ ft}$$

$$\text{Overhaul distance} = 638{\cdot}2 - 300 = \underline{338{\cdot}2 \text{ ft}}$$

$$\therefore \quad \text{Overhaul} = 338{\cdot}2 \times 2750 = \underline{344{\cdot}5} \text{ station yards.}$$

N.B. Instead of the above calculation the overhaul can be obtained direct as the sum of the two mass-haul curve areas acc_1 and dbd_1.

$$\text{Area } acc_1 = \frac{300\,000}{2700} \text{ station yd}$$

$$\text{Area } dbd_1 = \frac{630\,016}{2700} \text{ station yd}$$

$$\text{Total area} = \text{overhaul} = \frac{930\,016}{2700} = \underline{344{\cdot}5} \text{ station yards}$$

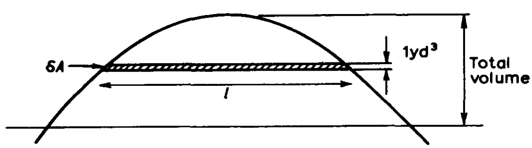


Fig. 10.46

Proof

Take any area cut off by a balancing line, Fig. 10.46.

Let a small increment of area δA = (say) 1 yd^3 and length of

haul be l.

$$\delta A = 1 \text{ yd}^3 \times l/100 \text{ station yd}$$

$$\therefore \quad A = n \times 1 \text{ yd}^3 \times \frac{\Sigma l}{n}$$

$$= \text{Total volume} \times \text{average haul distance.}$$

$$\therefore \quad \underline{\text{Area}} = \underline{\text{Total Haul}}$$

Exercises 10(c) (Earthwork volumes)

15. Calculate the cubic contents, using the prismoidal formula of the length of embankment of which the cross-sectional areas at 50 ft intervals are as follows:

Distance (ft)	0	50	100	150	200	250	300
Area (ft^2)	110	425	640	726	1590	1790	2600

Make a similar calculation using the trapezoidal method and explain why the results differ.

(I.C.E. Ans. 11 688 yd^3; 12 085 yd^3)

16. The following notes were taken from the page of a level book:

Reduced level	Remarks
45·85	At peg 10
44·10	30 ft to right at peg 10
44·75	30 ft to left at peg 10
46·35	At peg 11
42·85	30 ft to right at peg 11
48·35	30 ft to left at peg 11
46·85	At peg 12

Draw cross-sections to a scale of 1 in. = 10 ft at pegs 10 and 11, which are 100 ft apart on the centre line of a proposed branch railway, and thereafter calculate the volume of material excavated between the two pegs in forming the railway cutting. The width at formation level is 15 ft, and the sides of the cutting slope at 1½ horizontal to 1 vertical. The formation level of each peg is 30·5 ft.

(M.Q.B./M Ans. 2116 yd^3)

17. A level cutting is made on ground having a uniform cross-slope of 1 in 8. The formation width is 32 ft and the sides slope at 1 vertical to 1¾ horizontal. At 3 sections, spaced 66 ft apart, the depths to the centre line are 34, 28 and 20 ft.

Calculate (a) the side widths of each section (b) the volume of the cutting.

(N.R.C.T. Ans. 62·0; 96·6 ft; 53·3; 83·2 ft; 41·8, 65·2 ft; 11 600 yd^3)

18. Calculate the volume in cubic feet contained between three successive sections of a railway cutting, 50 ft apart. The width of formation is 10 ft, the sides slope 1 vertical to 2 horizontal and the heights at the top of the slopes in feet above formation level are as follows:

	Left	Centre	Right
1st Cross-section	13·6	12·0	14·0
2nd Cross-section	16·0	15·5	17·8
3rd Cross-section	18·3	16·0	16·0

(N.R.C.T. Ans. 63 670 ft^3)

19. The formation of a straight road was to be 40 ft wide with side slopes 1 vertically to 2½ horizontally in cutting. At a certain cross-section, the depth of excavation on the centre line was 10 ft and the cross-fall of the natural ground at right angles to the centre line was 1 vertically to 8 horizontally. At the next cross-section, 100 ft away, the depth on the centre line was 20 ft and the cross-fall similarly 1 in 10.

Assuming that the top edge of each slope was a straight line, find the volume of excavation between the two sections by the prismoidal formula and find the percentage error that would be made by using the trapezoidal formula.

(L.U. Ans. 11 853 ft^3 12·5 %)

20. A straight embankment is made on ground having a uniform cross-slope of 1 in 8. The formation width of the embankment is 30 ft and the side slopes are 1 vertical to 1½ horizontal. At three sections spaced 50 ft apart the heights of the bank at the centre of the formation level are 10, 15 and 18 ft. Calculate the volume of the embankment and tabulate data required in the field for setting out purposes.

(L.U. Ans. 2980 yd^3)

21. Cross-sections at 100 ft intervals along the centre line of a proposed straight cutting are levelled at 20 ft intervals from −60 ft to +60 ft and the following information obtained:

Distances (ft)	−60	−40	−20	0	+20	+40	+60
0	4·0	1·0	0·0	0·0	0·0	1·0	2·8
100	12·9	8·6	5·0	3·0	2·0	3·0	6·0
200	17·5	14·1	10·9	8·0	6·0	6·0	9·6
300	21·8	17·7	14·4	11·3	9·7	9·7	11·0
400	25·0	21·2	18·0	15·2	12·8	12·0	13·2

(Tabulated figures are levels in feet relative to local datum). The formation level is zero feet, its breadth 20 ft, and the side slopes 1 vertical to 2 horizontal. Find the volume of excavation in cubic yards over the section given.

(L.U. Ans. 5100 yd^3)

22. A minor road with a formation width of 15 ft is to be made up a plane slope of 1 in 10 so that it rises at 1 in 40. There is to be no cut or fill on the centre line, and the side slopes are to be 1 vertical to 2 horizontal. Calculate the volume of excavation per 100 ft of road. Derive formulae for calculating the side-widths and heights and the cross-sectional area of a 'two-level' section.

(N.R.C.T. Ans. 25 yd^3/100 ft)

23. The uniform slope of a hillside (which may be treated as a plane surface) was 1 vertically to 4 horizontally. On this surface a straight centre line AB was laid out with a uniform slope of 1 vertically to 9 horizontally. With AB as the centre line a path with a formation width of 10 ft was constructed with side slopes of 1 vertically to 2 horizontally. If the path was 500 ft in length and there was no cut or fill on the centre line, calculate the quantity of cutting in cubic feet.

(I.C.E Ans. 2530 ft^3)

24. The central heights of the ground above formation at three sections 100 ft apart are 10, 12 and 15 ft and the cross-falls at these sections 1 in 30, 1 in 40 and 1 in 20 (vertically to horizontally). If the formation width is 40 ft and the side slopes 1 vertically in 2 horizontally, calculate the volume of excavation in the 200 ft length

(a) if the centre line is straight,

(b) if the centre line is an arc of 400 ft radius.

(L.U. Ans. 158 270 ft^3; 158 270 ± 1068 ft^3)

25. The centre line of a highway cutting is on a curve of 400 ft radius, the original surface of the ground being approximately level. The cutting is to be widened by increasing the formation width from 20 to 30 ft, the excavation to be entirely on the inside of the curve and to retain the existing side slopes of 1½ horizontal to 1 vertical. If the depth of formation increases uniformly from 8 ft at ch. 600 to 17 ft at ch. 900, calculate the volume of earth to be removed in this 300 ft length.

(L.U. Ans. 1302 yd^3)

26. The contoured plan of a lake is planimetered and the following values obtained for the areas enclosed by the given underwater contours:

Contour (ft O.D.)	305	300	295	290	285
Area (ft^2)	38 500	34 700	26 200	7800	4900

The surface area of the water in the lake is 40 200 ft^2. The top water level and the lowest point in the lake are at 308·6 and 280·3 ft O.D. respectively. Find the quantity of water in the lake in millions of gallons.

(L.U. Ans. 3·73 m. gal)

27. The areas of ground within contour lines at the site of a reservoir are as follows:

Contour in ft above datum	Area (ft^2)
400	505 602
395	442 104
390	301 635
385	232 203
380	94 056
375	56 821
370	34 107
365	15 834
360	472

Taking 360 ft O.D. as the level of the bottom of the reservoir and 400 ft O.D. as the water level, estimate the quantity of water in gallons contained in the reservoir (assume 6·24 gal per ft^3).

(Ans. 45 276 500 gal)

28. Describe three methods of carrying out the field work for obtaining the volumes of earthworks.

Explain the conditions under which the 'end area' and 'prismoidal rule' methods of calculating volumes are accurate, and explain also the use of the 'prismoidal correction'.

The areas within the contour lines at the site of a reservoir are as follows:

Contour (ft)	Area (ft^2)
400	5 120 000
395	4 642 000
390	4 060 000
385	3 184 000
380	2 356 000
375	1 765 000
370	900 000
365	106 000
360	11 000

The level of the bottom of the reservoir is 360 ft. Calculate (a) the volume of water in the reservoir when the water level is 400 ft using the end area method, (b) the volume of water in the reservoir using the prismoidal formula (every second area may be taken as a mid-area), and (c) the water level when the reservoir contains 300 000 000 gallons.

(L.U. Ans. (a) 97·8925 m. ft^3 (b) 97·585 m. ft^3 388 ft)

29. A square level area $ABCD$ (in clockwise order) of 100 ft side is to be formed in a hillside which is considered to have a plane surface with a maximum gradient of 3 (horizontally) to 1 (vertically).

E is a point which bisects the side AD, and the area ABE is to be formed by excavation into the hillside, whilst the area $BCDE$ is to be formed on fill. The side slopes in both excavation and fill are to be 1 to 1, and adjacent side slopes meet in a straight line.

By means of contours at 2 ft intervals, plot the plan of the earthworks on graph paper to a scale of 50 ft to 1 inch. Hence compute the volume of excavation.

(I.C.E. Ans. $V \simeq 880$ yd^3)

30. A road having a formation width of 40 ft with side slopes of 1 in 1 is to be constructed. Details of two cross-sections of a cutting are as follows:

Chainage (ft)	Depth of Cutting on Centre Line (ft)	Side Slope Limits (ft) Left	Side Slope Limits (ft) Right
500	10·2	25·2	33·7
600	6·0	22·0	28·5

Assuming that these cross-sections are bounded by straight lines and that the undisturbed ground varies uniformly between them, compute the volume of excavation allowing for prismoidal excess.

If instead of being straight, the plan of the centre line had been a circular curve of radius R with the centre of curvature on the right, how would this have been taken into account in the foregoing calculations? Quote any formula that would have been used.

(I.C.E. Ans. 1370 yd^3)

31. A section of a proposed road is to run through a cutting from chainage 500 to 900, the formation level falling at 1 in 200 from chainage 500. The formation width is to be 30 ft and the side slopes are to be 1 vertical to 2 horizontal. The original ground surface is inclined uniformly at right-angles to the centre line at an inclination of 1 in 10.

With the information given below, calculate the volume of excavation in cubic yards, using the prismoidal formula.

Chainage	Formation Level	Ground Level at Centre Line
500	44·25 ft	51·11
600		50·82
700		50·93
800		51·09
900		50·77

(I.C.E. Ans. 5474 yd^3)

32. A road of 40 ft formation width is to be constructed with side slopes of 1 (vertical) to 1½ (horizontal) in excavation and 1 (vertical) to 2 (horizontal) in fill. Further details of two cross-sections are given

below where the cross fall of the undisturbed ground is 1 (vertical) to r (horizontal).

Chainage (ft)	Ground level on Centre-line (ft above datum)	Formation Level (ft above datum)	r
400	171·6	166·6	4
500	170·2	168·0	6

Assuming the road is straight between these two sections, compute the volumes of excavation and fill in 100 ft length neglecting prismoidal excess.

(I.C.E. Ans. 819 yd^3 cut, 11 yd^3 fill)

33. On a 1000 ft length of new road the earthwork volumes between sections at 100 ft intervals are as follows, the excavation being taken as positive and filling as negative:

Section No	0	1	2	3	4	5	6	7	8	9	10
Vol. (1000 yd^3)	3·7	9·1	15·0	13·9	6·4	1·4	−5·6	−19·4	−18·9	−5·6	

Draw the mass-haul curve and find

(i) the volume to be moved under the terms of the free-haul limit of 300 ft,

(ii) the volume to be moved in addition to (i),

(iii) the number of station-yards under (ii) where 1 station-yard equals 1 cubic yard moved 100 ft,

(iv) the average length of haul under (ii).

(L.U. Ans. 7500 yd^3; 42 000 yd^3; 23 500 station yds; 560 ft)

34. The following figures show the excavation (+) and filling (−) in cubic yards between successive stations 100 ft apart in a proposed road.

0	1	2	3	4	5	6
	+1500	+1100	+500	+100	−100	−1000
	7	8	9	10	11	12
	−2200	−2500	−1600	−400	+1800	+2800

State which of the following tenders is the lower and calculate the total mass haul in the 1200 ft length:

(a) Excavate, cart and fill at 9/6 per yd^3.

(b) Excavate, cart and fill at 9/- per yd^3, with a free-haul limit of 400 ft, plus 1/- per station yard for hauling in excess of 400 ft. (1 station yard = 100 ft × 1 yd^3).

(L.U. Ans. 28 500 station yards; (a) £3700, (b) £3750)

35. Volumes in yd^3 of excavation (positive) and fill (negative) between successive sections 100 ft apart on a 1300 ft length of a proposed railway are given in the following table:

Section	0	1	2	3	4	5	6
Volume		-1000	-2200	-1600	-500	+200	+1300

7	8	9	10	11	12	13
+2100	+1800	+1100	+300	-400	-1200	-1900

Draw a mass haul curve for this length. If earth may be borrowed at either end, which alternative would give the least haul? Show on the diagram the forward and backward free-hauls if the free-haul limit is 500 ft, and give these volumes.

(L.U. Ans. Borrow at 0 end, 1150 ft; 2900 yd^3; 2400 yd^3)

36. The volumes in yd^3 between successive sections 100 ft apart on a 900 ft length of a proposed road are given below; excavation is shown positive and fill negative

Section	0	1	2	3	4	5	6	7	8	9
Volume		+1700	-100	-3200	-3400	-1400	+100	+2600	+4600	+1100

Determine the maximum haul distance when earth may be wasted only at the 900 ft chainage end.

Show and evaluate on your diagram the overhaul if the free-haul limit is 300 ft.

(L.U. Ans. 510 ft; 4950 station yards)

Bibliography

BANNISTER, A. and RAYMOND, S., *Surveying* (Pitman)
DAVIES, R.E. FOOTE, F.S., and KELLY, J.W., *Surveying Theory and Practice* (McGraw Hill)
CLARK, D., *Plane and Geodetic Surveying, Vol.*1 (Constable)
THOMAS, W.N., *Surveying* (Edward Arnold)
CURTIN, W. and LANE, R.F., *Concise Practical Surveying* (English Universities Press)
RITTER-PAQUETTE, *Highway Engineering*, 2nd Ed. (Ronald Press)
SHARMA, R.C. and SHARMA, S.K. *Principles and Practice of Highway Engineering* (Asia Publishing House)
MINISTRY OF TECHNOLOGY, *Changing to the Metric System* (H.M.S.O.)

11 CIRCULAR CURVES

11.1 Definition

The *curve* can be defined by (a) the radius R or (b) the degree of the curve D. D can be expressed as the angle at the centre of the curve subtended by (i) a chord of 100 ft or (ii) an arc of 100 ft (the former is more generally adopted).

In Fig. 11.1,

$$\sin \tfrac{1}{2}D = \frac{\tfrac{1}{2} \times 100}{R} = \frac{50}{R}$$

or $$R = \frac{50}{\sin \tfrac{1}{2}D} \qquad (11.1)$$

If D is small,

$$\sin \tfrac{1}{2}D = \tfrac{1}{2}D \text{ radians}$$

$$R = \frac{206\,265 \times 50}{\tfrac{1}{2}D^\circ \times 3600} \simeq \underline{\frac{5730}{D^\circ}} \qquad (11.2)$$

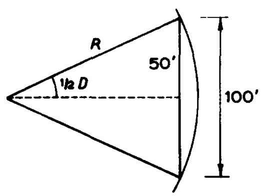


Fig. 11.1

11.2 Through Chainage

Through chainage represents the length of road or rail from some terminus (it does not necessarily imply Gunter's chain – it may be the engineer's chain). Pegs are placed at 'stations' (frequently at 100 ft intervals) and a point on the construction can be defined by reference to the 'station'.

When a curve is introduced, Fig. 11.2, the tangent point T_1 is said to be of chainage 46 + 25, i.e. 4625 ft from the origin. If the length of the curve was 400 ft, the chainage of T_2 would be expressed as (46 + 25) + (4 + 00), i.e. 50 + 25.

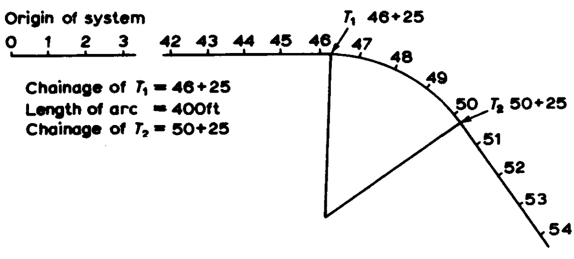


Fig. 11.2 Through chainage

11.3 Length of Curve (L)

$$L = 2\pi R \times \frac{\theta}{360} \tag{11.3}$$

$$\text{or} \quad L = R\,.\,\theta_{\text{rad}} \tag{11.4}$$

The second formula is better.

Example 11.1 If $\theta = 30^\circ 26'$ and $R = 100\,\text{ft}$,

$$\text{Length of arc} \quad L = \frac{2 \times 3{\cdot}142 \times 100 \times 30\frac{26}{60}}{360} = \underline{53{\cdot}12\,\text{ft}}$$

$$\text{or} \quad L = 100 \times 30^\circ 26'_{\text{rad}}$$

$$= 100 \times 0{\cdot}531\,16 = \underline{53{\cdot}12\,\text{ft}}$$

11.4 Geometry of the Curve

In Fig. 11.3, T_1 and T_2 are tangent points, I is the intersection point, ϕ is the deflection angle at intersection point, $IT_1 = IT_2$ = tangent length = $R \tan \frac{1}{2}\phi$.

$$T_1 T_2 = \text{Long chord} = 2\,.\,TX$$

$$= 2R \sin \tfrac{1}{2}\phi \tag{11.5}$$

$$T_1 A = \text{chord } c = 2R \sin \alpha \tag{11.6}$$

(if $c < R/20$ and α is small, arc $\simeq$ chord. i.e. $\sin \alpha = \alpha_{\text{rad}}$)

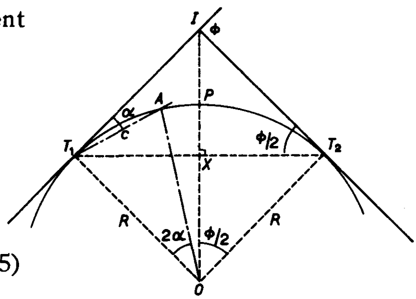


Fig. 11.3 Geometry of the curve

$$c = 2R\,\alpha_{\text{rad}} \tag{11.7}$$

$$\text{Deflection angle} \quad \alpha_{\text{rad}} = \frac{c}{2R} \tag{11.8}$$

$$\alpha_{\text{sec}} = \frac{206\,265\,c}{2R} \tag{11.9}$$

$$\alpha_{\text{mins}} = \frac{206\,265\,c}{2R \times 60} \tag{11.10}$$

$$= \frac{1718{\cdot}8\,c}{R} \tag{11.11}$$

$$IO = R \sec \tfrac{1}{2}\phi \tag{11.12}$$

$$IP = IO - PO = R \sec \tfrac{1}{2}\phi - R = R(\sec \tfrac{1}{2}\phi - 1) \tag{11.13}$$

$$PX = PO - XO = R - R\cos \tfrac{1}{2}\phi = R(1 - \cos \tfrac{1}{2}\phi) \tag{11.14}$$

$$= R \text{ versine } \tfrac{1}{2}\phi. \tag{11.15}$$

11.5 Special Problems

11.51 To pass a curve tangential to three given straights (Fig. 11.4)

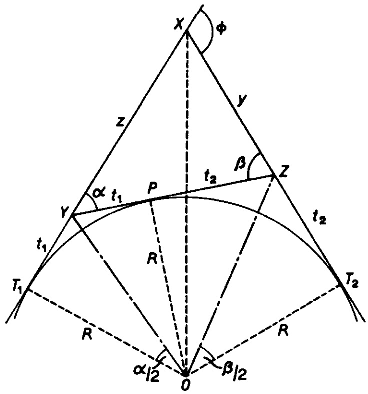


Fig. 11.4

$$YZ = t_1 + t_2 = x \qquad (1)$$
$$T_1X = XT_2$$

i.e. $t_1 + z = t_2 + y$

$$z - y = t_2 - t_1 \qquad (2)$$

(1) + (2) $\quad x + z - y = 2t_2$

i.e. $t_2 = \tfrac{1}{2}(x + z - y).$

$$\begin{aligned} R &= T_1X \cot \tfrac{1}{2}\phi \\ &= (t_1 + z) \cot \tfrac{1}{2}\phi = (t_2 + y) \cot \tfrac{1}{2}\phi \\ &= \left\{ \tfrac{1}{2}(x + z - y) + y \right\} \cot \tfrac{1}{2}\phi \\ &= \tfrac{1}{2}(x + y + z) \cot \tfrac{1}{2}\phi \end{aligned}$$

$\underline{R = s \cot \tfrac{1}{2}\phi}$ where $s = \tfrac{1}{2}$ perimeter of $\Delta\ XYZ$ (11.16)

Alternative solutions:

(a)
$$\text{Area } XOY = \tfrac{1}{2} Rz$$
$$XOZ = \tfrac{1}{2} Ry$$
$$YOZ = \tfrac{1}{2} Rx$$
$$XYZ = \text{areas } (XOY + XOZ - YOZ)$$
$$= \tfrac{1}{2} R(y + z - x)$$
$$= \tfrac{1}{2} R(x + y + z) - \tfrac{1}{2} R(x + x)$$
$$= Rs - Rx$$
$$= R(s - x)$$
$$\therefore \quad R = \frac{\text{area } \Delta\ XYZ}{s - x}. \qquad (11.17)$$

(b) If angles α and β are known or computed,
$$YP = R \tan \alpha/2$$
$$PZ = R \tan \beta/2$$
$$\therefore \quad YZ = YP + PZ = R(\tan \alpha/2 + \tan \beta/2)$$
$$\therefore \quad R = \frac{YZ}{\tan \alpha/2 + \tan \beta/2} \qquad (11.18)$$

Example 11.2 The co-ordinates of three stations A, B and C are as follows:-

A	E 1263·13 m	N 1573·12 m
B	E 923·47 m	N 587·45 m
C	E 1639·28 m	N 722·87 m.

The lines AB and AC are to be produced and a curve set out so that the curve will be tangential to AB, BC and AC. Calculate the radius of the curve.

In Fig. 11.5,

$$\text{Bearing } AB = \tan^{-1} \frac{923{\cdot}47 - 1263{\cdot}13}{587{\cdot}45 - 1573{\cdot}12}$$
$$= \tan^{-1} \frac{-339{\cdot}66}{-985{\cdot}67}$$
$$= \text{S } 19°\ 00'\ 50''\text{W}$$
$$= \underline{199°\ 00'\ 50''}.$$

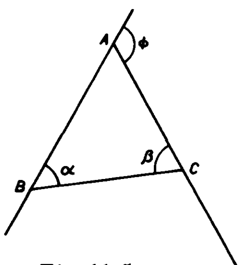


Fig. 11.5

$$\text{Length} \quad AB = 985{\cdot}67 \sec 19^\circ\, 00'\, 50'' = \underline{1042{\cdot}55}$$

$$\text{Bearing} \quad AC = \tan^{-1}\frac{1639{\cdot}28 - 1263{\cdot}13}{722{\cdot}87 - 1573{\cdot}12} = \tan^{-1}\frac{376{\cdot}15}{-850{\cdot}25}$$

$$= \text{S}\,23^\circ\, 51'\, 52''\text{E} = \underline{156^\circ\, 08'\, 08''}.$$

$$\text{Length} \quad AC = 850{\cdot}25 \sec 23^\circ\, 51'\, 52'' = \underline{929{\cdot}74}$$

$$\text{Bearing} \quad BC = \tan^{-1}\frac{1639{\cdot}28 - 923{\cdot}47}{722{\cdot}87 - 587{\cdot}45} = \tan^{-1}\frac{715{\cdot}81}{135{\cdot}42}$$

$$= \text{N}\,79^\circ\, 17'\, 14''\text{E} = \underline{079^\circ\, 17'\, 14''}$$

$$\text{Length} \quad BC = 135{\cdot}42 \sec 79^\circ\, 17'\, 14'' = \underline{728{\cdot}51}$$

$$\phi = 180 - (19^\circ\, 00'\, 50'' + 23^\circ\, 51'\, 52'') = 137^\circ\, 07'\, 18''$$

To find the radius

Using Eq. (11.16),

$$R = \tfrac{1}{2}(AB + BC + AC)\cot\tfrac{1}{2}\phi$$

$$= \tfrac{1}{2}(1042{\cdot}55 + 728{\cdot}51 + 929{\cdot}74)\cot 68^\circ\, 33'\, 39''$$

$$= 1350{\cdot}40 \cot 68^\circ\, 33'\, 39'' = \underline{530{\cdot}2\text{ m.}}$$

Alternatively, by Eq. (11.17),

$$R = \frac{\sqrt{\{s(s-a)(s-b)(s-c)\}}}{s-a}$$

$$= \frac{\sqrt{\{1350{\cdot}40(1350{\cdot}40 - 728{\cdot}51)(1350{\cdot}40 - 929{\cdot}74)(1350{\cdot}40 - 1042{\cdot}54)\}}}{1350{\cdot}40 - 728{\cdot}51}$$

$$= \underline{530{\cdot}2\text{ m.}}$$

Alternatively, by Eq. (11.18),

$$\alpha = 079^\circ\, 17'\, 14'' - 019^\circ\, 00'\, 50'' = 60^\circ\, 16'\, 24''$$

$$\beta = 336^\circ\, 08'\, 08'' - 259^\circ\, 17'\, 14'' = \underline{76^\circ\, 50'\, 54''}$$

$$\textit{check} \quad \phi = 137^\circ\, 07'\, 18''$$

$$R = \frac{728{\cdot}51}{\tan 30^\circ\, 08'\, 12'' + \tan 38^\circ\, 25'\, 27''} = \underline{530{\cdot}2\text{ m.}}$$

11.52 To pass a curve through three points (Fig. 11.6)

In Fig. 11.6,

$$\text{angle } COA = 2\theta$$

$$\text{angle } XOC = 180 - \theta$$

$$XC = \tfrac{1}{2}AC = R\sin(180 - \theta)$$

$$\therefore \quad \frac{AC}{\sin\theta} = 2R,$$

i.e. $$R = \frac{\frac{1}{2}AC}{\sin\theta} \qquad (11.19)$$

Similarly, the full sine rule is written

$$\frac{AB}{\sin C} = \frac{BC}{\sin A} = \frac{AC}{\sin B} = 2R \qquad (11.20)$$

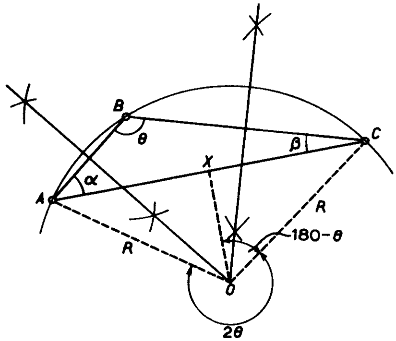


Fig. 11.6

Example 11.3 The co-ordinates of two points B and C with respect to A are

B	536·23 m N	449·95 m E
C	692·34 m N	1336·28 m E.

Calculate the radius of the circular curve passing through the three points.

$$\text{Bearing} \quad AB = \tan^{-1}\frac{449{\cdot}95}{536{\cdot}23} = \text{N}\,40°\text{E}$$

$$\text{Bearing} \quad BC = \tan^{-1}\frac{886{\cdot}33}{156{\cdot}11} = \text{N}\,80°\text{E}$$

$\therefore$ angle $ABC = \theta = 180 + 40 - 80 = 140°$.

$$\text{Length} \quad AC = \sqrt{(692{\cdot}34^2 + 1336{\cdot}28^2)} = 1505\text{ m}$$

$$R = \frac{1505}{2\sin 140°} = \frac{752{\cdot}5}{\sin 40} = \underline{1170{\cdot}7\text{ m}}.$$

Example 11.4 In order to find the radius of an existing road curve, three suitable points A, B, and C were selected on the centre line. The instrument was set up at B and the following tacheometrical readings taken on A and C, the telescope being horizontal and the staff held vertical in each case.

Staff at	Horizontal angle	Collimation	Stadia
A	0° 00′	4·03	5·39/2·67
C	195° 34′	6·42	8·04/4·80.

If the instrument had a constant multiplier of 100 and an additive constant of zero, calculate the radius of a circular arc ABC.

If the trunnion axis was 5·12 ft above the road at B, find the gradients of AB and BC. (L.U.)

In Fig. 11.7,

$AB = 100(5{\cdot}39 - 2{\cdot}67) = 272\text{ ft}$

$BC = 100(8{\cdot}04 - 4{\cdot}80) = 324\text{ ft.}$

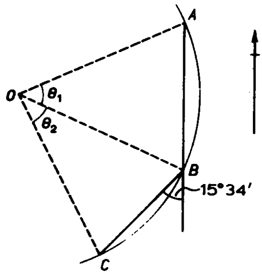


Fig. 11.7

In triangle ABC,

$$\tan\frac{A-B}{2} = \frac{324-272}{324+272}\tan\frac{195°34' - 180}{2}$$

$$= \frac{52\tan 7°47'}{596}$$

$$\frac{A-B}{2} = 0°\,41'$$

$$\frac{A+B}{2} = 7°\,47'$$

$$A = 8°\,28'$$

$$B = 7°\,06'$$

$$\frac{BC}{\sin A} = 2R$$

$$\therefore \quad R = \frac{324}{2\sin 8°\,28'} = \underline{1100{\cdot}3\text{ ft.}}$$

Difference in height $A - B$, $5{\cdot}12 - 4{\cdot}03 = 1{\cdot}09$ ft.
Gradient AB, 1·09 in 272 = 1 in 240.
Difference in height $B - C$, $6{\cdot}42 - 5{\cdot}12 = 1{\cdot}30$ ft.
Gradient BC, 1·30 in 324 = 1 in 249.

If the gradients are along the arc,

$$\theta_1 = 2\sin^{-1}\frac{324}{2 \times 1100{\cdot}3} = 16°\,56'$$

Length of arc $AB = 1100{\cdot}3 \times 16°\,56'_{\text{rad}} = 272{\cdot}7$ ft

$$\theta_2 = (2 \times 15°\,34') - 16°\,56' = 14°\,12'$$

Length of arc $BC = 1100{\cdot}3 \times 14°\,12'_{\text{rad}} = 352{\cdot}2$ ft.

Gradients along the arc are 1 in 251; 1 in 250.

Exercises 11 (a)

1. It is required to range a simple curve which will be tangential to three straight lines YX, PQ, and XZ, where PQ is a straight, joining the two intersecting lines YX and XZ. Angles $YPQ = 134° 50'$; $YXZ = 72° 30'$; $PQZ = 117° 40''$ and the distance $XP = 5{\cdot}75$ chains.

Compute the tangent distance from X along the straight YX and the radius of curvature.

(I.C.E. Ans. 8·273 chains; 6·066 chains)

2. A circular road has to be laid out so that it shall be tangent to each of the lines DA, AB, and BC.

Given the co-ordinates and bearings as follows, calculate the radius of the circle.

	Latitude	Departure
A	− 29·34 m	−128·76 m
B	−177·97 m	− 58·39 m

Bearing DA 114° 58′ 10″
CB 054° 24′ 10″

(Ans. 137·48 m)

3. A circular curve is tangential to three straight lines AB, BC, and CD, the whole circle bearings of which are 38°, 72° and 114° respectively. The length of BC is 630 ft. Find the radius and length of the curve and the distances required to locate the tangent points.

Also tabulate the data necessary for setting out the curve from the tangent point on BC with 100 ft chords and a theodolite reading to 30″ with clockwise graduations.

(L.U. Ans. $R = 913{\cdot}6$ ft; 1211·9 ft; 279·3 and 350·7 ft)

4. The co-ordinates of two stations Y and Z in relation to a station X are as follows:

	E (m)	N (m)
Y	215	576
Z	800	−750

Find by construction the radius of a circular curve passing through each of the stations X, Y, and Z.

(Ans. 790 m)

5. Three points A, B, and C lie on the centre line of an existing mine roadway. A theodolite is set up at B and the following observations were taken on to a vertical staff.

Staff at	Horizontal circle	Vertical circle	Staff readings Stadia	Staff readings Collimation
A	002° 10′ 20″	+ 2° 10′	6·83/4·43	5·63
C	135° 24′ 40″	− 1° 24′	7·46/4·12	5·79

If the multiplying constant is 100 and the additive constant zero, calculate:

(a) the radius of the circular curve which will pass through A, B and C,

(b) the gradient of the track laid from A to C if the instrument height is 5·16 ft.

(R.I.C.S./M Ans. 362·2 ft; 1 in 33·93)

11.53 To pass a curve through a given point P

(a) *Given the co-ordinates of P relative to I, i.e. IA and AP* (Fig. 11.8)

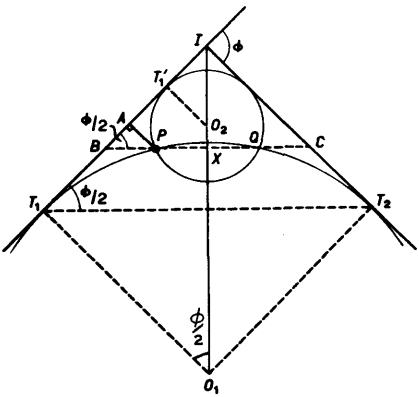


Fig. 11.8

In Fig. 11.8,

Assume BC passing through P is parallel to T_1T_2.

$$AB = AP \cot \tfrac{1}{2}\phi$$

$$BP = AP \operatorname{cosec} \tfrac{1}{2}\phi = QC$$

$$BX = (AB + AI) \cos \tfrac{1}{2}\phi$$

$$PX = BX - BP$$

$$BQ = BX + PX$$

$$T_1B = \sqrt{(BP \times BQ)} \qquad \text{(intersecting tangent and secant)}$$

$$T_1I = BI \pm T_1B \quad (\text{i.e. } T_1^1B = T_1B)$$

$$R = T_1I \cot \tfrac{1}{2}\phi \quad \text{or} \quad T_1^1I \cot \tfrac{1}{2}\phi$$

$$= \underline{(AB + AI \pm T_1B) \cot \tfrac{1}{2}\phi} \qquad (11.21)$$

(b) *Given polar co-ordinates of point P from intersection I* (Fig.11.9)

In Fig.11.9,

$$\alpha = 90 - (\frac{\phi}{2} + \theta)$$

$$OI = R \sec \tfrac{1}{2}\phi$$

$$\frac{OI}{OP} = \frac{R \sec \frac{1}{2}\phi}{R},$$

also by sine rule

$$= \frac{\sin(\alpha + \beta)}{\sin \alpha}$$

$$\therefore \; \underline{\sin(\alpha + \beta) = \sin \alpha \sec \tfrac{1}{2}\phi}$$

$$[= \sin 180 - (\alpha + \beta)] \qquad (11.22)$$

Thus there are 2 values for β.

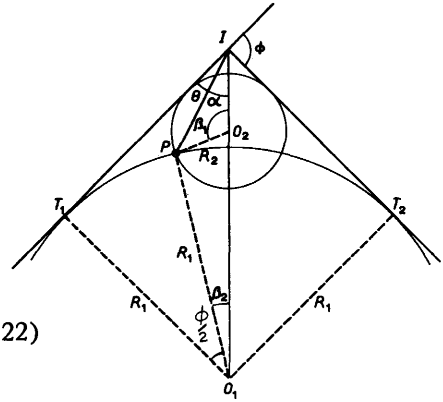


Fig. 11.9

$$\frac{R}{\sin \alpha} = \frac{IP}{\sin \beta}$$

$$\underline{R = IP \sin \alpha \operatorname{cosec} \beta} \qquad \text{(2 values)} \qquad (11.23)$$

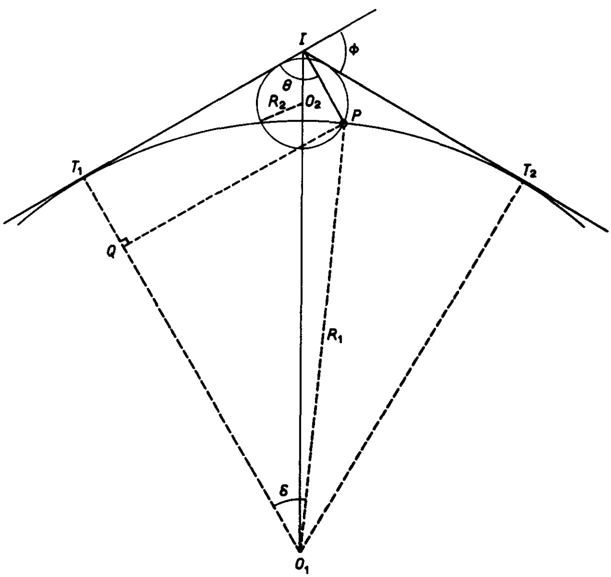


Fig. 11.10

If $\alpha = 0$, in Fig. 11.9, i.e. P lies on line IO,

$$OI = R \sec \tfrac{1}{2}\phi = IP \pm R$$

$$R(\sec \tfrac{1}{2}\phi \mp 1) = IP$$

$$R = \frac{IP}{\sec \frac{1}{2}\phi \pm 1} \qquad (11.24)$$

If $\theta = 90°$, in Fig. 11.10,

$$T_1O = R$$

$$QP = T_1I = R \sin\delta = R \tan \tfrac{1}{2}\phi$$

$$\therefore \quad \sin\delta = \tan \tfrac{1}{2}\phi$$

$$T_1Q = IP = R - R\cos\delta$$

$$= R(1 - \cos\delta)$$

$$\therefore \quad R = \frac{IP}{\text{vers}\,\delta} \qquad (11.25)$$

As δ has 2 values there are two values of R.

Example 11.5 A circular railway curve has to be set out to pass through a point which is 40 m from the intersection of the straights and equidistant from the tangent points.

The straights are deflected through 46° 40′. Calculate the radius of the curve and the tangent length.

Taking the maximum radius only, in Fig. 11.11,

$$OI = OP + PI$$

$$= R + x = R \sec \tfrac{1}{2}\phi$$

$$\therefore \quad R(1 - \sec \tfrac{1}{2}\phi) = -x$$

i.e., Eq. (11.24),

$$R = \frac{x}{\sec \frac{1}{2}\phi - 1} = \frac{40}{1{\cdot}089\,07 - 1}$$

$$= \underline{449{\cdot}1\text{ m}}.$$

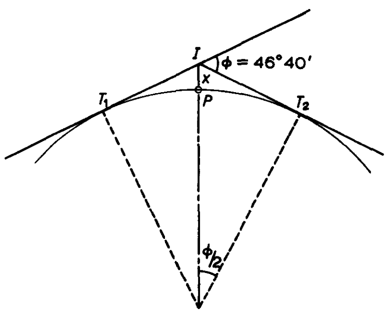


Fig. 11.11

$$T_1I = T_2I = R \tan \tfrac{1}{2}\phi$$

$$= 449{\cdot}1 \tan 23° \,20'$$

$$= \underline{193{\cdot}7\text{ m}}.$$

Example 11.6 The bearings of two lines AB and BC are $036° 36'$ and $080° 00'$ respectively.

At a distance of 276 metres from B towards A and 88 metres at right-angles to the line AB, a station P has been located.

Find the radius of the curve to pass through the point P and also touch the two lines. (R.I.C.S./M)

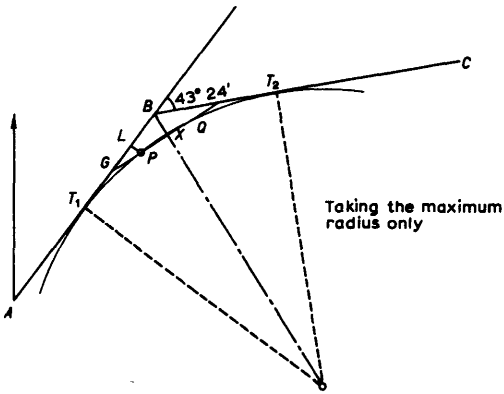


Fig. 11.12

In Fig. 11.12,

$$\phi = 080° 00' - 036° 36' = 43° 24'$$

In triangle GLP,

$$LG = LP \cot \tfrac{1}{2}\phi = 88 \cot 21° 42' = 221{\cdot}13 \text{ m}$$

$$GP = LP \operatorname{cosec} \tfrac{1}{2}\phi = 88 \operatorname{cosec} 21° 42' = 238{\cdot}00 \text{ m}$$

$$GX = (LG + LB) \cos \tfrac{1}{2}\phi = (221{\cdot}13 + 276{\cdot}00) \cos 21° 42' = 461{\cdot}90 \text{ m}$$

$$PX = GX - GP = 461{\cdot}90 - 238{\cdot}00 = 223{\cdot}90 \text{ m}$$

$$GQ = GX + PX = 461{\cdot}90 + 223{\cdot}90 = 685{\cdot}80 \text{ m}$$

$$T_1 G = \sqrt{(GP \times GQ)} = \sqrt{(238{\cdot}00 \times 685{\cdot}80)} = 404{\cdot}01 \text{ m}$$

$$T_1 B = GB + T_1 G = BL + LG + GT_1$$

$$= 276{\cdot}0 + 221{\cdot}13 + 404{\cdot}01 = 901{\cdot}14 \text{ m}$$

$$R = T_1 B \cot \tfrac{1}{2}\phi = 901{\cdot}14 \cot 21° 42'$$

$$= \underline{2264{\cdot}5 \text{ m.}}$$

Alternative solution

$$\theta = \tan^{-1} 88/276 = 17° 41'$$

$$\alpha = 90 - (\tfrac{1}{2}\phi + \theta) = 90 - (21° 42' + 17° 41') = 50° 37'$$

$$x = 276 \sec\theta = 289{\cdot}69\,\text{m}$$

Then, from Eq.(11.22),

$$\sin(\alpha + \beta) = \sin\alpha \sec\tfrac{1}{2}\phi = \sin 50° 37' \sec 21° 42'$$

$$\alpha + \beta = 56° 17' 30''$$

$$\alpha = 50° 37' 00''$$

$$\therefore \quad \beta = 5° 40' 30''$$

Therefore radius $R = x \sin\alpha \operatorname{cosec}\beta$

$$= 289{\cdot}69 \sin 50° 37' \operatorname{cosec} 5° 40' 30''$$

$$= \underline{2264{\cdot}5\,\text{m}}$$

Exercises 11(b) (Curves passing through a given point)

(N.B. In each case only the maximum radius is used)

6. In setting out a circular railway curve it is found that the curve must pass through a point 50 ft from the intersection point and equidistant from the tangents. The chainage of the intersection point is 280 + 80 and the intersection angle (i.e. deflection angle) is 28°.

Calculate the radius of the curve, the chainage at the beginning and end of the curve, and the degree of curvature.

(I.C.E. Ans. 1633 ft; 276 + 73; 284 + 73; 3° 30′)

7. Two straights intersecting at a point *B* have the following bearings: *BA* 270°; *BC* 110°. They are to be joined by a circular curve, but the curve must pass through a point *D* which is 150 ft from *B* and the bearing of *BD* is 260°. Find the required radius, the tangent distances, the length of the curve and the deflection angle for a 100 ft chord.

(L.U. Ans. 3127·4 ft; 551·4 ft; 1091·7 ft; 0° 55′)

8. The co-ordinates of the intersection point *I* of two railway straights, *AI* and *IB*, are 0,0. The bearing of *AI* is 90° and that of *IB* is 57° 14′. If a circular curve is to connect these straights and if this curve must pass through the point whose co-ordinates are − 303·1 ft E and + 20·4 ft N, find the radius of the curve.

Calculate also the co-ordinates of the tangent point on AI and the deflection angles necessary for setting out 100 ft chords from this tangent point. What would be the final chord length?

(L.U. Ans. 2000 ft; 588·0 ft E 0 ft N)

9. A straight BC deflects 24° right from a straight AB. These are to be joined by a circular curve which passes through a point D 200 ft from from B and 50 ft from AB.

Calculate the tangent length, the length of the curve, and the deflection angle for a 100 ft chord.

(L.U. Ans. 818·6; 806·5; 0° 45′)

10. A right-hand circular curve is to connect two straights AI and IB, the bearings of which are 70° 42′ and 130° 54′ respectively. The curve is to pass through a point X such that IX is 132·4 ft and the angle AIX is 34° 36′. Determine the radius of the curve.

If the chainage of the intersection point is 5261 ft, determine the tangential angles required to set out the first two pegs on the curve at through chainages of 50 ft.

(L.U. Ans. 754·1 ft; 0° 59′ 40″; 2° 53′ 40″)

11. The following readings were taken by a theodolite stationed at the point of intersection I of a circular curve of which A and B are respectively the first and second tangent points

A 14° 52′; B 224° 52′; C 344° 52′

It is required that the curve shall pass through the point C, which is near the middle of the curve, at a distance of 60 ft from I.

(a) Determine the radius of the curve.

(b) Calculate the running distances of the tangent points A and B and the point C, the distance at I being 200 + 72, in 100 ft units.

(c) Show in tabular form the running distances and tangential angles for setting out the curve between A and C.

(L.U. Ans. 1181 ft; 197 + 56; 203 + 74; 200 + 22)

11.54 Given a curve joining two tangents, to find the change required in the radius for an assumed change in the tangent length (Fig. 11.13)

In Fig. 11.13,

$$R_2 - R_1 = (t_3 - t_1)\cot\tfrac{1}{2}\phi \tag{11.26}$$

$$O_1O_2 = (R_2 - R_1)\sec\tfrac{1}{2}\phi \tag{11.27}$$

$$
\begin{aligned}
X_1X_2 &= IX_2 - IX_1 \\
&= (IO_2 - R_2) - (IO_1 - R_1) \\
&= (R_2 \sec \tfrac{1}{2}\phi - R_2) - (R_1 \sec \tfrac{1}{2}\phi - R_1) \\
&= R_2(\sec \tfrac{1}{2}\phi - 1) - R_1(\sec \tfrac{1}{2}\phi - 1) \\
&= (R_2 - R_1)(\sec \tfrac{1}{2}\phi - 1) \qquad (11.28)
\end{aligned}
$$

$$
\therefore \quad R_2 = \frac{X_1X_2}{\sec \frac{1}{2}\phi - 1} + R_1 \qquad (11.29)
$$

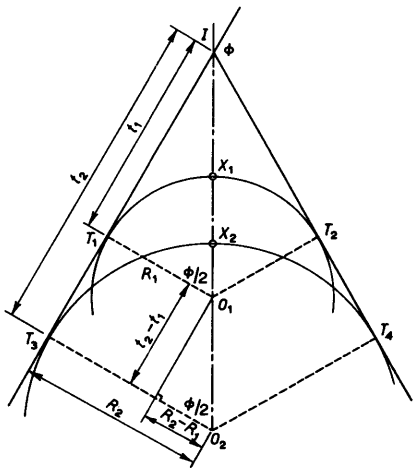


Fig. 11.13

Example 11.7 (a) A circular curve of 2000 ft radius joins two points A and C which lie on the two straights AB and BC. If the running chainage values of A and C are 1091 ft and 2895 ft respectively, calculate the distance of the midpoint of the curve from B.

(b) If the minimum clearance value of the curve from B is to be 200 ft, what radius would be required for the curve and what would be the chainage value for the new tangent points?

(R.I.C.S.)

(a) In Fig. 11.14,

$$
\begin{aligned}
\text{Length of curve} &= 2895 - 1091 \\
&= 1804\text{ ft} \\
&= R\phi_{\text{rad}}
\end{aligned}
$$

$$
\therefore \quad \phi_{\text{rad}} = \frac{\text{arc}}{R}
$$

$$= \frac{1804}{2000}$$

$$\phi = \underline{51° 41'}$$

In triangle T_1BO_1,

$$BO_1 = R \sec \tfrac{1}{2}\phi$$
$$= 2000 \sec 25° 50' 30''$$
$$= \underline{2222{\cdot}22 \text{ ft}}$$
$$BX_1 = 2222{\cdot}22 - 2000$$
$$= \underline{222{\cdot}22 \text{ ft}}$$

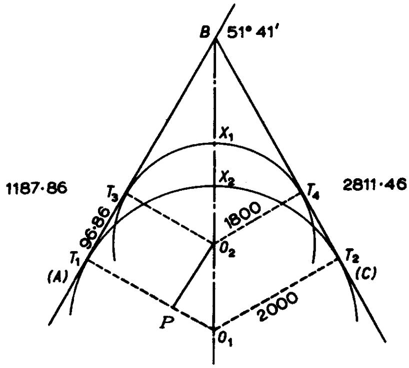


Fig. 11.14

In triangle T_3BO_2,

By Eq. (11.29),

$$R_1 = R_2 - \frac{X_1X_2}{\sec \frac{1}{2}\phi - 1}$$
$$= 2000 - \frac{222{\cdot}22 - 200}{1{\cdot}1111 - 1}$$
$$= 2000 - 200$$
$$= \underline{1800 \text{ ft}}$$

(b) By Eq. (11.27),

$$O_1O_2 = (R_2 - R_1) \sec \tfrac{1}{2}\phi$$
$$= (2000 - 1800) \sec 25° 50' 30''$$
$$= \underline{222{\cdot}22 \text{ ft}}$$
$$T_1T_3 = O_2P = O_1O_2 \sin \tfrac{1}{2}\phi$$
$$= 222{\cdot}22 \sin 25° 50' 30''$$
$$= \underline{96{\cdot}86 \text{ ft}}$$

Alternatively, by Eq. (11.26),

$$(t_2 - t_1) = (R_2 - R_1) \tan \tfrac{1}{2}\phi$$
$$= 200 \tan 25° 50' 30''$$
$$= \underline{96{\cdot}86 \text{ ft}}$$

$$\text{Chainage } T_3 = T_1 + (t_2 - t_1)$$
$$= 1091 + 96{\cdot}86$$
$$= \underline{1187{\cdot}86 \text{ ft}}$$

$$
\begin{aligned}
\text{Length of arc} &= 1800 \times \phi \\
&= 1800 \times 0{\cdot}902 \\
&= 1623{\cdot}60 \\
\therefore \quad \text{Chainage} \quad T_4 &= 1187{\cdot}86 + 1623{\cdot}60 \\
&= \underline{2811{\cdot}46 \text{ ft}}
\end{aligned}
$$

11.6 Location of Tangents and Curve

If no part of the curve or straights exists, setting out is related to a development plan controlled by traverse and/or topographical detail.

If straights exist

(1) Locate intersection point I.
(2) Measure deflection angle ϕ.
(3) Compute tangent length if radius R is known.
(4) Set off tangent points T_1 and T_2.

If the intersection point is inaccessible

Select stations A and B on the straights, Fig. 11.15.

Measure α and β; AB.

Solve triangle AIB.

$$
\begin{aligned}
AI &= AB \sin\beta \operatorname{cosec}\phi \\
BI &= AB \sin\alpha \operatorname{cosec}\phi \\
T_1I &= T_2I = R \tan\tfrac{1}{2}\phi \\
\therefore \quad T_1A &= T_1I - AI \\
T_2B &= T_1I - BI
\end{aligned}
$$

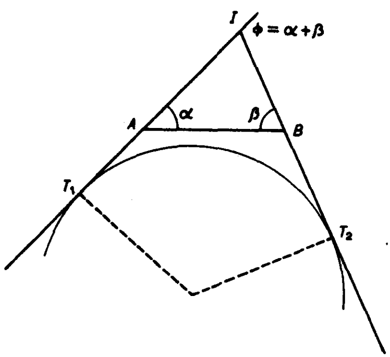


Fig. 11.15

For through chainage

Chainage of A known
Chainage of I = Chainage $A + AI$
Chainage of T_1 = Chainage $I - T_1I$
Chainage of T_2 = Chainage T_1 + arc T_1T_2

If the tangent point is inaccessible

If T_1 is not accessible set out where possible from T_2, Fig. 11.16. A check is possible by selecting station B on the curve and checking offset AB.

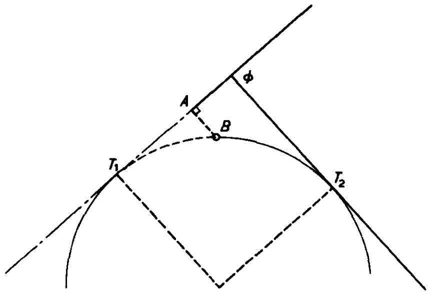


Fig. 11.16

N.B. To locate the curve elements a traverse may be required joining the straights. It is essential to leave permanent stations as reference points for ultimate setting out.

11.7 Setting out of Curves

11.71 By linear equipment only

(a) *Offsets from the long chord* (Fig. 11.17)

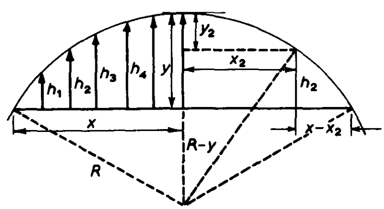


Fig. 11.17

If, in Fig. 11.17, the chord is sub divided into an even number of equal parts, the offsets $h_1 h_2$ etc. can be set out. Each side of the mid-point will be symmetrical.

Generally, $$(R - y)^2 = R^2 - x^2$$

$$y = R - \sqrt{(R^2 - x^2)} \qquad (11.30)$$

$$y_2 = R - \sqrt{(R^2 - x_2^2)} \qquad (11.31)$$

$$\therefore \quad h_2 = y - y_2 \qquad (11.32)$$

N.B. If ϕ is known, $y = R(1 - \cos \frac{1}{2}\phi)$, i.e. $y = R$ versine $\frac{1}{2}\phi$.

Example 11.8 A kerb is part of a 100 ft radius curve. If the chord joining the tangent points is 60 ft long, calculate the offsets from the chord at 10 ft intervals, Fig. 11.18.

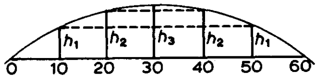


Fig. 11.18

As above (Eq. 11.30),

$$h_3 = y = R - \sqrt{(R^2 - x^2)}$$
$$= 100 - \sqrt{(100^2 - 30^2)}$$
$$= 100 - \sqrt{(100 - 30)(100 + 30)}$$
$$= \underline{4{\cdot}61}.$$

By Eqs.11.31/11.32, $h_2 = h_3 - y_2$

$$y_2 = 100 - \sqrt{(100^2 - 10^2)}$$
$$= 0{\cdot}50$$
$$h_2 = 4{\cdot}61 - 0{\cdot}50 = \underline{4{\cdot}11}$$
$$h_1 = h_3 - y_1$$
$$y_1 = 100 - \sqrt{(100^2 - 20^2)}$$
$$= 2{\cdot}02$$
$$h_1 = 4{\cdot}61 - 2{\cdot}02 = \underline{2{\cdot}59}.$$

(b) *Offsets from the straight* (Fig. 11.19)

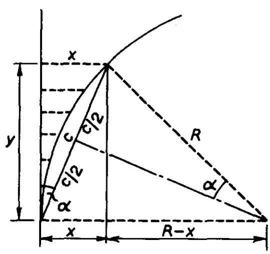


Fig. 11.19

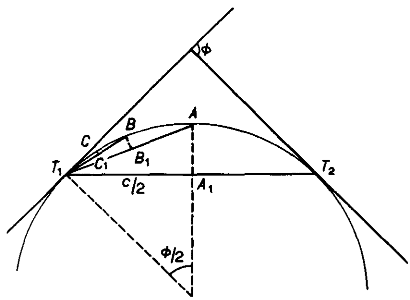


Fig. 11.20

As above, $(R - x)^2 = R^2 - y^2$

i.e., by Eq. (11.30), $x = R - \sqrt{(R^2 - y^2)}$

Alternatively,

$$\sin\alpha = \frac{c}{2R} = \frac{x}{c} \tag{11.33}$$

$$\therefore \quad x = \frac{c^2}{2R} \tag{11.34}$$

If α is small, $c \simeq y$.

$$\therefore \quad x = \frac{y^2}{2R} \tag{11.35}$$

(c) *Offsets from the bisection of the chord* (Fig. 11.20).

From Eq. (11.30), $AA_1 = R - \sqrt{\{R^2 - \left(\frac{c}{2}\right)^2\}}$

Alternatively,

$$AA_1 = R(1 - \cos\tfrac{1}{2}\phi)$$
$$BB_1 = R(1 - \cos\tfrac{1}{4}\phi)$$
$$CC_1 = R(1 - \cos\tfrac{1}{8}\phi)$$

(d) *Offsets from the bisection of successive chords; centre of curve fixed* (Fig. 11.21).

As above,

$$\sin\alpha = \frac{c}{2R}$$

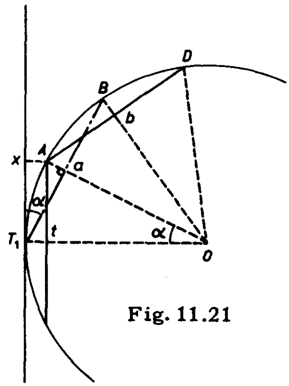


Fig. 11.21

Assuming equal chords,

$$Tt = Aa = Bb = Dd \text{ etc.}$$
$$= R(1 - \cos\alpha) = R \text{ vers}\,\alpha\,.$$

Alternatively, $$Tt = R - \sqrt{\{R^2 - \left(\frac{c}{2}\right)^2\}}$$

Lay off $T_1x = \dfrac{c}{2}$

Lay off $xA = Tt$

$Aa = Tt$ along line AO

T_1B on line Ta produced

$Bb = Tt$ along line BO

(e) *Offsets from chords produced.*

(i) *Equal chords* (Fig. 11.22)

$$A_2A = c\sin\alpha$$

$$= c \times \frac{c}{2R} = \frac{c^2}{2R} \qquad \text{(Eq. 11.34)}$$

If α is small, $A_2A \simeq A_1A$. This can be controlled if the length of chord is limited,

i.e. $$c < \frac{R}{20}$$

$B_1B = 2A_1A$ if α is small.

$$\therefore \quad B_1B = \frac{c^2}{R} \tag{11.36}$$

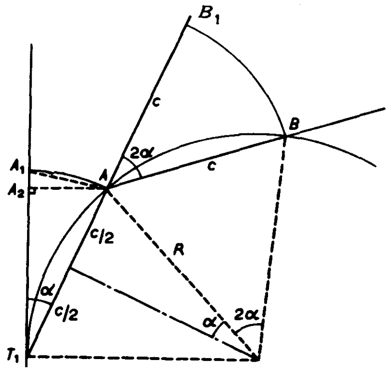


Fig. 11.22

(ii) *Unequal chords (through chainage)*

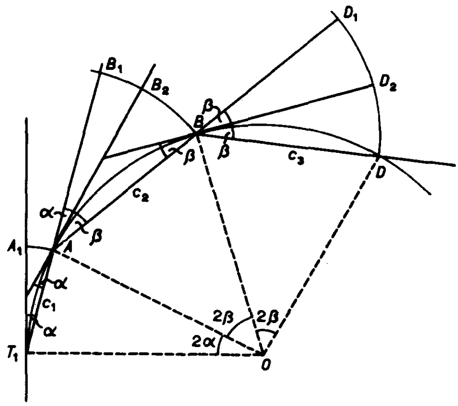


Fig. 11.23

In Fig. 11.23,

$$\text{Offset from sub-chord } O_1 = A_1A = \frac{c_1^2}{2R} \qquad \text{(Eq. 11.34)}$$

$$\text{Offset from 1st full chord } O_2 = B_1B = B_1B_2 + B_2B$$

$$= O_1 \times \frac{c_2}{c_1} + \frac{c_2^2}{2R}$$

$$= \frac{c_1c_2}{2R} + \frac{c_2^2}{2R} = \frac{c_2(c_1 + c_2)}{2R} \tag{11.37}$$

$$\text{Offset from 2nd full chord} = O_3 = D_1D = D_1D_2 + D_2D$$
$$= 2D_2D = 2B_2B$$
$$= \frac{c_2^2}{R} \quad (11.38)$$

By Eq. (11.37),
$$O_3 = \frac{c_3(c_2 + c_3)}{2R} \quad \text{but } c_3 = c_2$$
$$\therefore \quad = \frac{2c_3^2}{2R} = \frac{c_3^2}{R}$$

Generally,
$$O_n = \frac{c_n(c_n + c_{n-1})}{2R} \quad (11.39)$$

11.72 By linear and angular equipment

(a) *Tangential deflection angles* (Fig. 11.24)

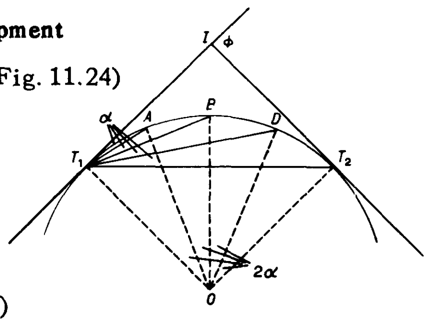


Fig. 11.24

By Eq. (11.33), $\sin\alpha = \dfrac{c}{2R}$

If α is small $(c < R/20)$

$$\sin\alpha = \alpha_{rad}$$

$$\therefore \quad \alpha_{sec} = \frac{206\,265\,c}{2R} \quad (11.40)$$

$$\alpha_{min} = \frac{206\,265\,c}{2R \times 60} = \frac{1718{\cdot}8\,c}{R} \quad (11.41)$$

For equal chords $2\alpha = \dfrac{\phi}{n}$ (11.42)

where n = no. of chords required,

For sub-chords,

$$\alpha_s = \alpha \times \frac{\text{sub-chord}}{\text{standard chord}} \quad (11.43)$$

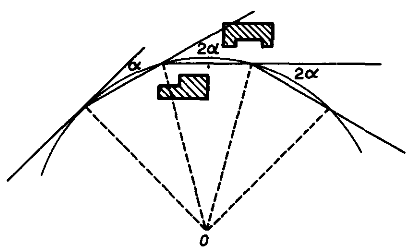


Fig. 11.25

(b) *Deflection angles from chord produced* (Fig. 11.25)

The theodolite is successively moved round the curve. This method is applicable where sights from T_1 are restricted. It is the method applied underground.

11.73 By angular equipment only (Fig. 11.26)

Deflection angles are set out from each tangent point, e.g., A is the intersection of α from T_1 with 3α from T_2.

N.B. $n\alpha = \frac{1}{2}\phi$ (11.44)

Generally, if the deflection angle from T_1 is α_1
the deflection angle β from $T_2 = 360 - \frac{1}{2}\phi + \alpha_1$ (11.45)

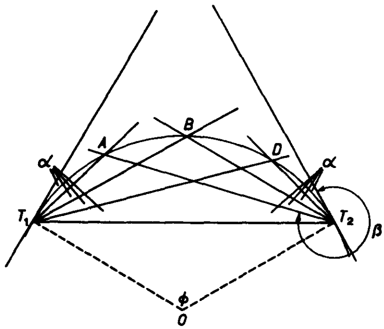


Fig. 11.26

Example 11.9 In a town planning scheme, a road 30 ft wide is to intersect another road 40 ft wide at 60°, both being straight. The kerbs forming the acute angle are to be joined by a circular curve of 100 ft radius and those forming the obtuse angle by one of 400 ft radius.

Calculate the distances required for setting out the four tangent points.

Describe how to set out the larger curve by the deflection angle method and tabulate the angles for 50 ft chords. (L.U.)

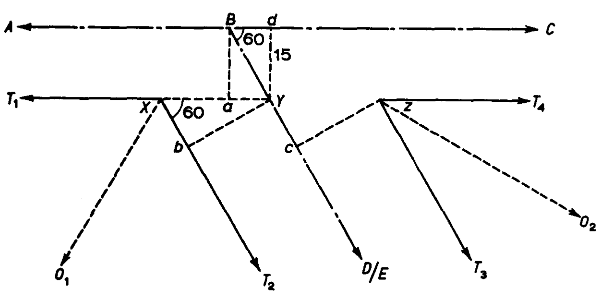


Fig. 11.27

$T_1X = T_2X = 400 \tan 30 = 230{\cdot}94$ ft

$T_3Z = T_4Z = 100 \tan 60 = 173{\cdot}21$ ft

In triangle XYb,

$$XY = \frac{20}{\sin 60} = YZ = 23{\cdot}09$$

$$Xb = 20 \tan 30 = Yc = 11{\cdot}55$$

In triangle BYa,

$$BY = \frac{15}{\sin 60} = 17{\cdot}33$$

$$aY = Bd = 15 \tan 30^\circ = 8{\cdot}66$$

Distances to tangent points measured along centre lines:

$$AB = T_1X + XY - aY = 230{\cdot}94 + 23{\cdot}09 - 8{\cdot}66 = 245{\cdot}37 \text{ ft}$$

$$BC = T_4Z + XY + aY = 173{\cdot}21 + 23{\cdot}09 + 8{\cdot}66 = 204{\cdot}96 \text{ ft}$$

$$BD = T_2X - Xb + BY = 230{\cdot}94 - 11{\cdot}55 + 17{\cdot}33 = 236{\cdot}72 \text{ ft}$$

$$BE = T_3Z + Xb + BY = 173{\cdot}21 + 11{\cdot}55 + 17{\cdot}33 = 202{\cdot}09 \text{ ft}$$

Deflection angles, 50 ft chords

$$\sin \alpha_1 = \frac{50}{2 \times 400}$$

$$\therefore \quad \alpha_1 = 3^\circ 35'$$

By approximation,

$$\alpha_{sec} = \frac{206\,265 \times 50}{2 \times 400} = 12\,892 \text{ sec}$$

$$= \underline{3^\circ 34' 52''}$$

$$\therefore \quad \alpha_1 = 3^\circ 35'$$

$$\alpha_2 = 7^\circ 10'$$

$$\alpha_3 = 10^\circ 45'$$

$$\alpha_4 = 14^\circ 20'$$

$$\alpha_5 = 17^\circ 55'$$

$$\alpha_6 = 21^\circ 30'$$

$$\alpha_7 = 25^\circ 05'$$

$$\alpha_8 = 28^\circ 40'$$

$$\alpha_9 = 30^\circ 00' \quad (\text{sub-chord} = 2R \sin 1^\circ 20'$$

$$= 2 \times 400 \sin 1^\circ 20'$$

$$= \underline{18{\cdot}62 \text{ ft}})$$

Example 11.10 A curve of radius 1000 ft is to be set out connecting 2 straights as shown in Fig. 11.28.

Point X is inaccessible and BC is set out and the data shown obtained. Assuming the chainage at B is 46 + 47·3 ft, calculate sufficient data to set out the curve by deflection angles from the tangent by chords of 50 ft based on through chainage.

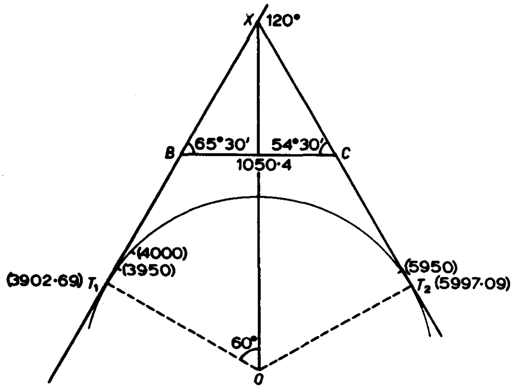


Fig. 11.28

(derived values in brackets)

In triangle BXC, (Fig. 11.28)

$$BX = 1050{\cdot}4 \sin 54°30' \operatorname{cosec} 60° = 987{\cdot}44 \text{ ft}$$
$$XC = 1050{\cdot}4 \sin 65°30' \operatorname{cosec} 60° = 1103{\cdot}69 \text{ ft}$$

In triangle OT_1X,

Tangent length $\quad T_1X = 1000 \tan 60°$

$= \underline{1732{\cdot}05 \text{ ft}}$

$\therefore \quad BT_1 = T_1X - BX$

$= 1732{\cdot}05 - 987{\cdot}44$

$= \underline{744{\cdot}61 \text{ ft}}$

Similarly $\quad CT_2 = 1732{\cdot}05 - 1103{\cdot}69$

$= \underline{628{\cdot}36 \text{ ft}}$

Chainage $\quad T_1 = 4647{\cdot}30 - 744{\cdot}61$

$= \underline{3902{\cdot}69 \text{ ft}}$

Length of curve $= R\phi = 1000 \times 2{\cdot}094\,40$

$= \underline{2094{\cdot}40 \text{ ft}}$

Chainage $\quad T_2 = 3902{\cdot}69 + 2094{\cdot}40$

$= \underline{5997{\cdot}09 \text{ ft}}$

Length of chords,

$$C_1 = 3950 - 3902{\cdot}69 = 47{\cdot}31\text{ ft}$$
$$C_2 = C_{n-1} = 50{\cdot}0\text{ ft}$$
$$C_n = 5997{\cdot}09 - 5950 = 47{\cdot}09\text{ ft}$$

Deflection angles (Eq. 11.40)

$$\text{for } C_2 = 50\text{ ft}, \quad \alpha_2 = \frac{206\,265 \times 50}{2 \times 1000} = 5156{\cdot}62'' = \underline{1°25'57''}$$
(say 1°26′00″)

$$\text{for } C_1 = 47{\cdot}3\text{ ft}, \quad \alpha_1 = \frac{5156{\cdot}6 \times 47{\cdot}3}{50} = 4879'' = \underline{1°21'19''}$$
(say 1°21′20″)

$$\text{for } C_n = 47{\cdot}1\text{ ft}, \quad \alpha_n = \frac{5156{\cdot}6 \times 47{\cdot}1}{50} = 4856'' = \underline{1°20'56''}$$
(say 1°21′00″)

Check $4879 + 40 \times 5156{\cdot}62 + 4856 = \underline{216\,000''}$
$= \underline{60°00'00''}$

Example 11.11 Assuming that Example 11.21 is to be set out using angular values only, calculate the deflection angle from T_1 and T_2 suitable for use with a 20″ theodolite, Fig. 11.29.

Angles at T_1 may be calculated as follows from the previous calculations.

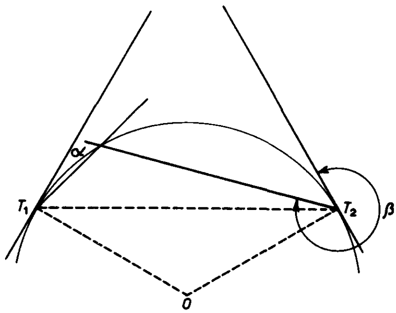


Fig. 11.29

			Accumulative deflection angle	
α_1	= 1°21′19″	say	1°21′20″	
$+\alpha_2$	= 1°25′57″		2°47′20″	
$+\alpha_3$	= 1°25′57″		4°13′20″	
$+\alpha_4$	= 1°25′57″		5°39′20″	
$+\alpha_5$	= 1°25′57″		7°05′00″	etc.

Angle at T_2, by Eq. (11.44):

$$\begin{aligned} \beta_1 &= 360 - 60 + \alpha_1 &&= 301°21'20'' \\ \beta_2 &= 300 + \alpha_1 + \alpha_2 &&= 302°47'20'' \\ \beta_3 &= 300 + \alpha_1 + \alpha_2 + \alpha_3 &&= 304°13'20'' \qquad \text{etc.} \end{aligned}$$

Example 11.12 Fig. 11.30 shows the centre lines of existing and proposed roadways in an underground shaft siding. It is proposed to connect roadways $A-B$ and no. 1 shaft $-C$ respectively by curves BD and CD, each 100 ft radius, and to drive a haulage road from D in the direction DE on a line tangential to both curves. B and C are tangent points of the respective curves and D is a tangent point common to both curves.

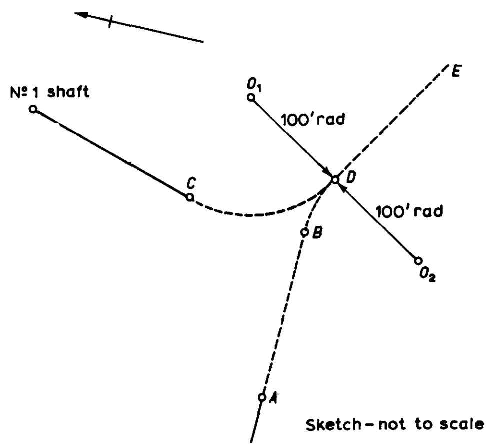


Fig. 11.30

Co-ordinates of A N 2311·16 ft, E 2745·98 ft.
Co-ordinates of no. 1 shaft N 2710·47 ft, E 3052·71 ft.
Bearing of no. 1 shaft $-C$ 186°30′00″.
Distance of no. 1 shaft $-C$ 355 ft.
Bearing of roadway $A-B$ 87°23′50″.

Calculate the distance from A to the tangent point B of the curve BD, the co-ordinates of B and the bearing of the proposed roadway DE.

(M.Q.B./S)

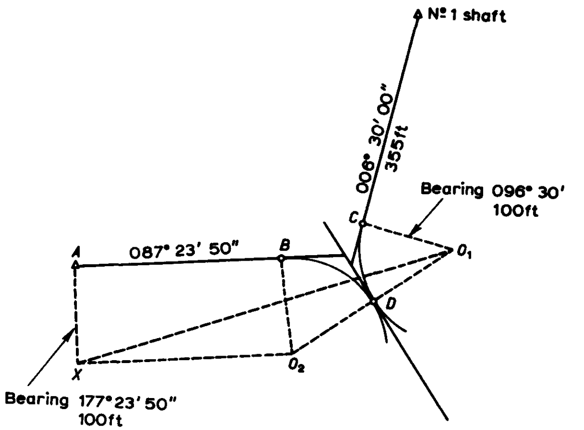


Fig. 11.31

Construction (Fig. 11.31) Draw AX 100 ft perpendicular to AB. Join XO_2; O_1O_2.

Co-ordinates of C

$E_{no.\,1}$	3052·71
ΔE_{1-C} 355 sin 6°30′	−40·19
E_C	3012·52
$N_{no.\,1}$	2710·47
ΔN_{1-C} 355 cos 6°30′	−352·72
N_C	2357·75

Co-ordinates of O_1

E_C	3012·52
ΔE_{C-O_1} 100 sin 85°30′	+99·36
E_{O_1}	3111·88
N_C	2357·75
ΔN_{C-O_1} 100 cos 85°30′	−11·32
N_{O_1}	2346·43

Co-ordinates of X

E_A	2745·98
ΔE_{AX} 100 sin 2°36′10″	+ 4·54
E_X	2750·52
N_A	2311·16
ΔN_{AX} 100 cos 2°36′10″	−99·90
N_X	2211·26

$$\text{Bearing} \quad XO_1 = \tan^{-1}\frac{3111{\cdot}88 - 2750{\cdot}52}{2346{\cdot}43 - 2211{\cdot}26} = \tan^{-1}\frac{361{\cdot}36}{135{\cdot}17}$$

$$= \text{N } 69°29'29'' \text{ E}$$

$$\text{Length} \quad XO_1 = 361{\cdot}36 \operatorname{cosec} 69°29'29''$$

$$= 385{\cdot}81 \text{ ft}$$

In triangle XO_1O_2,

$$XO_2 = \frac{XO_1 \sin O_1}{\sin O_2}$$

$$\text{Bearing} \quad XO_2 = 87°23'50''$$

$$XO_1 = 69°29'29'' \qquad \therefore \quad \text{Angle } X = 17°54'21''$$

$$\sin O_2 = \frac{O_1X \sin X}{2R} = \frac{385{\cdot}81 \sin 17°54'21''}{200}$$

$$O_2 = (36°22'33'') \text{ or } 180 - 36°22'33''$$

$$= 143°38'27''$$

$$\text{Thus} \quad XO_2 = \frac{385{\cdot}81 \sin(180 - 143°38'27'' - 17°54'21'')}{\sin 143°38'27''}$$

$$= \frac{385{\cdot}81 \sin 18°27'12''}{\sin 36°22'33''} = \underline{205{\cdot}91 \text{ ft} = AB}$$

Co-ordinates of B

E_A	2745·98
ΔE_{AB} 205·91 sin 87°23′50″	+205·70
E_B	2951·68
N_A	2311·16
ΔN_{AB} 205·91 cos 87°23′50″	+ 9·35
N_B	2320·51

Bearing XO_2 = AB	=	$087°23'50''$
Angle XO_2O_1	=	$143°38'27''$
		$231°02'17''$
		$-180°$
Bearing O_2O_1		$051°02'17''$
		$+90°$
Bearing DE		$141°02'17''$

Ans. AB 205·91 ft.
Co-ordinates of B, E 2951·68 N 2320·51.
Bearing of DE, $141°02'17''$.

Exercises 11(c)

12. AB and CD are straight portions of two converging railways which are to be connected by a curve of 1500 ft radius. The point of intersection of AB and CD produced is inaccessible. X and Y are points on AB and CD respectively which are not intervisible and the notes of a theodolite traverse from one to the other are as follows:

Line	Horizontal	Angle	Distance (ft)
Xa	AXa	$260°10'$	160
ab	Xab	$169°00'$	240
bc	abc	$210°30'$	300
cY	bcY	$80°00'$	180
YC	cYC	$268°40'$	-

Calculate the apex angle and the position of the start and finish of the curve relative to X and Y.

(M.Q.B./S Ans. $91°40'$; T_1X 1234·42 ft; T_2Y 780·09 ft)

13. To locate the exact position of the tangent point T_2 of an existing 500 ft radius circular curve in a built-up area, points a and d were selected on the straights close to the estimated positions of the two tangent points T_1 and T_2 respectively, and a traverse $abcd$ was run between them.

Station	Length (ft)	Deflection Angle
a		$9°54'$ R
b	ab 178	$19°36'$ R
c	bc 231	$30°12'$ R
d	cd 203	$5°18'$ R

The angles at a and d were relative to the straights. Find the distance T_2d.

(L.U. Ans. 34·1 ft)

14. Two straights AI and BI meet at I on the far side of a river. On the near side of the river, a point E was selected on the straight AI, and a point F on the straight BI, and the distance from E to F measured and found to be 3·40 chains.

The angle, AEF, was found to be 165°36′ and the angle BFE 168°44′. If the radius of a circular curve joining the straights is 20 chains, calculate the distance along the straights from E and F to the tangent points.

(I.C.E. Ans. 3·005 ch; 2·604 ch)

15. The centre line of a proposed railway consists of two straights joined by a 3° curve. The angle of deflection between the straights is 26° and the chainage (increasing from left to right) of their intersection is 7367 ft.

Calculate the deflection angles from the tangent for setting out the circular curve from the first tangent point by pegs at every 100 ft chainage and check on to the second tangent point.

(I.C.E. Ans. peg 70 – 1° 06·6′ 71 – 2° 36·6′ 72 – 4° 06·6′
73 – 5° 36·6′ 74 – 7° 06·6′ 75 – 8° 36·6′
76 – 10° 06·6′ 77 – 11° 36·6′ T.P. 13° 00′)

16. Outline three different methods for setting out a circular curve of several hundred feet radius using a chain and tape and without using a theodolite. Sketch a diagram for each method and quote any formulae used in the calculations associated with each method.

The centre line of a certain length of a proposed road consists of two straights with a deflection angle of 30°00′ and joined by a circular curve of 1000 ft radius; the chainage of the tangent point on the first straight is 3630 ft. The curve is to set out by deflection angles from this tangent point using a theodolite which reads to 20″, and pegs are required at every 100 ft of through chainage and at the second tangent point.

Calculate these deflection angles and any other data that could be used in the field for checking the position of the second tangent point.

(I.C.E. Ans. 2°00′20″; 4°52′20″; 7°44′00″
10°36′00″; 13°28′00″; 15°00′00″)

17. The tangent length of a simple curve was 663·14 ft and the deflection angle for a 100 ft chord 2°18′.

Calculate the radius, the total deflection angle, the length of the curve and the final deflection angle.

(L.U. Ans. 1245·5 ft; 56°03′50″; 1218·7 ft; 28°01′55″)

18. (a) What is meant by the term 'degree of curve' (D)?

State the advantages, if any, of defining a circular curve in this way. Show how the degree of curve is related to the radius of the curve (R).

(b) Two straights EF and FG of a proposed road intersect at point F. The bearings of the straights are:

EF 76°12′
FG 139°26′

The chainage of E is 11 376·0 (113+76·0) and the distance EF is 2837·6 (28+37·6).

Calculate the chainages of the tangent points and prepare a table of the deflection angles from the tangent point on GF for pegs at whole 100 ft chainages for a 5° curve connecting the two straights.

Explain with the aid of a diagram how you would set out this curve.

(R.I.C.S./L Ans. R = 1146 ft; 10 670·3; 11 935·4)

19. A circular curve XT of 1000 ft radius joins two straights AB and BC which have bearings of 195°10′ and 225°40′ respectively. At what chainage from X measured along the curve, will the curve be nearest to point B?

If this point of nearest approach to B be point W what is the bearing of WB?

(R.I.C.S./G Ans. 266·2 ft; 120°25′)

20. (a) The lines AB and BC are to be joined by a circular curve of radius 3000 ft. The point B of intersection of the lines is inaccessible.

The following values have been measured on the ground:

angle AMN = 146°05′, angle MNC = 149°12′, MN = 2761 ft;

and the chainage of M is 25 342 ft (measurement from A).

Calculate the length of the curve and the chainage of the beginning and end of the curve.

(b) Deduce formulae for setting out intermediate points on the curve by using steel band, linen tape, optical square and ranging rods only.

(R.I.C.S./L Ans. 3388·6; 25 004·7; 28 393·3)

21. $ABCD$ is a plot of land, being part of a block.

It is required to round off the corner by a circular curve tangential to the boundaries at B and C. What is the radius of the curve to the nearest tenth of a foot?

Angle	BAD = 90°	AB =	100 ft
Angle	ADC = 52°38′	AD =	140 ft

(R.I.C.S./L Ans. R = 28·2 ft)

22. A railway boundary CD in the form of a circular arc is intersected by a farm boundary BA in E. Calculate this point of intersection E.

The radius of the curve is 500 ft and the co-ordinates in feet are:

A	E 7525·7,	N 21951·7
B	E 7813·4	N 20163·3
C	E 8009·3	N 21179·6
D	E 7101·2	N 21074·3

(Ans. E 7610·5 N 21424·7)

23. A 750 ft length of straight connects two circular curves which both deflect right. The first is of radius 1000 ft, the second is of radius 800 ft and deflection angle $27° 35'$. The combined curve is to be replaced by a single circular curve between the same tangent points.

Find the radius of this curve and the deflection angle of the first curve.

(L.U. Ans. R = 1666·5 ft; $31°33'$)

24. In a level seam two roadways AB and DC are connected by roadways AD and BC. Point B is 820 ft due East of A, D is 122 ft due North of A, and C is 264 ft due North of B. It is proposed to drive a circular curve connecting BA and DC and tangential to BA, AD and DC. Calculate the radius of the curve and the distances from A and D to the tangent point of the curve on the lines AB and DC respectively.

(M.Q.B./S Ans. 66·24 ft; 66·24 ft; 55·76 ft)

25. The co-ordinates of two points A and B are:

	E	N
A	0·0	399·60
B	998·40	201·40

A straight line AC bears $110° 30'$ and intersects at C a straight line BC bearing $275° 50'$. The chainage of A is 2671·62 ft. Calculate the lengths of AC and CB.

The two straights are to be joined by a curve of 500 ft radius. Calculate the chainage of the tangent points and of B.

(L.U. Ans. 377·97 ft; 647·71 ft; 2985·24 ft; 3113·23 ft; 3696·59 ft)

11.8 Compound Curves

Compound curves consist of two or more consecutive circular arcs of different radii, having their centres on the same side of the curve.

There are seven components in a compound curve made up of two arcs:

Two radii, R_1 and R_2.

Two tangent lengths, t_1 and t_2.

An angle subtended by each arc, α_1 and α_2.

A deviation angle at the intersection point I, $\phi = \alpha_1 + \alpha_2$.

At least 4 values must be known.

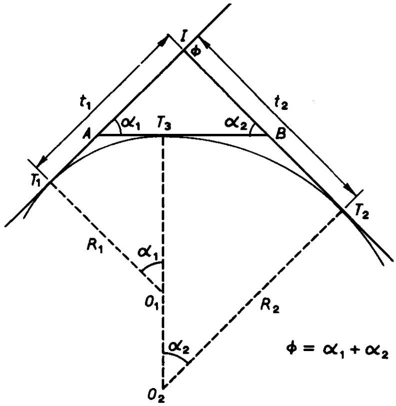


Fig. 11.32

In Fig. 11.32,

$$AI = \frac{AB \sin\alpha_2}{\sin\phi} = \frac{(R_1 \tan\frac{1}{2}\alpha_1 + R_2 \tan\frac{1}{2}\alpha_2)\sin\alpha_2}{\sin\phi}$$

$$T_1I = T_1A + AI = t_1$$

$$t_1 = R_1 \tan\tfrac{1}{2}\alpha_1 + \frac{(R_1 \tan\frac{1}{2}\alpha_1 + R_2 \tan\frac{1}{2}\alpha_2)\sin\alpha_2}{\sin\phi}$$

$$t_1 \sin\phi = R_1 \tan\tfrac{1}{2}\alpha_1 \sin\phi + R_1 \tan\tfrac{1}{2}\alpha_1 \sin\alpha_2 + R_2 \tan\tfrac{1}{2}\alpha_2 \sin\alpha_2$$

$$= R_1 \tan\tfrac{1}{2}\alpha_1(\sin\phi + \sin\alpha_2) + R_2 \frac{\sin\frac{1}{2}\alpha_2\, 2\sin\frac{1}{2}\alpha_2 \cos\frac{1}{2}\alpha_2}{\cos\frac{1}{2}\alpha_2}$$

$$= R_1 \frac{\sin\frac{1}{2}(\phi - \alpha_2)\, 2\sin\frac{1}{2}(\phi + \alpha_2)\cos\frac{1}{2}(\phi - \alpha_2)}{\cos\frac{1}{2}(\phi - \alpha_2)} + 2R_2 \sin^2\tfrac{1}{2}\alpha_2$$

$$= R_1(\cos\alpha_2 - \cos\phi) + R_2(1 - \cos\alpha_2)$$

$$= R_1\{(1 - \cos\phi) - (1 - \cos\alpha_2)\} + R_2(1 - \cos\alpha_2)$$

$$= (R_2 - R_1)(1 - \cos\alpha_2) + R_1(1 - \cos\phi)$$

$$\underline{t_1 \sin\phi = (R_2 - R_1)\text{ versine }\alpha_2 + R_1 \text{ versine }\phi} \qquad (11.46)$$

Similarly, it may be shown that

$$\underline{t_2 \sin\phi = (R_1 - R_2)\text{ versine }\alpha_1 + R_2 \text{ versine }\phi} \qquad (11.47)$$

An alternative solution is shown involving more construction but less trigonometry, Fig. 11.33.

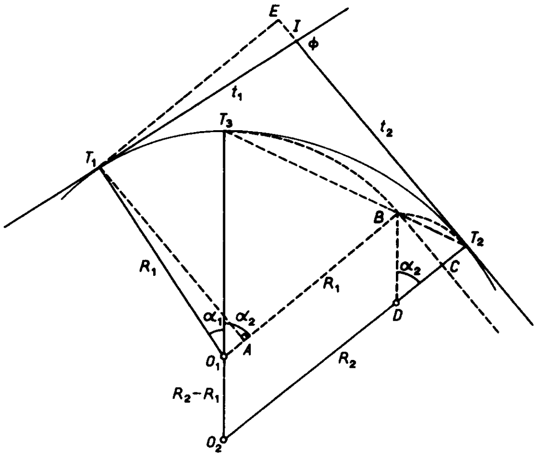


Fig. 11.33

Construction

Produce arc T_1T_3 of radius R_1 to B

Draw O_1B parallel to O_2T_2

BC parallel to IT_2

BD parallel to T_3O_2

T_1A perpendicular to O_1B

T_1E perpendicular to T_2I (produced)

N.B. (a) T_3BT_2 is a straight line.

(b) $T_3O_1O_2$ is a straight line.

(c) $BD = O_1O_2 = DT_2 = R_2 - R_1$.

$$T_1E = AB + CT_2$$
$$= (O_1B - O_1A) + (DT_2 - DC)$$

i.e.

$$t_1 \sin\phi = R_1 - R_1\cos(\alpha_1 + \alpha_2) + \{(R_2 - R_1) - (R_2 - R_1)\cos\alpha_2\}$$
$$= R_1(1 - \cos\phi) + (R_2 - R_1)(1 - \cos\alpha_2)$$
$$\underline{t_1 \sin\phi = R_1 \text{ versine } \phi + (R_2 - R_1) \text{ versine } \alpha_2} \qquad (11.48)$$

By similar construction, Fig. 11.34,

$$FT_2 = GH - HJ$$
$$= (O_2H - O_2G) - (HK - JK)$$
$$\underline{t_2 \sin\phi = R_2 \text{ vers } \phi + (R_1 - R_2) \text{ vers } \alpha_1} \qquad (11.49)$$

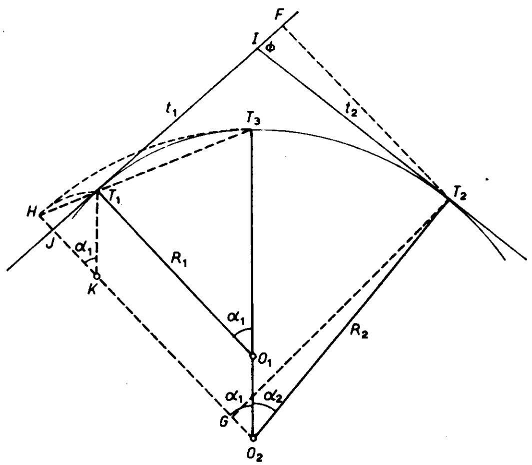


Fig. 11.34

Example 11.13. Given $R_1 = 20$ m, $R_2 = 40$ m, $T_1I = 20{\cdot}5$ m, $\phi = 80° 30'$, it is required to find length T_2I.

From Eq. (11.48),

$$\text{versine } \alpha_2 = \frac{T_1I \sin\phi - R_1 \text{ versine } \phi}{R_2 - R_1}$$

$$= \frac{20{\cdot}5 \sin 80° 30' - 20 \text{ vers } 80° 30'}{40 - 20}$$

$$= \frac{3{\cdot}520}{20}$$

$$\underline{\alpha_2 = 34° 31'}$$

then $\underline{\alpha_1 = 45° 59'}$ i.e. $\phi - \alpha_2$

From Eq. (11.49),

$$T_2I = \frac{R_2 \text{ vers } \phi - (R_2 - R_1) \text{ vers } \alpha_1}{\sin\phi}$$

$$= \frac{40 \text{ vers } 80° 30' - 20 \text{ vers } 45° 59'}{\sin 80° 30'}$$

$$= \underline{27{\cdot}67 \text{ m}}$$

Alternative solution from first principles (Fig. 11.35)

Construction

Join O_1I

Draw O_1P parallel to O_2T_2

O_1Q parallel to IT_2

In triangle T_1IO_1,

$$\theta = \tan^{-1}\frac{R_1}{T_1I} = \tan^{-1}\frac{20}{20 \cdot 5}$$

$$= 44° 18'$$

$$\beta = 180 - (\theta + \phi) = 55° 12'$$

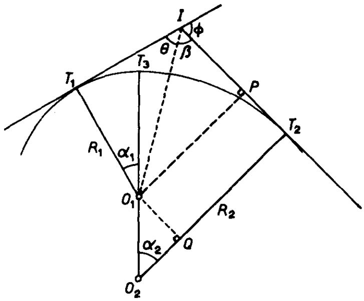


Fig. 11.35

In triangle IPO_1,

$$IP = O_1I \cos\beta$$

$$= R_1 \operatorname{cosec}\theta \cos\beta$$

$$= 20 \operatorname{cosec} 44° 18' \cos 55° 12'$$

$$= 16 \cdot 35$$

$$O_1P = O_1I \sin\beta$$

$$= 20 \operatorname{cosec} 44° 18' \sin 55° 12' = 23 \cdot 52$$

$$O_2Q = O_2T_2 - QT_2 = R_2 - O_1P$$

$$= 40 - 23 \cdot 52$$

$$= 16 \cdot 48$$

In triangle O_1O_2Q,

$$\alpha_2 = \cos^{-1}\frac{O_2Q}{O_1O_2} = \cos^{-1}\frac{16 \cdot 48}{20} = 34° 31'$$

$$\therefore \quad \alpha_1 = 45° 59'$$

$$O_1Q = O_1O_2 \sin\alpha_2 = 20 \sin 34° 31' = 11 \cdot 32 \text{ m}$$

$$IT_2 = IP + PT_2 = IP + O_1Q = 16 \cdot 35 + 11 \cdot 32,$$

$$= \underline{27 \cdot 67 \text{ m}}$$

Example 11.14. *AB* and *DC* are the centre lines of two straight portions of a railway which are to be connected by means of a compound curve *BEC*, *BE* is one circular curve and *EC* the other. The radius of the circular curve *BE* is 400 ft.

Given the co-ordinates in ft: *B* N 400 E 200, *C* N 593 E 536, and the directions of *AB* and *DC* NE 25° 30′ and NW 76° 30′ respectively, Calculate (a) the co-ordinates of *E*, (b) the radius of the circular curve *EC*. (R.I.C.S./M)

Formulae are not very suitable and the method below shows an alternative from first principles.

$$\text{Bearing } BC = \tan^{-1}\frac{536 - 200}{593 - 400} = 60° 07'$$

$$BC = 193 \sec 60° 07'$$

$$\phi = 180 - 76° 30' - 25° 30' = 78° 00'$$

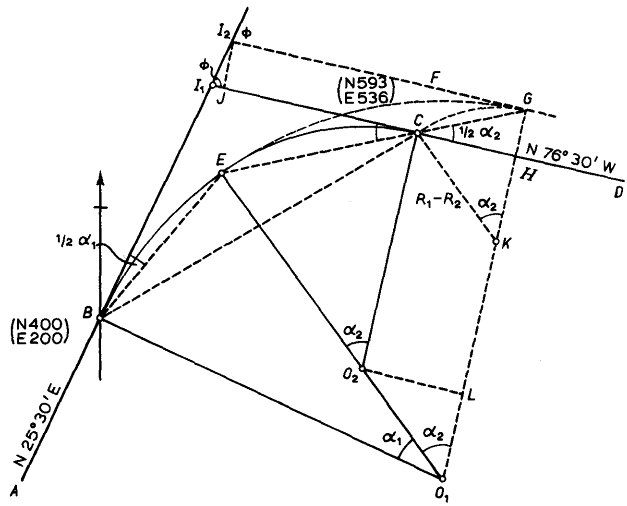


Fig. 11.36

In triangle BI_1C,

$$\hat{B} = 60°07' - 25°30' = 34°37'$$

$$\hat{C} = 78°00' - 34°37' = 43°23'$$

$$BI_1 = BC \sin c \operatorname{cosec} \phi$$

$$= 193 \sec 60°07' \sin 43°23' \operatorname{cosec} 78° = 272{\cdot}0 \text{ ft}$$

$$CI_1 = BC \sin B \operatorname{cosec} \phi = 225{\cdot}0 \text{ ft}$$

$$BI_2 = 400 \tan \frac{78}{2} = I_2G = 323{\cdot}9 \text{ ft}$$

$$I_1I_2 = 323{\cdot}9 - 272 = 51{\cdot}9 \text{ ft}$$

$$I_1J = 51{\cdot}9 \cos 78° = 10{\cdot}8 \text{ ft}$$

$$I_2J = 51{\cdot}9 \sin 78° = 50{\cdot}8 \text{ ft}$$

$$FG = CH = O_2L = I_2G + I_1J - I_1C = 323{\cdot}9 + 10{\cdot}8 - 225{\cdot}0$$

$$= 109{\cdot}7$$

In triangle CGH,

$$\frac{\alpha_2}{2} = \tan^{-1} GH/CH$$

$$\alpha_2 = 2 \tan^{-1} I_2J/CH = 2 \tan^{-1} \frac{50{\cdot}8}{109{\cdot}7}$$

$$= 2 \times 24°50' = 49°40'$$

$$\alpha_1 = 78° - 49°40' = 28°20'$$

In triangle O_1O_2L,

$$O_1O_2 = O_2L \operatorname{cosec} \alpha_2 = 109{\cdot}7 \operatorname{cosec} 49^\circ 40' = 143{\cdot}9$$

$$\therefore \quad R_2 = 400 - 143{\cdot}9 = \underline{256{\cdot}1}$$

To find the co-ordinates of E,

$$\text{Bearing } BE = 25^\circ 30' + 14^\circ 10' = 39^\circ 40'$$

$$BE = 2 \times 400 \sin 14^\circ 10'$$

$$\text{Partial Lat.} = BE \cos 39^\circ 40' = 150{\cdot}71$$

$$\text{Total Lat.} = \underline{\text{N } 550{\cdot}7}$$

$$\text{Partial Dep.} = BE \sin 39^\circ 40' = 124{\cdot}98$$

$$\text{Total Dep.} = \underline{\text{E } 325{\cdot}0}$$

Check

From Eq. (11.48),

$$t_1 \sin \phi = 272 \sin 78^\circ = 266{\cdot}06 \text{ ft}$$

$$R_1 \text{ versine} \phi = 400(1 - \cos 75^\circ) = 316{\cdot}83 \text{ ft}$$

$$(R_2 - R_1) \text{ versine} \alpha_2 = (256{\cdot}1 - 400)(1 - \cos 49^\circ 40') = -50{\cdot}76 \text{ ft}$$

$$\therefore \quad R_1 \text{ vers} \phi + (R_2 - R_1) \text{ vers} \alpha_2 = 316{\cdot}83 - 50{\cdot}76 = \underline{266{\cdot}07 \text{ ft}}$$

Example 11.15. Undernoted are the co-ordinates in ft of points on the respective centre lines of two railway tracks ABC and DE

Co-ordinates.			
	A	0	0
	B	E 525·32	N 52·82
	C	E 827·75	N 247·29
	D	E 10·89	S 108·28
	E	E 733·23	S 35·65

The lines AB and DE are straight and B and C are tangent points joined by circular curve. It is proposed to connect the two tracks at C by a circular curve starting at X on the line DE, C being a tangent point common to both curves. Calculate the radius in chains of each curve, the distance DX and the co-ordinates of X.

(M.Q.B./S)

In Fig. 11.37,

$$\text{Bearing } AB = \tan^{-1} + 525{\cdot}32/+52{\cdot}82$$

$$= \text{N } 84^\circ 15' 30'' \text{ E}$$

$$\text{Bearing } DE = \tan^{-1}\frac{733{\cdot}23 - 10{\cdot}89}{-35{\cdot}65 + 108{\cdot}28}$$
$$= \text{N } 84^\circ 15' 30'' \text{ E}$$

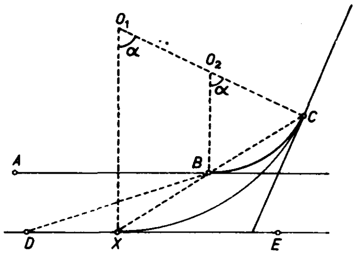


Fig. 11.37

As bearings are alike, C, B and X must lie on the same straight line.

$$\text{Bearing } BC = \text{bearing } XC$$
$$= \tan^{-1}\frac{827{\cdot}75 - 525{\cdot}32}{247{\cdot}29 - 52{\cdot}82}$$
$$= \text{N } 57^\circ 15' 29'' \text{ E}$$

$$\text{Length } BC = \Delta\text{N sec bearing}$$
$$= (247{\cdot}29 - 52{\cdot}82) \sec 57^\circ 15' 29''$$
$$= 359{\cdot}56$$

$$\text{Angle } (\alpha) \text{ subtended at centre} = 2 \times \text{angle } BXE$$
$$= 2 \times (84^\circ 15' 30'' - 57^\circ 15' 29'')$$
$$= 54^\circ 00' 02''$$

$$\text{Radius } O_2C = \tfrac{1}{2}BC \operatorname{cosec} \tfrac{1}{2}\alpha$$
$$= \tfrac{1}{2} \times 359{\cdot}56 \operatorname{cosec} 27^\circ 00' 01'' = 396 \text{ ft} = \underline{6 \text{ chains}}$$

$$\text{Bearing } DB = \tan^{-1}\frac{525{\cdot}32 - 10{\cdot}89}{52{\cdot}82 + 108{\cdot}28} = \text{N } 72^\circ 36' 41'' \text{ E}$$

$$\text{Length } DB = 161{\cdot}10 \sec 72^\circ 36' 41'' = 539{\cdot}06$$

$$\text{Bearing } DE = 084^\circ 15' 30''$$
$$DB = 072^\circ 36' 41'' \quad \therefore \text{ Angle } BOX = 11^\circ 38' 49''$$
$$BC = 057^\circ 15' 29'' \quad \therefore \text{ Angle } DBX = 15^\circ 21' 12''$$

In triangle DBX,

$$DX = \frac{BD \sin B}{\sin X} = \frac{539{\cdot}06 \sin 15^\circ 21' 12''}{\sin 27^\circ 00' 01''} = 314{\cdot}38 \text{ ft}$$

$$BX = \frac{BD \sin D}{\sin X} = \frac{539{\cdot}06 \sin 11^\circ 38' 49''}{\sin 27^\circ 00' 01''} = 239{\cdot}71 \text{ ft}$$

Co-ordinates of X

Line DX N $84^\circ 15' 30''$ 314·38 ft

$$\therefore \Delta\text{E} = 314{\cdot}38 \sin 84^\circ 15' 30'' = +312{\cdot}80$$
$$\text{E}_D = 10{\cdot}89$$
$$\text{E}_X = \underline{323{\cdot}69}$$
$$\Delta\text{N} = 314{\cdot}38 \cos 84^\circ 15' 30'' = +\ 31{\cdot}45$$
$$\text{N}_D = -108{\cdot}28$$

$$N_X = \underline{-76{\cdot}83}$$

$$CX = BX + BC$$

$$= 239{\cdot}71 + 359{\cdot}56 = 599{\cdot}27 \text{ ft}$$

$$\text{Radius } O_1C = \tfrac{1}{2}CX \operatorname{cosec} 27^\circ 00' 01''$$

$$= \frac{1}{2} \times 599{\cdot}27 \operatorname{cosec} 27^\circ 00' 01''$$

$$= 659{\cdot}986 \text{ ft}$$

$$\simeq 660{\cdot}0 \text{ ft} = \underline{10 \text{ chains}}$$

Ans. O_2C = 6 chains radius

O_1C = 10 chains radius

DX = 314·38 ft

Co-ordinates of X E 323·69, S 76·83

Exercises 11 (d) (Compound curves)

26. A main haulage road AD bearing due north and a branch road DB bearing N 87° E are to be connected by a compound curve formed by two circular curves of different radii in immediate succession. The first curve of 200 ft radius starts from a tangent point A, 160 ft due South of D and is succeeded by a curve of 100 ft radius which terminates at a tangent point on the branch road DB.

Draw a plan of the roadways and the connecting curve to the scale 1/500 and show clearly all construction lines.

Thereafter calculate the distance along the branch road from D to the tangent point of the second curve and the distance from A along the line of the first curve to the tangent point common to both curves.

(M.Q.B./S Ans. 120·2 ft; 145·4 ft)

27. The intersection point I between two railway straights $T'XI$ and IYT'' is inaccessible. Accordingly, two arbitrary points X and Y are selected in the straights and the following information is obtained:

Line	Whole circle bearing		
$T'XI$	49° 25′		
IYT''	108° 40′	Length XY	= 1684·0 ft
XY	76° 31′	Chainage of X	= 8562·3 ft

The straights are to be connected by a compound circular curve such that the arc $T'C$, of radius 3000 ft, is equal in length to the arc CT'', of radius 2000 ft, C being the point of compound curvature. Make the necessary calculations to enable the points T', C and T'' to be pegged initially, and show this information on a carefully dimensional sketch.

Determine also the through chainages of these points.

(L.U. $T'I$ = 1489·1 ft chainage T' 8115·9 ft
$T''I$ = 1235·5 ft chainage T'' 10 597·7 ft
C 9356·8 ft)

11.9 Reverse Curves

There are four cases to consider:

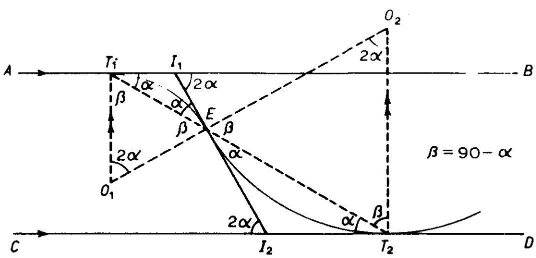
(1) *Tangents parallel* (a) Radii equal (b) Radii unequal.

(2) *Tangents not parallel* (a) Radii equal (b) Radii unequal.

Tangents parallel (*Cross-overs*) (Fig. 11.38)

$T_1 T_2$, $O_1 O_2$ and $I_1 I_2$ are all straight lines intersecting at E, the common tangent point,

Fig. 11.38



In Fig. 11.38,

$$\text{Angle } BT_1E = T_1EI_1 = \frac{1}{2}T_1O_1E = \frac{1}{2}BI_1I_2 = \alpha$$

$$\text{Angle } CT_2E = T_2EI_2 = \frac{1}{2}T_2O_2E = \frac{1}{2}CI_2I_1 = \alpha$$

Triangle T_1I_1E is similar to triangle ET_2I_2

and Triangle T_1EO_1 is similar to triangle EO_2T_2.

Tangents not parallel (Fig. 11.39)

N.B. T_1T_2 does not cut O_1O_2 at E.

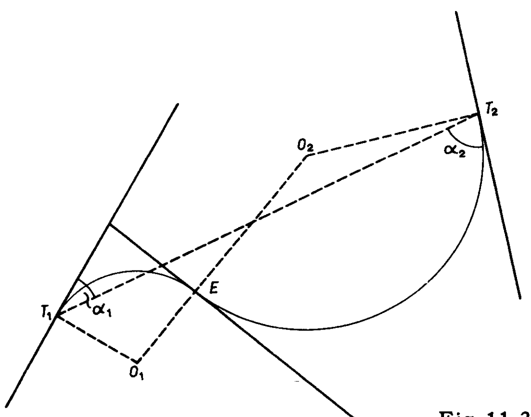


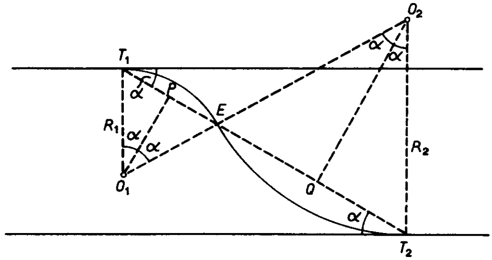
Fig. 11.39

N.B. These solutions are intended only as a guide to possible methods of approach.

Tangents parallel (Fig. 11.40)

Bisect T_1E and T_2E. Draw perpendiculars PO_1 and QO_2.

Fig. 11.40



$$T_1E = 2T_1P = 2R_1 \sin\alpha$$

$$T_2E = 2T_2Q = 2R_2 \sin\alpha = T_1T_2 - T_1E$$

$$2R_2 \sin\alpha = T_1T_2 - 2R_1 \sin\alpha$$

$$R_2 = \frac{T_1T_2}{2\sin\alpha} - R_1 \qquad (11.50)$$

If $R_1 = R_2$,

$$R = \frac{T_1T_2}{4\sin\alpha} \qquad (11.51)$$

Example 11.16 Two parallel railway lines are to be connected by a reverse curve, each section having the same radius. If the centre lines are 30 ft apart and the distance between the tangent points is 120 ft, what will be the radius of cross-over?

In Fig. 11.41,

$T_1A = 30$ ft

$T_1T = 120$ ft

In triangle T_1T_2A,

$$\sin\alpha = \frac{T_1A}{T_1T_2} = \frac{30}{120}$$

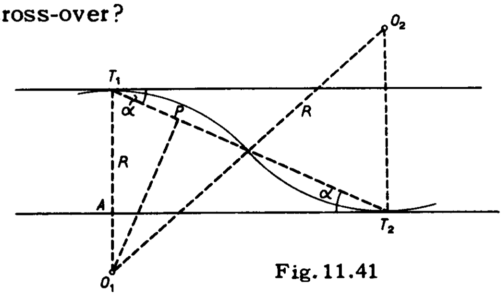


Fig. 11.41

In triangle T_1PO_1, $R = T_1O_1 = \dfrac{T_1P}{\sin\alpha} = \dfrac{T_1T_2}{4\sin\alpha} = \dfrac{120}{4 \times 30/120} = \underline{120 \text{ ft}}$

Tangents not parallel, radii equal (Fig. 11.42)

Construction

Draw O_1S parallel to T_1T_2, PO_1 perpendicular to T_1T_2, O_2Q perpendicular to T_1T_2

In triangle T_1PO_1,

$$T_1P = R\sin\alpha_1$$
$$PO_1 = R\cos\alpha_1 = QS$$

In triangle O_2T_2Q,

$$T_2Q = R\sin\alpha_2$$
$$O_2Q = R\cos\alpha_2$$
$$\begin{aligned} O_2S &= O_2Q + QS \\ &= R\cos\alpha_2 + R\cos\alpha_1 \\ &= R(\cos\alpha_1 + \cos\alpha_2) \end{aligned}$$

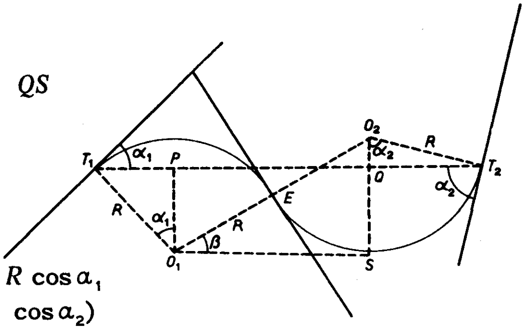


Fig. 11.42

$$\begin{aligned} \therefore\quad \beta &= \sin^{-1}\frac{O_2S}{O_1O_2} \\ &= \sin^{-1}\frac{R(\cos\alpha_1 + \cos\alpha_2)}{2R} \end{aligned}$$

$$\begin{aligned} O_1S &= 2R\cos\beta \\ &= T_1T_2 - (T_1P + QT_2) \\ &= T_1T_2 - R(\sin\alpha_1 + \sin\alpha_2) \end{aligned}$$

$$R = \frac{T_1T_2}{2\cos\beta + \sin\alpha_1 + \sin\alpha_2} \qquad (11.52)$$

Example 11.17 Two underground roadways AB and CD are to be connected by a reverse curve of common radii with tangent points at B and C. If the bearings of the roadways are AB S 83° 15′ E and CD CD S 74° 30′ E and the co-ordinates of B E 1125·66 ft N 1491·28 ft, C E 2401·37 ft N 650·84 ft, calculate the radius of the curve.

(M.Q.B./S)

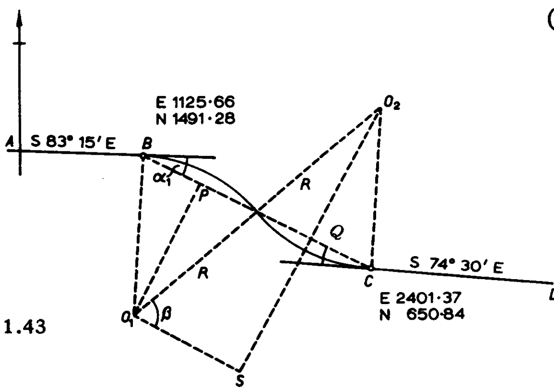


Fig. 11.43

The bearings of the tangents are different and thus the line BC does not intersect with O_1O_2 at the common tangent point although at this scale the plotting might suggest this.

In Fig. 11.43,

$$\text{Bearing } BC = \tan^{-1}\frac{2401{\cdot}37 - 1125{\cdot}66}{650{\cdot}84 - 1491{\cdot}28}$$

$$= \tan^{-1}\frac{1275{\cdot}71}{-840{\cdot}44}$$

$$= \text{S } 56°\,37'\,23''\text{ E}$$

Length BC = $840{\cdot}44 \sec 56°\,37'\,23''$ = $1527{\cdot}67$ ft

Bearing BC S 56° 37′ 23″ E

AB S 83° 15′ 00″ E $\therefore\ \alpha_1 = 26°\,37'\,37''$

CD S 74° 30′ 00″ E $\alpha_2 = 17°\,52'\,37''$

$$\beta = \sin^{-1}\tfrac{1}{2}(\cos 26°\,37'\,37'' + \cos 17°\,52'\,37'')$$
$$= 67°\,20'\,39''$$

$$O_1S = 2R\cos 67°\,20'\,39''$$

$$BP = R\sin\alpha_1$$
$$= R\sin 26°\,37'\,37''$$

$$QC = R\sin\alpha_2$$
$$= R\sin 17°\,52'\,37''$$

$$BC = BP + O_1S + QC$$

$$\therefore\ 1527{\cdot}67 = R\sin 26°\,37'\,37'' + 2R\cos 67°\,20'\,39'' + R\sin 17°\,52'\,37''$$

By Eq. (11.50),

$$R = \frac{1527{\cdot}67}{\sin 26°\,37'\,37'' + 2\cos 67°\,20'\,39'' + \sin 17°\,52'\,37''}$$
$$= \underline{1001{\cdot}31\text{ ft}}$$

Tangents not parallel, radii not equal (Fig. 11.44)

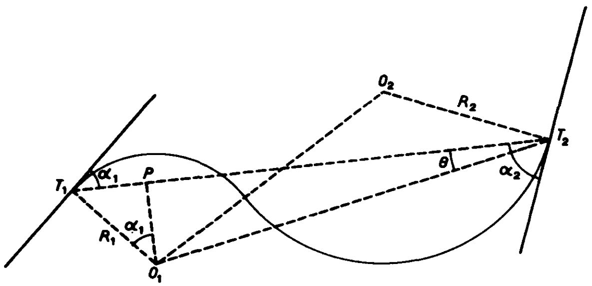


Fig. 11.44

Construction

Join O_1T_2. Draw O_1P perpendicular to T_1T_2

$$T_1P = R_1\sin\alpha_1$$
$$O_1P = R_1\cos\alpha_1$$

$$PT_2 = T_1T_2 - T_1P = T_1T_2 - R_1 \sin\alpha_1$$

$$\theta = \tan^{-1}\frac{O_1P}{PT_2} = \tan^{-1}\frac{R_1 \cos\alpha_1}{T_1T_2 - R_1 \sin\alpha_1} \tag{11.53}$$

$$O_1T_2 = \frac{O_1P}{\sin\theta} = \frac{R_1 \cos\alpha_1}{\sin\theta}$$

In triangle $O_1O_2T_2$,

$$O_1O_2^2 = O_2T_2^2 + O_1T_2^2 - 2O_2T_2\,O_1T_2 \cos O_1\hat{T}_2O_2$$

$$(R_1 + R_2)^2 = R_2^2 + \frac{R_1^2 \cos^2\alpha_1}{\sin^2\theta} - \frac{2R_2R_1 \cos\alpha_1 \cos\{90 - (\alpha_2 - \theta)\}}{\sin\theta}$$

$$R_1^2 + 2R_1R_2 + R_2^2 = R_2^2 + \frac{R_1^2 \cos^2\alpha_1 - 2R_1R_2 \cos\alpha_1 \sin(\alpha_2 - \theta)\sin\theta}{\sin^2\theta}$$

$$R_1(R_1 + 2R_2) = \frac{R_1(R_1 \cos^2\alpha_1 - 2R_2 \cos\alpha_1 \sin(\alpha_2 - \theta) \sin\theta)}{\sin^2\theta}$$

$$2R_2 \sin\theta\{\sin\theta + \cos\alpha_1 \sin(\alpha_2 - \theta)\} = R_1(\cos^2\alpha_1 - \sin^2\theta)$$

$$R_2 = \frac{R_1(\cos^2\alpha_1 - \sin^2\theta)}{2 \sin\theta\{\sin\theta + \cos\alpha_1 \sin(\alpha_2 - \theta)\}} \tag{11.54}$$

Example 11.18 Two straights AB and CD are to be joined by a circular reverse curve with an initial radius of 200 ft, commencing at B.

From the co-ordinates given below, calculate the radius of the second curve which joins the first and terminates at C.

		E	N
Co-ordinates (ft)	A	103·61	204·82
	B	248·86	422·62
	C	866·34	406·61
	D	801·63	141·88

(R.I.C.S./M)

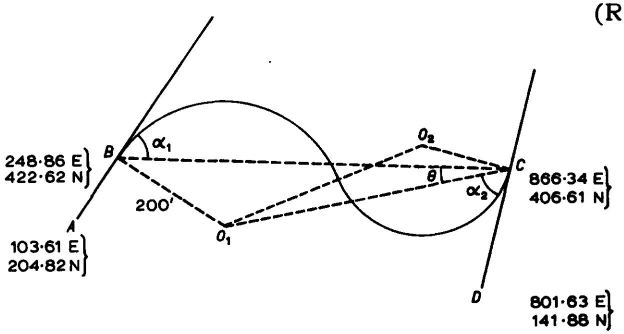


Fig. 11.45

In Fig. 11.45,

$$\text{Bearing } AB = \tan^{-1}\frac{(248{\cdot}86-103{\cdot}61)}{(422{\cdot}62-204{\cdot}82)} = \text{N } 33^\circ 42'00'' \text{ E} = \underline{033^\circ 42'00''}$$

$$DC = \tan^{-1}\frac{(866{\cdot}34-801{\cdot}63)}{(406{\cdot}61-141{\cdot}88)} = \text{N } 13^\circ 44'00'' \text{ E} = \underline{013^\circ 44'00''}$$

$$BC = \tan^{-1}\frac{(866{\cdot}34-248{\cdot}86)}{(406{\cdot}61-422{\cdot}62)} = \text{S } 88^\circ 30'50'' \text{ E} = 091^\circ 29'10''$$

$$\text{Length } BC = 617{\cdot}48/\cos 88^\circ 30'50'' = 617{\cdot}69 \text{ ft} \quad (T_1 T_2)$$

$$\text{Angle } \alpha_1 = 91^\circ 29'10'' - 033^\circ 42'00'' = 57^\circ 47' 10''$$

$$\alpha_2 = 91^\circ 29'10'' - 013^\circ 44'00'' = 77^\circ 45' 10''$$

By Eq. (11.53),

$$\theta = \tan^{-1}\frac{R_1 \cos\alpha_1}{T_1T_2 - R_1 \sin\alpha_1} = \tan^{-1}\frac{200 \cos 57^\circ 47' 10''}{617{\cdot}48 - 200 \sin 57^\circ 47' 10''}$$

$$= \underline{13^\circ 22' 20''}$$

By Eq. (11.54),

$$R_2 = \frac{200\,(\cos^2 57^\circ 47'10'' - \sin^2 13^\circ 22'20'')}{2 \sin 13^\circ 22'20'' \{\sin 13^\circ 22'20'' + \cos 57^\circ 47'10'' \sin(77^\circ 45'10'' - 13^\circ 22'20}$$

$$= \underline{140{\cdot}0 \text{ ft}}$$

Exercises 11(e) (Reverse curves)

28. A reverse curve is to start at a point A and end at C with a change of curvature at B. The chord lengths AB and BC are respectively 661·54 ft and 725·76 ft and the radius likewise 1200 and 1500 ft.

Due to irregular ground the curves are to be set out using two theodolites and no tape and chain.

Calculate the data for setting out and describe the procedure in the field.

(L.U. Ans. Total deflection angles, 16°; 14°;
Setting out by 1° 11′ 37″ deflections)

29. Two roadways AB and CD are to be connected by a reverse curve of common radius, commencing at B and C.

The co-ordinates of the stations are as follows:

A	21 642·87 m E	37 160·36 m N
B	21 672·84 m E	37 241·62 m N
C	21 951·63 m E	37 350·44 m N

If the bearing of the roadway CD is N 20° 14′ 41″ E, calculate the radius of the curve.

(Ans. 100·0 m)

30. CD is a straight connecting two curves AC and DB. The curve AC touches the lines AM and CD; the curve DB touches the lines CD and BN.

Given:		Bearing MA	=	165° 13′
		BN	=	135° 20′
		Radius of curve AC	=	750 m
		DB	=	1200 m

Co-ordinates	A	E + 1262·5 m	N − 1200·0 m
	B	0	0

Calculate the co-ordinates of C and D.

(Ans. C E + 1562·54 m N − 1358·58 m; D E + 57·32 m N − 53·09 m)

31. Two parallel lines which are 780 m apart are to be joined by a reverse curve ABC which deflects to the right by an angle of 20° from the first straight.

If the radius of the first arc AB is 1400 m and the chainage of A is 2340 m, calculate the radius of the second arc and the chainages of B and C. (Ans. 1934 m; 2828 m; 3503·8 m)

32. Two straight railway tracks 300 ft apart between centre lines and bearing N 12° E are to be connected by a reverse or 'S' curve, starting from the tangent point A on the centre line of the westerly track and turning in a north-easterly direction to join the easterly track at the tangent point C. The first curve AB has a radius of 400 ft and the second BC has a radius of 270 ft. The tangent point common to both curves is at B.

Calculate (a) the co-ordinates of B and C relative to the zero origin at A (b) the lengths of the curves AB and BC.

(M.Q.B./S Ans. (a) B E 244·53 N 288·99 ft
C E 409·58 N 484·05 ft
(b) 394·30 ft; 266·15 ft)

Bibliography

THOMAS, W.N., *Surveying* (Edward Arnold)
CLARK, D., *Plane and Geodetic Surveying* (Constable)
DAVIES R.E., FOOTE F.S., and KELLY, J.W., *Surveying Theory and Practice* (McGraw-Hill)
SANDOVER, J.A., *Plane Surveying* (Edward Arnold)
BANNISTER, A., and RAYMOND, S., *Surveying* (Pitman)
JAMESON, A.H., *Advanced Surveying* (Pitman)
SALMON, V.G., *Practical Surveying and Fieldwork* (Griffin)
JENKINS, R.B.M., *Curve Surveying* (Macmillan)
KISSAM, P., *Surveying for Civil Engineers* (McGraw-Hill)
SEARLES, W.H. and IVES, H.C., rev. by P. KISSAM, *Field Engineering*, 22nd Ed. (John Wiley)
HIGGINS, A.L., *Higher Surveying* (Macmillan)

12 VERTICAL AND TRANSITION CURVES

12.1 Vertical Curves

Where it is required to smooth out a change of gradient some form of parabolic curve is used. There are two general forms, (a) convex or summit curves, (b) concave, i.e. sag or valley curves.

The properties required are:

(a) Good riding qualities, i.e. a constant change of gradient and a uniform rate of increase of centrifugal force.

(b) Adequate sighting over summits or in underpasses.

The simple parabola is normally used because of its simplicity and constant change of gradient, but recently the cubic parabola has come into use, particularly for valley intersections. It has the advantage of a uniform rate of increase of centrifugal force and less filling is required.

Gradients are generally expressed as 1 in x, i.e. 1 vertical to x horizontal. For vertical curve calculations % gradients are used:

$$1 \text{ in } x = \frac{100}{x}\%$$

$$\text{e.g.} \quad 1 \text{ in } 5 = 20\%$$

Gradients rising left to right are considered $+ve$.

Gradients falling left to right are considered $-ve$.

The *grade angle* is usually considered to be the deflection angle = difference in % grade. By the convention used later, the value is given as $q\% - p\%$, Fig. 12.1.

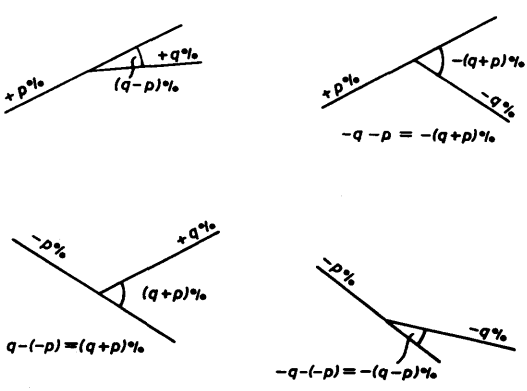


Fig. 12.1

12.2 Properties of the Simple Parabola

$$y = ax^2 + bx + c \tag{12.1}$$

$$\frac{dy}{dx} = 2ax + b$$

$$x = -\frac{b}{2a} \text{ for max or min} \tag{12.2}$$

$$\frac{d^2y}{dx^2} = 2a, \text{ i.e. constant rate of change of gradient} \tag{12.3}$$

If a is $+ve$, a valley curve is produced.

If a is $-ve$, a summit curve is produced.

The value of b determines the maximum or minimum position along the x axis.

The value of c determines where the curve cuts the y axis.

The difference in elevation between a vertical curve and a tangent to it is equal to half the rate of change of the gradient × the square of the horizontal distance from the point of tangency, Fig. 12.2.

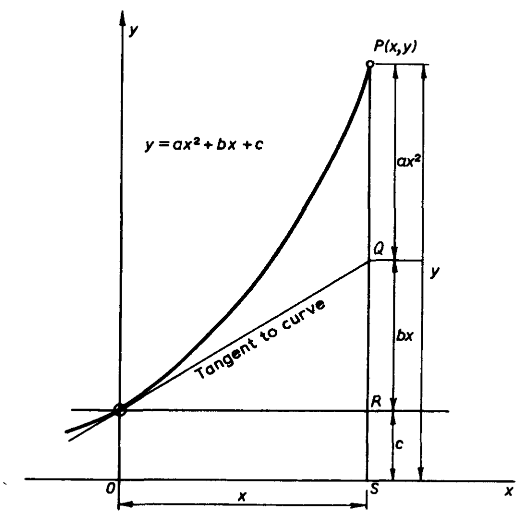


Fig. 12.2

If
$$y = ax^2 + bx + c,$$
$$\frac{dy}{dx} = 2ax + b \quad \text{(grade of tangent)}$$

When $x = 0$, grade of tangent $= +b$
value of $y = +c$

In Fig. 12.2, if the grade of the tangent at $x = 0$ is b,

$$\begin{aligned} QR &= b \times x \text{ i.e. } +bx \\ RS &= \qquad\qquad +c \\ \therefore\quad PQ &= \qquad\qquad ax^2 \\ \text{as}\quad y &= PQ + QR + RS \\ &= ax^2 + bx + c \end{aligned}$$

The horizontal lengths of any two tangents from a point to a vertical curve are equal, Fig. 12.3.

$PB = ax_1^2 = ax_2^2 \qquad (1)$

$\therefore x_1 = x_2$

i.e. the point of intersection B of two gradients is horizontally midway between the tangent points A and C.

A chord to a vertical curve has a rate of grade equal to the tangent at a point horizontally midway between the points of intercept, Fig. 12.3.

From (1),

$AA_1 = CC_1 = ax^2$

AA_1 is parallel to CC_1

$\therefore ACC_1A_1$ is a parallelogram

$\therefore AC$ is parallel to A_1C_1.

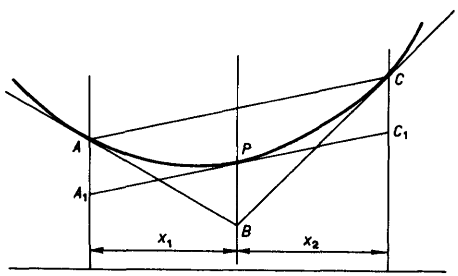


Fig. 12.3

12.3 Properties of the Vertical Curve

From the equation of the curve, $y = ax^2 + bx + c$ as seen previously.

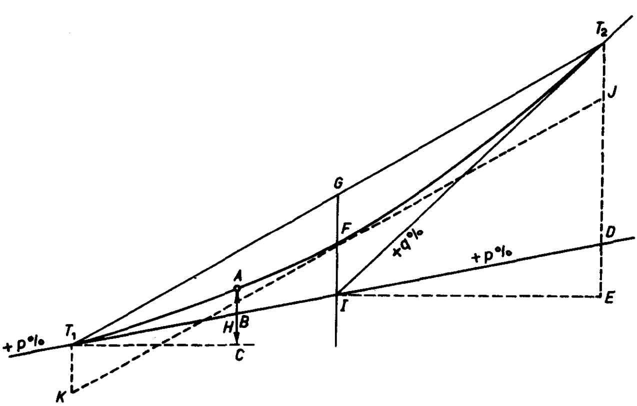


Fig. 12.4

In Fig. 12.4,

$$AB = ax^2$$
$$BC = bx$$

(N.B. For summit curves, a is negative)

$\therefore$ the level on the curve at $A = ax^2 + bx$

$$DT_2 = al^2 = ET_2 - ED$$
$$= \frac{ql}{200} - \frac{pl}{200} = \frac{l(q-p)}{200}$$

Half the rate of change of gradient

(N.B. a will be negative when $p > q$)

$$a = \frac{q-p}{200l} \tag{12.4}$$

Length of curve

$$l = \frac{q-p}{200\,a} \tag{12.5}$$

It is common practice to express the rate of change of gradient $2a$ as a % per 100 ft.

$\therefore$ the horizontal length of curve may be expressed as

$$l = \frac{100 \times \text{grade angle}}{\text{\% rate of change of grade per 100 ft}}$$
$$= \frac{100\,(q-p)}{2a\%\,\text{per }100} \tag{12.6}$$

Distance from the intersection point to the curve

$$IF = KT_1 = FG = JT_2$$
$$= a\left(\frac{l}{2}\right)^2$$
$$= \frac{l^2}{4} \times \frac{q-p}{200l} = \frac{l(q-p)}{800} \tag{12.7}$$

Maximum or minimum height on the curve

$$H = ax^2 + bx$$
$$= \frac{(q-p)x^2}{200\,l} + \frac{px}{100}$$

for max or min

$$\therefore \quad \frac{dH}{dx} = \frac{(q-p)\,x}{100\,l} + \frac{p}{100} = 0$$

then

$$x = \frac{-pl}{q-p} = \frac{pl}{p-q} \tag{12.8}$$

Example 12.1 A parabolic vertical curve of length 300 ft is formed at a summit between grades of 0·7 per cent up and 0·8 per cent down. The length of the curve is to be increased to 400 ft, retaining as much as possible of the original curve and adjusting the gradients on both sides to be equal. Determine the gradient. (L.U.)

From Eq. (12.4),

$$a = \frac{q-p}{200\,l}$$

$$= \frac{-0{\cdot}8 - 0{\cdot}7}{200 \times 300} = -\frac{1{\cdot}5}{60\,000}$$

If the gradients are to be made equal, $p = q$.

$$\therefore \quad p + q = 200\,al$$

$$2p = 200 \times 400 \times \frac{1{\cdot}5}{60\,000}$$

$$\underline{p = 1\%}$$

12.4 Sight Distances (s)

12.41 Sight distances for summits

(1) $s > l$

In Fig. 12.5,

Let l = horizontal length of vertical curve
s = sight distance
h_1 = AC = height of eye above road at A
h_2 = OL = height of object above road at O
d_1 = distance of vehicle from tangent point T_1
d_2 = distance of object from tangent point T_2

$$T_2J = \frac{lp}{200} + \frac{lq}{200} = \frac{l}{200}(p+q)$$

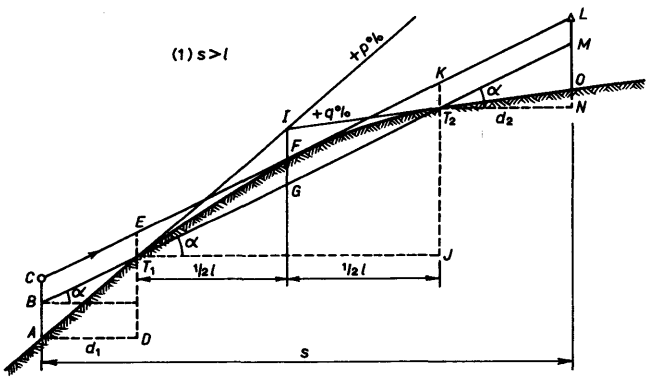


Fig. 12.5 Principles of sight distances

$$\tan\alpha = \frac{T_2J}{T_1J} = \frac{p+q}{200}$$

$$AB = \frac{d_1 p}{100} - d_1\tan\alpha = \frac{d_1 p}{100} - \frac{d_1(p+q)}{200} = \frac{d_1}{200}(2p-p-q)$$

$$= \frac{d_1}{200}(p-q)$$

Similarly,

$$MO = d_2\tan\alpha - \frac{d_2 q}{100} = \frac{d_2(p+q)}{200} - \frac{d_2 q}{100} = \frac{d_2}{200}(p+q-2q)$$

$$= \frac{d_2}{200}(p-q)$$

$$h_1 = AB + BC$$

$$= \frac{d_1}{200}(p-q) + a\left(\frac{l}{2}\right)^2$$

but a is negative, i.e. $a = \dfrac{-(q-p)}{200\,l}$

$$\therefore \quad h_1 = \frac{d_1}{200}(p-q) - \frac{(q-p)}{200\,l} \times \frac{l^2}{4}$$

$$= \frac{d_1}{200}(p-q) + \frac{l(p-q)}{800} = \frac{p-q}{800}[4d_1 + l]$$

Similarly,

$$h_2 = OM + BC = \frac{p-q}{800}[4d_2 + l]$$

$$s - l = d_1 + d_2 = \frac{800h_1 - l(p-q)}{4(p-q)} + \frac{800h_2 - l(p-q)}{4(p-q)}$$

$$= \frac{400(h_1+h_2) - l(p-q)}{2(p-q)}$$

$$2s - 2l = \frac{400(h_1+h_2)}{p-q} - l$$

$$l = 2s - \frac{400(h_1+h_2)}{p-q} \tag{12.9}$$

If $s = l$,

$$l = 2l - \frac{400(h_1+h_2)}{p-q}$$

$$l = \frac{400(h_1+h_2)}{p-q} \tag{12.10}$$

(2) $s < l$

In Fig. 12.6, $\quad h_1 = a d_1^2 \quad \therefore\ d_1 = \dfrac{\sqrt{h_1}}{\sqrt{a}}$

$$h_2 = a d_2^2 \qquad \therefore \; d_2 = \frac{\sqrt{h_2}}{\sqrt{a}}$$

$$s = \frac{1}{\sqrt{a}}[\sqrt{h_1} + \sqrt{h_2}]$$

But $$a = \frac{p - q}{200\,l}$$

$$\therefore \quad s^2 = \frac{200l}{p - q}[\sqrt{h_1} + \sqrt{h_2}]^2$$

$$\therefore \quad l = \frac{s^2(p - q)}{200(\sqrt{h_1} + \sqrt{h_2})^2} \tag{12.11}$$

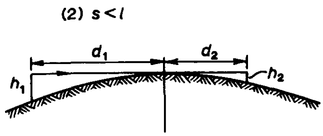


Fig. 12.6

N.B. If $h_1 = h_2 = h$:

(i) $s > l$ $$l = 2s - \frac{400 \times 2h}{p - q}$$

$$= 2s - \frac{800\,h}{p - q} \tag{12.12}$$

(ii) $s = l$ $$l = \frac{800\,h}{p - q} \tag{12.13}$$

(iii) $s < l$ $$l = \frac{s^2(p - q)}{200\,.\,[2(\sqrt{h})]^2}$$

$$= \frac{s^2(p - q)}{800\,h} \tag{12.14}$$

12.42 Sight distances for valley curves

Underpasses

Given: clearance height H,
height of driver's eye above road h_1,
height of object above road h_2,
sight length s,
gradients $p\%$ and $q\%$.

(1) $s > l$

Let the depth of the curve below the centre of the chord AD (the distance between the observer and the object) be M, Fig. 12.7.

$$\tan\alpha = \frac{GL}{\frac{1}{2}s} = \frac{DE}{s}$$

$$GL = GJ + JI + IL = M + \frac{al^2}{4} + \frac{sp}{200}$$

By Eq. (12.4), $$a = \frac{q - p}{200\,l}$$

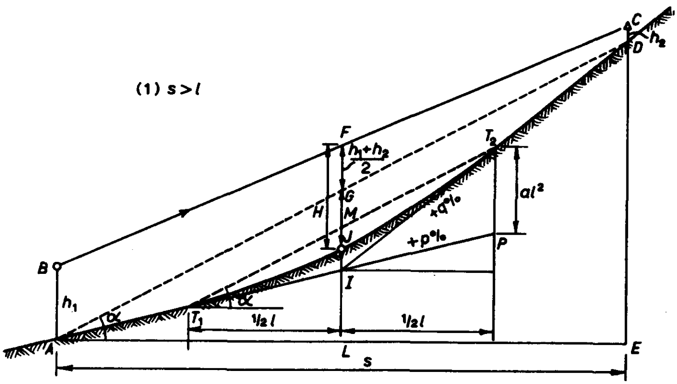


Fig. 12.7 Sight distances for valley curves

and from above,

$$DE = 2GL = \frac{sp}{200} + \frac{sq}{200} = \frac{s}{200}(p+q)$$

Then
$$GL = M + \frac{l(q-p)}{4 \times 200} + \frac{sp}{200} = \frac{s(p+q)}{400}$$

$$\therefore \quad l = \left[\frac{s(p+q)}{400} - \frac{sp}{200} - M\right] \div \frac{q-p}{800}$$

$$= \frac{2sp + 2sq - 4sp - 800M}{q-p}$$

$$= \frac{2s(q-p) - 800M}{q-p}$$

$$= 2s - \frac{800M}{q-p}$$

If $AL = LE$, then

$$M = H - \frac{h_1 + h_2}{2}$$

and
$$l = 2s - \frac{800}{q-p}\left[H - \frac{h_1+h_2}{2}\right] \qquad (12.15)$$

(2) $s \leqslant l$

In Fig. 12.8, let the depth of the curve below the centre of the chord $T_1 T_2 = M$

$$M = \frac{as^2}{4} = \frac{(q-p)s^2}{800l}$$

$$\therefore \quad l = \frac{s^2(q-p)}{800M}$$

$$= \frac{s^2(q-p)}{800\left(H - \frac{h_1+h_2}{2}\right)} \tag{12.16}$$

If $s = l$,

$$l = \frac{800\left(H - \frac{h_1+h_2}{2}\right)}{q-p} \tag{12.17}$$

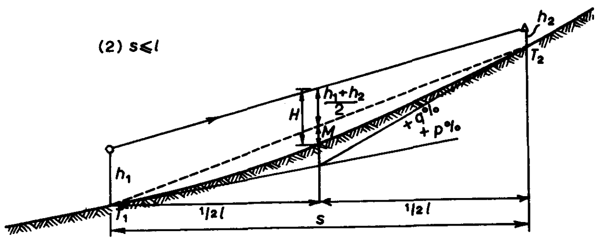


Fig. 12.8

12.43 Sight distance related to the length of the beam of a vehicle's headlamp

(1) $s > l$

In Fig. 12.9, the height of the beam $= h$ is at A,
the beam hits the road at T_2,
the angle of the beam is θ° above the horizontal axis of the vehicle.

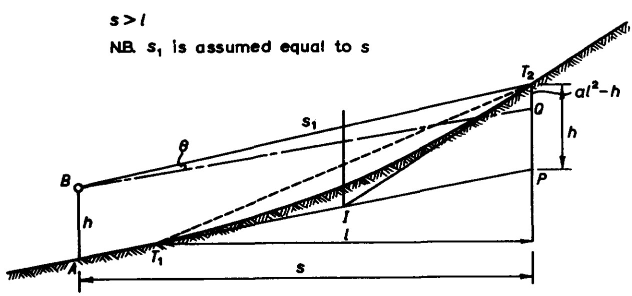


Fig. 12.9 Sight distance related to the beam of a headlamp

In triangle BT_2Q,

$$T_2Q = s\theta$$

i.e. $$al^2 - h = s\theta$$

But
$$a = \frac{q-p}{200\,l}$$

$$\therefore \qquad \frac{(q-p)\,l^2}{200\,l} = s\theta + h$$

$$l = \frac{200(s\theta + h)}{q-p} \tag{12.18}$$

Practice suggests that
$$\theta = 1°$$
$$h = 2{\cdot}5\text{ ft}$$

Then
$$l = \frac{200(0{\cdot}017\,5\,s + 2{\cdot}5)}{q-p} = \frac{3{\cdot}5s + 500}{q-p} \tag{12.19}$$

(2) $s \leqslant l$

As before, $as^2 - h = s\theta$

$$\frac{(q-p)s^2}{200\,l} = s\theta + h$$

$$l = \frac{s^2(q-p)}{200(s\theta + h)} \tag{12.20}$$

If $\theta = 1°$ and $h = 2{\cdot}5$ ft

$$l = \frac{s^2(q-p)}{3{\cdot}5s + 500} \tag{12.21}$$

Example 12.2. The sag vertical curve between gradients of 3 in 100 downhill and 2 in 100 uphill is to be designed on the basis that the headlamp sight distance of a car travelling along the curve equals the minimum safe stopping distance at the maximum permitted car speed. The headlamps are 2·5 ft above the road surface and their beams tilt upwards at an angle of 1° above the longitudinal axis of the car. The minimum safe stopping distance is 500 ft.

Calculate the length of the curve, given that it is greater than the sight distance. (L.U.)

$$p = -3\%$$
$$q = +2\%$$
$$s = 500\text{ ft}$$

By Eq. (12.21),

$$l = \frac{s^2(q-p)}{3{\cdot}5s + 500} = \frac{500^2 \times (2+3)}{500 \times 3{\cdot}5 + 500} = \frac{500 \times 5}{3{\cdot}5 + 1}$$

$$= \frac{2500}{4{\cdot}5} = \underline{555{\cdot}5\text{ ft}}$$

12.5 Setting-out Data

Gradients are generally obtained by levelling at chainage points, e.g. A, B, C and D, Fig. 12.10.

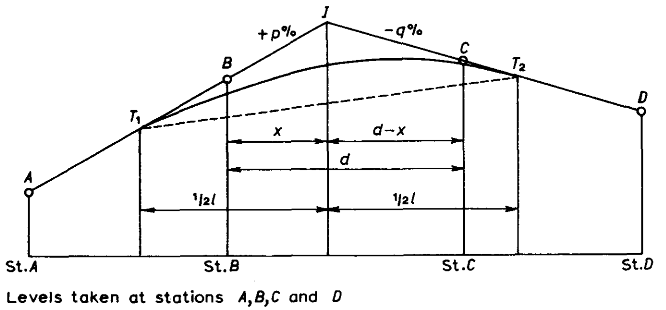


Fig. 12.10 Setting out data

Chainage and levels of I, T_1 and T_2.

$$\text{Level of } I = \text{level of } B + \frac{px}{100}$$

$$= \text{level of } c + \frac{q(d-x)}{100} \qquad (12.22)$$

Solving the equation gives the value of x.

$$\text{Chainage of } I = \text{chainage of } B + x \qquad (12.23)$$

$$\text{Chainage of } T_1 = \text{chainage of } I - \tfrac{1}{2}l \qquad (12.24)$$

$$\text{Chainage of } T_2 = \text{chainage of } I + \tfrac{1}{2}l \qquad (12.25)$$

$$\text{Level of } T_1 = \text{level of } I - \frac{lp}{200} \qquad (12.26)$$

$$\text{Level of } T_2 = \text{level of } I - \frac{lq}{200} \qquad (12.27)$$

Levels on the curve (Fig. 12.11)

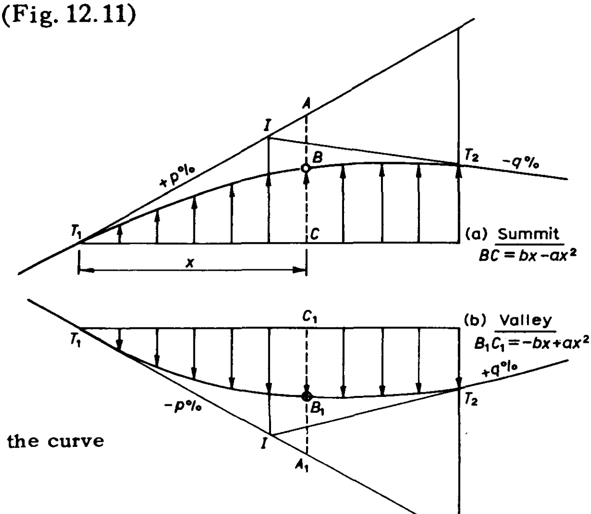


Fig. 12.11 Levels on the curve

$$\text{Levels on the tangent at } A = \text{level of } T_1 + bx \qquad (12.28)$$

where $b = \pm p/100$.

$$\text{Difference in level between tangent and curve} = \pm ax^2 \qquad (12.29)$$

$$\therefore \quad \text{Levels on the curve at } B = \text{level of tangent} \pm \text{difference in level between curve and tangent}$$

$$= T_1 \pm AC \mp AB \qquad (12.30)$$

Check on computation

(a) Tangent level is obtained by successive addition of the difference in level per station.

(b) Successive curve levels should check back to the level obtained by the spot level derived from $y = \mp ax^2 \pm bx$

(c) The final value of the check on T_2 will prove that the tangent levels have been correctly computed, though the curve levels may not necessarily be correct.

In order to define the shape of the curve, the values of a and b in the formula must be in some way obtained.

p and q will always be known.

$$\therefore \quad b = p/100$$

Either l, s or a must be given, and if the value of s is required, the height of the vehicle (h) must be known. For general purposes this is taken as 3·75 ft, and Ministry of Transport Memoranda on recommended visibility distances are periodically published.

Example 12.3. As part of a dual highway reconstruction scheme, a line of levels were taken at given points on the existing surface.

	Reduced level	Chainage
A	104·63	20 + 75
B	109·13	22 + 25
C	107·29	25 + 50
D	103·79	27 + 25

If the curve, based on a simple parabola, is designed to give a rate of change of gradient of 0·6% per 100 ft, calculate:

(a) the length of the curve l,

(b) the chainage and level of the intersection point,

(c) the chainage and level of the tangent points,

(d) the level of the first three chainage points on the curve (i.e. stations 100 ft apart based on through chainage),

(e) the length of the line of sight s to a similar vehicle of a driver 3 ft 9 in. above the road surface.

(N.B. $s < l$)

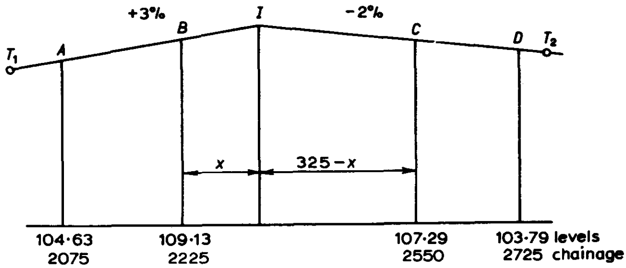


Fig. 12.12

(a) Gradient AB = (109·13 − 104·63) in (2225 − 2075)

$$= 4{\cdot}50 \text{ ft in } 150 \text{ ft}$$

i.e. +3%

CD = (107·29 − 103·79) in (1725 − 2550)

$$= 3{\cdot}50 \text{ ft in } 175 \text{ ft}$$

i.e. −2%

By Eq. (12.6), Length of curve $= \dfrac{100(3+2)}{0{\cdot}6} = \underline{833{\cdot}33 \text{ ft}}$

(b) By Eqs. (12.22/23),

$$\text{Level of } I = 109{\cdot}13 + \frac{3x}{100}$$

$$= 107{\cdot}29 + \frac{2(325-x)}{100}$$

Solving for x, $\underline{x = 93{\cdot}20 \text{ ft}}$

$$\underline{325 - x = 231{\cdot}80 \text{ ft}}$$

$$\therefore \quad \text{Level of } I = 109{\cdot}13 + \frac{3 \times 93{\cdot}2}{100} = 111{\cdot}93$$

$$\text{also} \quad = 107{\cdot}29 + \frac{2 \times 231{\cdot}8}{100} = 111{\cdot}93 \quad (\textit{check})$$

Chainage of I = Chainage of $B + x$

$$= (22 + 25) + 93{\cdot}20$$

$$= \underline{23 + 18{\cdot}20}$$

(c) By Eqs. (12.24/25),

Chainage of T_1 = 2318·20 − $l/2$

$$= 2318{\cdot}20 - 416{\cdot}67 = \underline{1901{\cdot}53 \text{ ft}}$$

Chainage of T_2 = 2318·20 + 416·67 = $\underline{2734{\cdot}87 \text{ ft}}$

By Eqs. (12.26/27),

$$\text{Level of } T_1 = 111{\cdot}93 - \frac{3 \times 833{\cdot}33}{200} = \underline{99{\cdot}43 \text{ ft}}$$

$$\text{Level of } T_2 = 111{\cdot}93 - \frac{2 \times 833{\cdot}33}{200} = \underline{103{\cdot}60 \text{ ft}}$$

(d) *Setting-out data*

Point	Chainage	Length (x)	Tangent Level	ax^2 $(0{\cdot}3 \times 10^{-4}x^2)$	Formation Level
T_1	1901·5	0	99·43	0	99·43
			+ 2·955		
	2000·0	98·5	102·385	−0·291	102·09
			+ 3·000		
	2100·0	198·5	105·385	−1·181	104·20
			+ 3·000		
	2200·0	298·5	108·385	−2·675	105·71

(e) Line of sight (s)

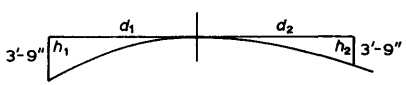


Fig. 12.13

In Fig. 12.13,

$$h_1 = a d_1^2 \qquad d_1 = \sqrt{\frac{h_1}{a}}$$

$$h_2 = a d_2^2 \qquad d_2 = \sqrt{\frac{h_2}{a}}$$

$$\therefore \quad s = d_1 + d_2 = \sqrt{\frac{h_1}{a}} + \sqrt{\frac{h_2}{a}}$$

but $\quad h_1 = h_2 \qquad a = 0{\cdot}3 \times 10^{-4}$

$$\therefore \quad s = 2\sqrt{\frac{h}{a}} = 200\sqrt{\frac{3{\cdot}75}{0{\cdot}3}}$$

$$= \underline{707 \text{ ft}}$$

Example 12.4. A 6% downgrade on a proposed road is followed by a 1% upgrade. The chainage and reduced level of the intersection point of the grades is 2010 ft and 58·62 ft respectively. A vertical parabolic curve, not less than 250 ft long, is to be designed to connect the two grades. Its actual length is to be determined by the fact that at chainage 2180 ft, the reduced level on the curve is to be 61·61 ft to provide adequate headroom under the bridge at that point.

Calculate the required length of the curve and also the chainage and reduced level of its lowest point. (R.I.C.S.)

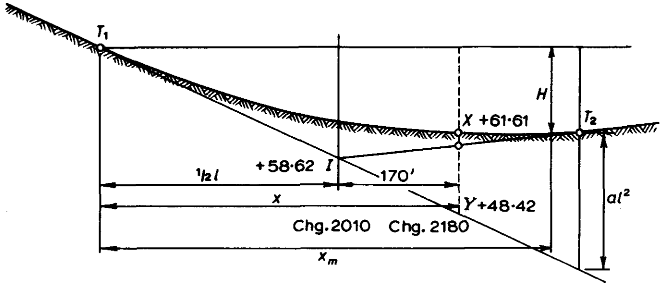


Fig. 12.14

In Fig. 12.14,

$$\text{let} \quad x - \frac{l}{2} = 170\,\text{ft}$$

$$\text{level at } I = 58{\cdot}62$$

$$\text{tangent level at } Y = 58{\cdot}62 - (170 \times 0{\cdot}06)$$

$$= 58{\cdot}62 - 10{\cdot}20 = 48{\cdot}42\,\text{ft}$$

$$\text{curve level at } X = 61{\cdot}61\,\text{ft}$$

$$\therefore \quad XY = 13{\cdot}19\,\text{ft}$$

Thus,

$$ax^2 = 13{\cdot}19$$

$$\text{i.e.} \quad a = \frac{13{\cdot}19}{x^2}$$

Also,

$$al^2 = \frac{pl}{200} + \frac{ql}{200} = \frac{l}{200}(p+q)$$

$$\therefore \quad a = \frac{p+q}{200\,l} = \frac{7}{200\,l}$$

Thus

$$\frac{13{\cdot}19}{x^2} = \frac{7}{200\,l}$$

$$l = \frac{7x^2}{200 \times 13{\cdot}19} = \frac{7x^2}{2638}$$

Thus

$$x - \frac{7x^2}{5276} = 170$$

i.e.

$$7x^2 - 5276\,x = 170 \times 5276$$

Solving for x,

$$x = \frac{+5276 \pm \sqrt{\{5276^2 - 4 \times 7 \times 170 \times 5276\}}}{14}$$

$$= \frac{5276 \pm 1649{\cdot}9}{14} = 494{\cdot}7 \text{ or } 259{\cdot}0\ \text{ft}$$

i.e.
$$\frac{l}{2} = 324{\cdot}7 \quad \text{or} \quad 89{\cdot}0 \text{ ft}$$

$$\underline{l = 650 \text{ ft}} \quad \text{(length must not be less than 250 ft)}$$

To find the minimum height on the curve,

$$H = aX^2 - bX$$
$$= \frac{(p+q)X^2}{200\,l} - \frac{pX}{100}$$
$$\frac{dH}{dX} = \frac{2(p+q)X}{200\,l} - \frac{p}{100} = 0$$
$$\therefore \quad X = \frac{pl}{p+q}$$
$$= \frac{6 \times 650}{7} = \underline{557{\cdot}1 \text{ ft}}$$

$$\text{Chainage of minimum height} = 2010 + 557{\cdot}1 - \frac{650}{2}$$
$$= \underline{2242{\cdot}1 \text{ ft}}$$

Level at minimum height:

$$\text{Level at } T_1 = 58{\cdot}62 + 0{\cdot}06 \times (325) = 78{\cdot}12 \text{ ft}$$
$$\text{Level of tangent} = 78{\cdot}12 - 0{\cdot}06 \times (557{\cdot}1) = 44{\cdot}70$$
$$aX^2 = \frac{7 \times 557{\cdot}1^2}{200 \times 650} = 16{\cdot}71 \text{ ft}$$
$$\therefore \quad \text{Level on curve} = 44{\cdot}70 + 16{\cdot}71$$
$$= \underline{61{\cdot}41 \text{ ft}}$$

Example 12.5. On the application of the cubic parabola for valley curves.

A valley curve of length 400 ft is to be introduced into a road to link a descending gradient of 1 in 30 and an ascending gradient of 1 in 25. It is composed of two cubic parabolas, symmetrical about the bisector of the angle of intersection of the two straights produced.

The chainage of the P.I. of the straights is 265 + 87 ft, and its reduced level 115·36 ft.

Calculate:

(i) The reduced levels of the beginning, the mid-point and the end of the curve.

(ii) The chainage and reduced level of the lowest point of the curve.

(iii) The reduced level at chainage 267 + 00 ft.

(The formula for the cubic parabola relating to the curve of total length L and terminal radius R is $Y = \frac{x^3}{6RL}$) (N.U.)

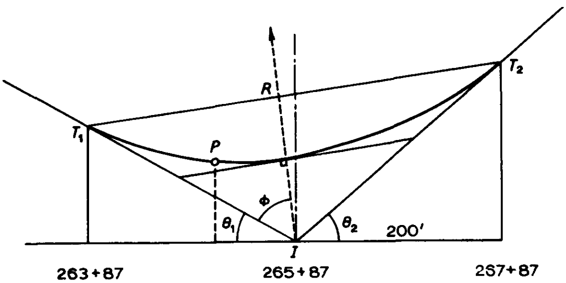


Fig. 12.15

From the gradients, Fig. 12.15, $\theta_1 = \cot^{-1}30 = 1° 54' 33''$

$$\theta_2 = \cot^{-1}25 = 2° 15' 27''$$

$$\phi = \tfrac{1}{2}[180 - (\theta_1 + \theta_2)] = 87° 55' 00''$$

From the equation, $Y = \frac{x^3}{6RL}$

$$\frac{dy}{dx} = \frac{3x^2}{6RL} = \cot\phi$$

and if $x = L = 400/2$, then $R = 100 \tan\phi$

$$= 100 \times 27{\cdot}49 = \underline{2749 \text{ ft}}$$

At l, $y = \frac{L^2}{6R} = \frac{40\,000}{6 \times 2749} = 2{\cdot}43$

(i) As the gradients are small, the values of y are assumed vertical.

$\therefore$ Level of curve at the mid point $= 115{\cdot}36 + 2{\cdot}43 = \underline{117{\cdot}79 \text{ ft}}$

Level at $T_1 = 115{\cdot}36 + \frac{200}{30} = 115{\cdot}36 + 6{\cdot}67 = \underline{122{\cdot}03 \text{ ft}}$

Level at $T_2 = 115{\cdot}36 + \frac{200}{25} = 115{\cdot}36 + 8{\cdot}00 = \underline{123{\cdot}36 \text{ ft}}$

(ii) Level at the lowest point P (relative to T_1) $= \frac{-x}{30} + \frac{x^3}{6RL}$

$$\frac{dL}{dx} = -\frac{1}{30} + \frac{x^2}{2RL} = 0 \quad \text{(for min value)}$$

$$x^2 = \frac{2RL}{30} = \frac{2 \times 2749 \times 200}{30}$$

$$\therefore \quad x = 191 \text{ ft from } T_1$$

The chainage of the lowest point P

$$= \text{chainage of } T_1 + x$$

$$= (263 + 87) + (1 + 91) = \underline{265 + 78 \text{ ft}}$$

$$\text{Level of } P = -\frac{191}{30} + \frac{191^3}{6 \times 2749 \times 200} + 122{\cdot}03$$

$$= -6{\cdot}37 + 2{\cdot}11 + 122{\cdot}03 = \underline{117{\cdot}77 \text{ ft}}$$

(iii) At chainage 267 + 00, i.e. 87 ft from T_2,

$$y = \frac{87^3}{6 \times 2749 \times 200} = 0{\cdot}199 \text{ ft, i.e. } \underline{0{\cdot}20 \text{ ft}}$$

$$\therefore \quad \text{Level at } 267 + 00 = 123{\cdot}36 - \frac{87}{25} + 0{\cdot}20$$

$$= \underline{120{\cdot}08 \text{ ft}}$$

Exercises 12(a)

1. An uphill gradient of 1 in 100 meets a downhill gradient of 0·45 in 100 at a point where the chainage is 61+00 and the reduced level is 126 ft. If the rate of change of gradient is to be 0·18 % per 100 ft, prepare a table for setting out a vertical curve at intervals of 100 ft.

(I.C.E. Ans. 121·97, 122·88, 123·61, 124·16, 124·53, 124·72, 124·73, 124·56, 124·19)

2. (a) A rising gradient of 1 vertically to 200 horizontally, is to be joined by a rising gradient of 1 in 400 by a 400 ft long parabolic curve. If the two gradients meet at a level of 365·00 ft A.O.D., tabulate the levels on the curve at 50 ft intervals.

(b) Recalculate on the basis that the first gradient is falling and the second likewise falling in the same direction.

(L.U. Ans. 364·00, 364·24, 364·47, 364·68, 364·87, 365·05, 365·22, 365·37, 365·50;
366·00, 365·76, 365·53, 365·32, 365·13, 364·95, 364·78, 364·63, 364·50)

3. A rising gradient, g_1, is followed by another rising gradient g_2 (g_2 less than g_1). These gradients are connected by a vertical curve having a constant rate of change of gradient. Show that at any point on the curve, the height y above the first tangent point A is given by $y = g_1 x - \dfrac{(g_1 - g_2)x^2}{2L}$ where x is the horizontal distance of the point from A_1 and L is the horizontal distance between the tangent points.

Draw up a table of heights above A for 100 ft pegs from A, when $g_1 = +5\%$, $g_2 = +2\%$ and $L = 1000$ ft. At what horizontal distance from A is the gradient $+3\%$?

(I.C.E. Ans. 4·85, 9·40, 13·65, 17·60, 21·25, 24·60, 27·65, 30·40, 32·85, 35·00, 667 ft)

4. A rising gradient of 1 in 100 meets a falling gradient of 1 in 150 at a level of 210·00. Allowing for headroom and working thickness, the vertical parabolic curve joining the two straights is to be at a level of 208·00 at its midpoint.

Determine the length of the curve and the levels at 100 ft intervals from the first tangent point.

(L.U. Ans. 960 ft; 200·40, 206·11, 206·85, 207·42, 207·81, 208·00, 208·03, 208·07, 207·94, 207·63, 207·15, 206·80)

5. On a straight portion of a new road, an upward gradient of 1 in 100 was connected to a gradient of 1 in 150 by a vertical parabolic summit curve of length 500 ft. A point P, at chainage 59 100 ft on the first gradient, was found to have a reduced level of 45·12 ft, and at point Q, at chainage 60 000 ft on the second gradient, of 44·95 ft.

(a) Find the chainages and reduced levels of the tangent points to the curve.

(b) Tabulate the reduced levels of the points on the curve at intervals of 100 ft from P at its highest point.

Find the minimum sighting distance to the road surface for each of the following cases:

(c) the driver of a car whose eye is 4 ft above the surface of the road.

(d) the driver of a lorry for whom the similar distance is 6 ft.

(Take the sighting distance as the length of the tangent from the driver's eye to the road surface.)

(L.U. Ans. (a) 59 200, 46·12, 59 700, 46·94
(b) 46·12, 46·95, 47·45, 47·62 (highest point) 47·45, 46·94.
(c) 605 ft (d) 845 ft

6. A rising gradient of 1 in 500 meets another rising gradient of 1 in 400 at a level of 264·40 ft, and a second gradient 600 ft long then meets a falling gradient of 1 in 600. The gradients are to be joined by two transition curves, each 400 ft long.

Calculate the levels on the curves at 100 ft intervals.

(L.U. Ans. 264·00, 264·21, 264·42, 264·65, 264·90, 265·15, 265·40, 265·61, 265·70, 265·69, 265·58)

7. A falling gradient of 4% meets a rising gradient of 5% at chainage 2450·0 ft and level 216·42 ft.

At chainage 2350, the underside of a bridge has a level of 235·54 ft. The two gradients are to be joined by a vertical parabolic curve giving 14 ft clearance under the bridge. List the levels at 50 ft intervals along the curve.

(L.U. Ans. 224·42, 222·70, 221·54, 220·95, 220·92, 221·45, 222·54, 224·20, 226·42)

8. The surface of a length of a propsed road of a rising gradient of 2% is followed by a falling gradient of 4% with the two gradients joined by a vertical parabolic summit curve 400 ft long. The two gradients produced meet at a reduced level of 95·00 ft O.D.

Compute the reduced level of the curve at the ends, at 100 ft intervals, and at the highest point.

What is the minimum distance at which a driver whose eye is 3 ft 9 in. above the road surface would be unable to see an obstruction 4 inches high?

(I.C.E. Ans. 91·00, 92·25, 92·00, 90·25, 87·00 ft A.O.D. highest point 92·33 ft AOD; sight distance 290 ft)

9. An existing length of road consists of a rising gradient of 1 in 20, followed by a vertical parabolic summit curve 300 ft long, and then a falling gradient of 1 in 40. The curve joins both gradients tangentially and the reduced level of the highest point on the curve is 173·07 ft above datum.

Visibility is to be improved over the stretch of road by replacing this curve with another parabolic curve 600 ft long.

Find the depth of excavation required at the midpoint of the curve. Tabulate the reduced levels of points at 100 ft intervals on the new curve.

What will be the minimum visibility on the new curve for a driver whose eyes are 4·0 ft above the road surface?

(I.C.E. Ans. 2·81 ft; 160·57, 164·95, 168·08, 169·95, 170·61, 169·99, 168·07 ft A.O.D.; minimum visibility 253 ft)

10 A vertical curve 400 ft long of the parabolic type is to join a falling gradient of 1 in 200 to a rising gradient of 1 in 300. If the level of the intersection of the two gradients is 101·20 ft, give the levels at 50 ft intervals along the curve.

If the headlamp of a car was 1·25 ft above the road surface, at what distance will the beam strike the road surface when the car is at the start of the curve? Assume that the beam is horizontal when the car is on a level surface.

(L.U. Ans. 102·20, 101·98, 101·80, 101·68, 101·62, 101·60, 101·64, 101·72, 101·86; 347 ft)

11. A vertical parabolic curve 500 ft long connects an upward gradient of 1 in 100 to a downward gradient of 1 in 50. If the tangent point T_1 between the first gradient and the curve is taken as datum, calculate the levels of points at intervals of 100 ft along the curve, until it meets

the second gradient at T_2.

Calculate also the level of the summit giving the horizontal distance of this point from T_1.

If an object 3 in. high is lying on the road between T_1 and T_2 at 10 ft from T_2 and a car is approaching from the direction of T_1, calculate the position of the car when the driver first sees the object, if his eyes are 4 ft above the road surface.

(L.U. Ans. 0·70, 0·80, 0·30, −0·80, −2·50, 0·84, 166·67 ft 33·6 ft from T_1)

12. A parabolic vertical curve of length L is formed at a summit between an uphill gradient of $a\%$ and a downhill gradient of $b\%$. As part of a road improvement, the uphill gradient is reduced to $c\%$ and the downhill gradient increased to $d\%$, but as much as possible of the original curve is retained.

Show that the length of the new vertical curve is

$$L \times \frac{(c+d)}{(a+b)} \qquad \text{(L.U.)}$$

13. The algebraic difference in the gradient of a sag vertical curve L ft long is a ft/ft. The headlamps of a car travelling along this curve are 2·5 ft above the road surface and their beams tilt upwards at an angle of 1° above the longitudinal axis of the car.

Show that if s, the sight distance in feet, is less than L, then,

$$L = \frac{as^2}{5 + 0{\cdot}035\,s} \qquad \text{(L.U.)}$$

12.6 Transition Curves

12.61 Superelevation (θ)

A vehicle of weight W (lbf) or (kgf) on a curve of radius r (ft) or (m), is travelling at a velocity v (ft/s) or (m/s) or V (mile/h). Fig. 12.16 shows the centrifugal force Wv^2/gr which must be resisted by either (a) the rails in the case of a railway train, or (b) adhesion between the road and the vehicle's tyres, unless superelevation is applied, when the forces along the plane are equalised.

Then
$$\frac{Wv^2\cos\theta}{gr} = W\sin\theta \tag{12.31}$$

i.e.
$$\tan\theta = \frac{v^2}{gr} \tag{12.32}$$

If θ is small,

$$\theta_{rad} = \frac{v^2}{gr} \quad \text{(known as the centripetal ratio)} \tag{12.33}$$

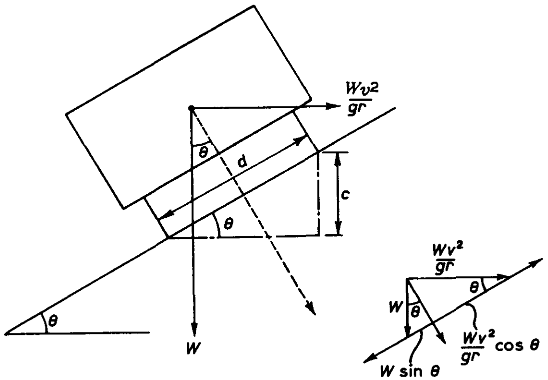


Fig. 12.16

12.62 Cant (c)

If d is the width of the track, then the cant c is given as

$$c = d\sin\theta \tag{12.34}$$

$$= \frac{d\,v^2}{gr}\cos\theta \tag{12.35}$$

$$\simeq \frac{d\,v^2}{gr} \quad \text{(if } \theta \text{ is small)} \tag{12.36}$$

N.B. $c \propto v^2$
$\propto 1/r$

On railways c is usually limited to 6 in. with a 4 ft 8½in. gauge.
On roads $\tan\theta$ is usually limited to 0·1.

12.63 Minimum curvature for standard velocity

Without superelevation on roads, side slip will occur if the side thrust is greater than the adhesion, i.e. if $Wv^2/gr > \mu W$, where μ = the coefficient of adhesion, usually taken as 0·25.

Thus the limiting radius $$r = \frac{v^2}{\mu g} \tag{12.37}$$

If the velocity V is given in mile/h,

$$r = \frac{V^2}{15\,\mu}\,\text{ft} \tag{12.38}$$

12.64 Length of transition

Various criteria are suggested:

(1) An arbitrary length of say 200 ft.

(2) The total length of the curve divided into 3 equal parts, 1/3 each transition; 1/3 circular.

(3) An arbitrary gradient of 1 in. in s ft, e.g. 1 in. in 25 ft – the steepest gradient recommended for railways.

(4) At a limited rate of change of radial acceleration (W.H. Short, 1908, fixed 1 ft/s^3 as a suitable value for passenger comfort.)

(5) At an arbitrary time rate i.e. 1 to 2 in per second.

N.B. (4) is the most widely adopted.

12.65 Radial acceleration

The radial acceleration increases from zero at the start of the transition to v^2/r at the join with the circular curve.

The time taken $$t = \frac{l}{v},$$ where l = the length of transition.

∴ the rate of gain of radial acceleration

$$a = \frac{v^2}{r} \div \frac{l}{v}$$

$$= \frac{v^3}{rl} \text{ ft/s}^3 \qquad (12.39)$$

Thus the length of the curve l is given as

$$l = \frac{v^3}{ar} \qquad (12.40)$$

If a is limited to 1 ft/s³

then $$l = \frac{v^3}{r} \text{ (ft)} \qquad (12.41)$$

$$= \frac{3{\cdot}155\, V^3}{r} \text{ (ft)} \qquad (12.42)$$

On sharp radius curves, the value of a would be too great, so the superelevation is limited and the speed must be reduced.

From Eq. (12.32), $$\tan\theta = \frac{v^2}{gr}$$

then $$v = \sqrt{(g \,.\, r \tan\theta)} \qquad (12.43)$$

but $$l = \frac{v^3}{ar}$$

$$\therefore \qquad l = (g \tan\theta)^{3/2} \frac{\sqrt{r}}{a} \qquad (12.44)$$

i.e. when c is limited, $l \propto \sqrt{r}$.

For railways with a maximum superelevation of 6 in.,

$$\sin\theta = \frac{6 \text{ in.}}{4 \text{ ft } 8\tfrac{1}{2} \text{ in.}} = 0{\cdot}1008$$

$$\therefore \qquad \theta = 5^\circ 47'$$

$$\tan\theta = 0{\cdot}1013.$$

Thus the maximum speed should be:

$$v = \sqrt{(32{\cdot}2 \times 0{\cdot}1013\, r)}$$

$$= 1{\cdot}806\sqrt{r} \quad \text{ft/s} \qquad (12.45)$$

$$\simeq 2\sqrt{r} \quad \text{ft/s} \qquad (12.46)$$

If $a = 1 \text{ ft/s}^3$,

$$l = \frac{v^3}{r} = \frac{(2\sqrt{r})^3}{r}$$

$$= 8\sqrt{r} \qquad (12.47)$$

If l and r are given in Gunter chains,

$$66\, l = 8\sqrt{(66\, r)}$$

$$l \simeq \sqrt{r} \quad \text{chains} \qquad (12.48)$$

12.7 The Ideal Transition Curve

If the centrifugal force $F = Wv^2/gr$ is to increase at a constant rate, it must vary with time and therefore, if the speed is constant, with distance.

i.e. $$F \propto l \propto \frac{Wv^2}{gr}$$

$$\therefore \qquad l \propto \frac{1}{r} \quad \text{i.e. } rl = \text{constant} = RL$$

where R = the radius of the circular curves

L = the total length of the transition.

In Fig. 12.17,

$$\delta l = r\delta\phi$$

$$d\phi = \frac{1}{r} dl$$

but $$rl = RL = \text{constant } k$$

$$d\phi = \frac{l}{k}\, dl$$

Integrating, $$\phi = \frac{l^2}{2k} + c$$

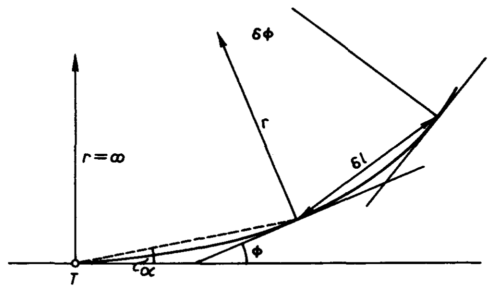


Fig. 12.17 The ideal transition curve

$$\text{but} \quad \phi = 0 \quad \text{when } l = 0$$

$$\therefore \quad c = 0$$

$$\therefore \quad \phi = \frac{l^2}{2RL} \tag{12.49}$$

$$l = M\sqrt{\phi}$$

$$\text{where } M = \sqrt{2RL} \tag{12.50}$$

This is the intrinsic equation of the clothoid, to which the cubic parabola and lemniscate are approximations often adopted when the deviation angle is small, Fig. 12.18.

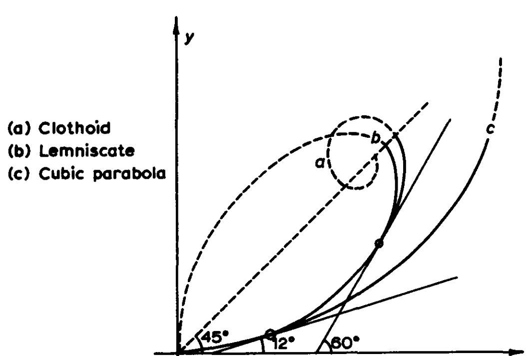


Fig. 12.18

(a) Clothoid $l = M\sqrt{\phi}$

(b) Lemniscate $c^2 = a^2 \sin 2\theta$

(c) Cubic parabola $y = \dfrac{x^3}{6RX}$ $\left(y = \dfrac{l^3}{6RL} \text{ cubic spiral}\right)$

12.8 The Clothoid

$$l = \sqrt{(2RL)}\sqrt{\phi}$$
$$\phi = l^2/2RL$$

where R = the minimum radius (i.e. the radius of the circular curve).

As the variable angle ϕ cannot be measured from one position, it is difficult to set out in this form.

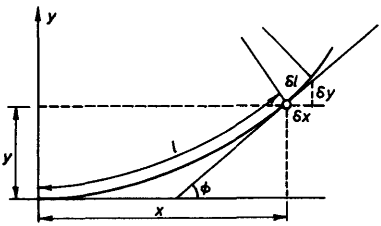


Fig. 12.19 Clothoid

12.81 To find Cartesian co-ordinates

$$\frac{dx}{dl} = \cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\cdots \tag{12.51}$$

$$= 1 - \frac{l^4}{2!(2RL)^2} + \frac{l^8}{4!(2RL)^4}\cdots \tag{12.52}$$

Integrating,

$$x = l\left[1 - \frac{l^4}{5\times 2!(2RL)^2} + \frac{l^8}{9\times 4!(2RL)^4}\cdots\right] \tag{12.53}$$

$$= l\left[1 - \frac{\phi^2}{5\times 2!} + \frac{\phi^4}{9\times 4!}\cdots\right] \tag{12.54}$$

For ϕ_{max} $l = L$

then
$$x \simeq l\left[1 - \frac{l^2}{40R^2}\right] \tag{12.55}$$

Similarly,

$$\frac{dy}{dl} = \sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}\cdots \tag{12.56}$$

$$= \frac{l^2}{2RL} - \frac{l^6}{3!(2RL)^3} + \frac{l^{10}}{5!(2RL)^5}\cdots \tag{12.57}$$

Integrating,

$$y = l\left[\frac{l^2}{3(2RL)} - \frac{l^6}{7\times 3!(2RL)^3} + \frac{l^{10}}{11\times 5!(2RL)^5}\cdots\right] \tag{12.58}$$

$$= l\left[\frac{\phi}{3} - \frac{\phi^3}{7\times 3!} + \frac{\phi^5}{11\times 5!}\cdots\right] \tag{12.59}$$

For ϕ_{max},
$$y \simeq \frac{l^2}{6R}\left[1 - \frac{l^2}{56R^2}\right] \tag{12.60}$$

12.82 The tangential angle α

$$\text{Tan}\,\alpha = \frac{y}{x} = \frac{\phi}{3} + \frac{\phi^3}{105} + \ldots \tag{12.61}$$

but
$$\alpha = \tan\alpha - \tfrac{1}{3}\tan^3\alpha + \tfrac{1}{5}\tan^5\alpha \ldots \tag{12.62}$$

By substitution,
$$\alpha = \frac{\phi}{3} - \frac{8\phi^3}{2835} \ldots \tag{12.63}$$

i.e. $\alpha = \dfrac{\phi}{3} - k$ (a rapidly decreasing quantity) (12.64)

Thus, if ϕ is small,
$$\alpha = \frac{\phi}{3} \tag{12.65}$$

Jenkins* shows that if $\phi < 6°$, no correction is required; and if $\phi < 20°$ no correction $> 20''$ is required.

12.83 Amount of shift (s)

The shift is the displacement of the circular curve from the tangent, i.e. DF, Fig. 12.20.

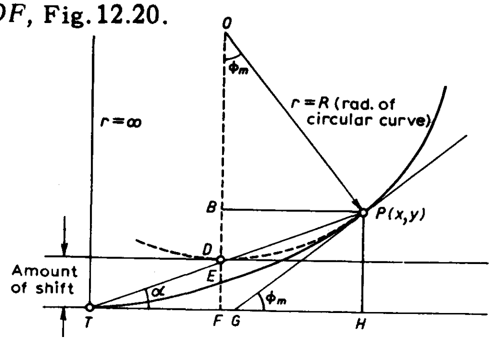


Fig. 12.20 Amount of shift

By Eq. (12.59), $PH = BF = y = L\left[\dfrac{\phi_m}{3} - \dfrac{\phi_m^3}{7 \times 3!} + \ldots\right]$

$$DF = BF - BD = y - R(1 - \cos\phi) \tag{12.66}$$

Expanding $\cos\phi$ and putting $\phi_{max} = \dfrac{L^2}{2RL} = \dfrac{L}{2R}$ (12.67)

$$DF = \frac{L^2}{24R}\left[1 - \frac{\phi_m}{28} + \ldots\right] \tag{12.68}$$

* R.B.M. Jenkins, *Curve Surveying* (Macmillan).

12.9 The Bernouilli Lemniscate

Polar equation $c^2 = a^2 \sin 2\alpha$

Identical to clothoid for deviation angles up to 60° (radius decreases up to 135°), Fig. 12.21,

$$c \text{ is maximum} = a \text{ when } \alpha = 45^\circ$$

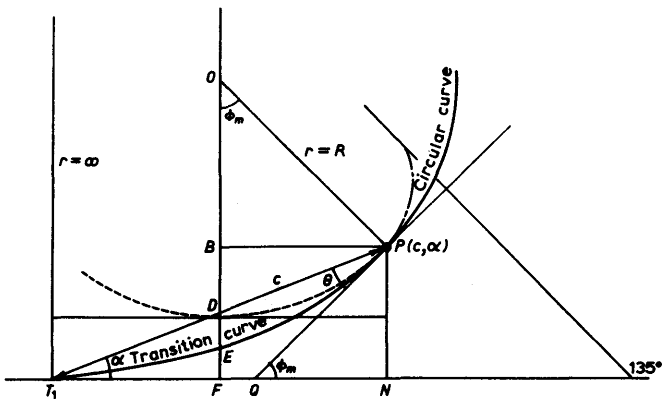


Fig. 12.21 Principles of the Bernouilli lemniscate

Referring to Fig. 12.22,

$$c^2 = a^2 \sin 2\alpha \tag{12.69}$$

$$2c\frac{dc}{d\alpha} = 2a^2 \cos 2\alpha$$

$$\div 2c^2 \text{ gives} \quad \frac{1}{c}\frac{dc}{d\alpha} = \frac{2a^2 \cos 2\alpha}{2a^2 \sin 2\alpha} = \cot 2\alpha$$

If θ = angle T_1PQ and P_1 and P_2 are 2 neighbouring points,

$$\cot\theta = \frac{P_1M}{P_2M} \qquad (P_2MP_1 = 90^\circ)$$

$$= \frac{\delta c}{c\,\delta\alpha}$$

$$\text{i.e.} \quad \frac{1}{c}\frac{dc}{d\alpha} = \cot\theta$$

$$\therefore \theta = 2\alpha$$

$$\underline{\phi = \theta + \alpha = 3\alpha} \tag{12.70}$$

as shown in the clothoid and the cubic parabola.

$$P_1P_2 = \delta l = \frac{P_2M}{\sin\theta} = \frac{c\,\delta\alpha}{\sin\theta}$$

$$= \frac{c\,\delta\alpha}{\sin 2\alpha}$$

$$\therefore \frac{dl}{d\alpha} = \frac{c}{\sin 2\alpha} = \frac{c}{c^2/a^2} = \frac{a^2}{c}$$

Now $dl = rd\phi = 3rd\alpha$

$$\therefore \frac{dl}{d\alpha} = 3r = \frac{a^2}{c}$$

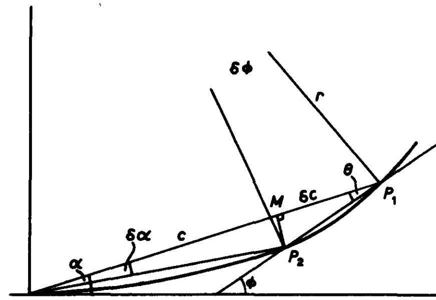


Fig. 12.22

Thus $\quad a^2 = 3rc \qquad (12.71)$

i.e., from Eq. (12.69) $\quad c^2 = 3Rc\sin 2\alpha$ if the lemniscate approximates to the circle radius R

$$\underline{c = 3R\sin 2\alpha} \qquad (12.72)$$

In Fig. 12.21

$$\begin{aligned} OF &= OB + BF \\ &= r\cos\phi + c\sin\alpha \\ &= r\cos\phi + 3r\sin\alpha\sin 2\alpha \\ &= r[\cos 3\alpha + 3\sin\alpha\sin 2\alpha] \qquad (12.73) \end{aligned}$$

$$\begin{aligned} T_1F &= T_1N - FN \\ &= c\cos\alpha - r\sin\phi \\ &= 3r\sin 2\alpha\cos\alpha - r\sin 3\alpha \\ &= r[3\sin 2\alpha\cos\alpha - \sin 3\alpha] \qquad (12.74) \end{aligned}$$

12.91 Setting out using the lemniscate

Using Eq. (12.72),

$$c = 3R\sin 2\alpha$$

Shift

$$\begin{aligned} DF &= OF - OD \\ &= R[\cos 3\alpha + 3\sin\alpha\sin 2\alpha] - R \quad \text{where } \alpha = \frac{\phi}{3} \\ &= R[\cos 3\alpha + 3\sin\alpha\sin 2\alpha - 1] \qquad (12.75) \end{aligned}$$

Equal values of α enable values of c to be computed, or equal chords c, $2c$, $3c$, etc.

$$\sin 2\alpha = \frac{c}{3R} \qquad (12.76)$$

If α is small,
$$\alpha'' = \frac{206\,265\,c}{6R} \tag{12.77}$$
$$\alpha_{max} = \frac{206\,265\,L}{6R} \tag{12.78}$$

(The same value as with the cubic spiral)

For offset values,
$$y = c \sin \alpha \tag{12.79}$$
$$x = c \cos \alpha \tag{12.80}$$

If the chord lengths are required between adjacent points on the curve, the length of the chord
$$c' = \sqrt{(x^2 + y^2)} \tag{12.81}$$

12.10 The cubic parabola

This is probably the most widely used in practice because of its simplicity. It is almost identical with the clothoid and lemniscate for deviation angles up to 12°. The radius of curvature reaches a minimum for deviation angles of 24° 06′ and then increases. It is therefore not acceptable beyond this point.

Let
$$y = \frac{x^3}{k} \tag{12.82}$$
$$\frac{dy}{dx} = \frac{3x^2}{k} \tag{12.83}$$
$$\frac{d^2y}{dx^2} = \frac{6x}{k} \tag{12.84}$$

By the calculus the curvature (ρ) is given as
$$\rho = \frac{1}{r} = \frac{d^2y/dx^2}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}} \tag{12.85}$$

If ϕ is small, dy/dx is small and $(dy/dx)^2$ is neglected.
$$\therefore \quad \frac{1}{r} = \frac{d^2y}{dx^2} = \frac{6x}{k} \tag{12.86}$$
$$\therefore \quad k = 6rx = 6RX \text{ at the end of the curve} \tag{12.87}$$

Equation of the cubic parabola
$$y = \frac{x^3}{6RX} \tag{12.88}$$

If the deviation angle ϕ is small, $x \simeq l$, $X \simeq L$.

Equation of the cubic spiral
$$y = \frac{l^3}{6RL} \tag{12.89}$$

N.B. This is the first term in the clothoid series.

$$\phi \simeq \tan\phi = \frac{dy}{dx} = \frac{3x^2}{6RX} = \frac{x^2}{2RX} \tag{12.90}$$

$$\alpha \simeq \tan\alpha = \frac{y}{x} = \frac{x^2}{6RX} \tag{12.91}$$

$$\therefore \alpha = \frac{1}{3}\phi \tag{12.92}$$

as in the first term of the clothoid series.

In Fig. 12.21,

$$PB = R\sin\phi \simeq R\phi = X/2 \tag{12.93}$$

i.e. the shift bisects the length.

$$DB = R(1 - \cos\phi) \tag{12.94}$$

Shift

$$DF = y - R(1 - \cos\phi) \tag{12.95}$$

$$= \frac{X^3}{6RX} - 2R\sin^2\frac{1}{2}\phi = \frac{X^2}{6R} - \frac{2R\phi^2}{4}$$

$$= \frac{X^2}{6R} - \frac{X^2}{8R}$$

$$= \frac{X^2}{24R} \simeq \frac{L^2}{24R} \tag{12.96}$$

the first term in the Clothoid series .

As E is on transition and $TF = FH = \frac{1}{2}X$,

$$EF = \frac{\left(\frac{X}{2}\right)^3}{6RX} = \frac{X^3}{48RX}$$

$$= \frac{X^2}{48R} = \frac{1}{2}DF \tag{12.97}$$

i.e. the transition bisects the shift.

12.11 The Insertion of Transition Curves

The insertion of transition curves into the existing alignment of straights is done by one of the following alternatives.

(1) The radius of the existing circular curve is reduced by the amount of 'shift', Fig. 12.23. The centre O is retained.

$$R_1 - R_2 = \text{shift }(s) = \frac{L^2}{24R}$$

$$T_1'T_1 \simeq \frac{1}{2}L$$

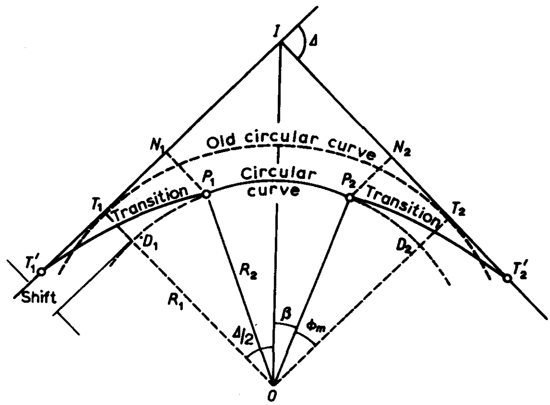


Fig. 12.23

(2) The radius and centre O are retained and the tangents are moved outwards to allow transition, Fig. 12.24.
(N.B. Part of the original curve is retained.)

$$I_1I_2 = OI_2 - OI_1$$
$$= \frac{R+S}{\cos\frac{\Delta}{2}} - \frac{R}{\cos\frac{\Delta}{2}}$$
$$= \frac{S}{\cos\frac{\Delta}{2}}$$

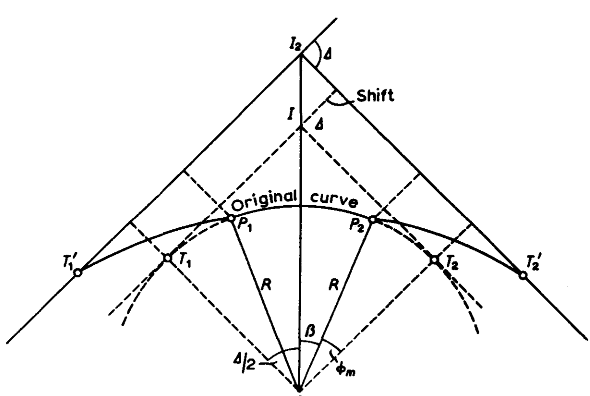


Fig. 12.24

(3) The radius of the curve is retained, but the centre O is moved away from the intersection point, Fig. 12.25.

$$O_1O_2 = \frac{\text{shift}}{\cos \frac{1}{2}\Delta}$$

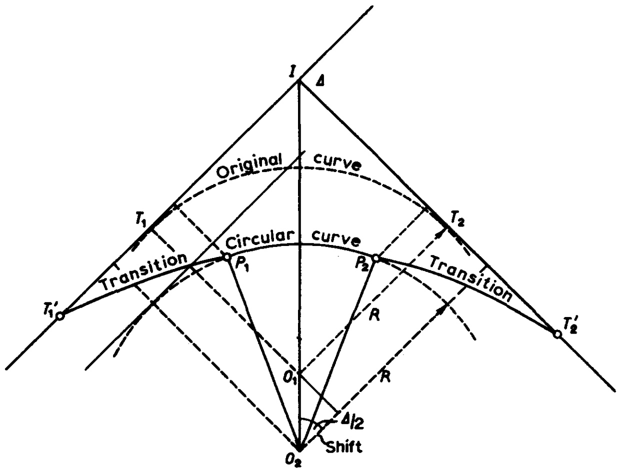


Fig. 12.25

(4) Tangent, radius and part of the existing curve are retained, but a compound circular curve is introduced to allow shift, Fig. 12.26.

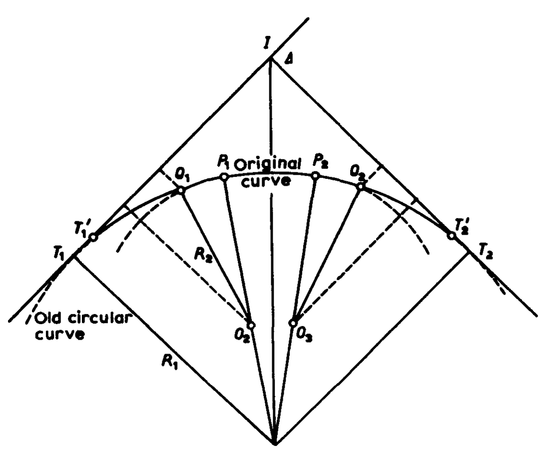


Fig. 12.26

(5) A combination of any of these forms.

12.12 Setting-out Processes

Location of tangent points (Fig. 12.27)

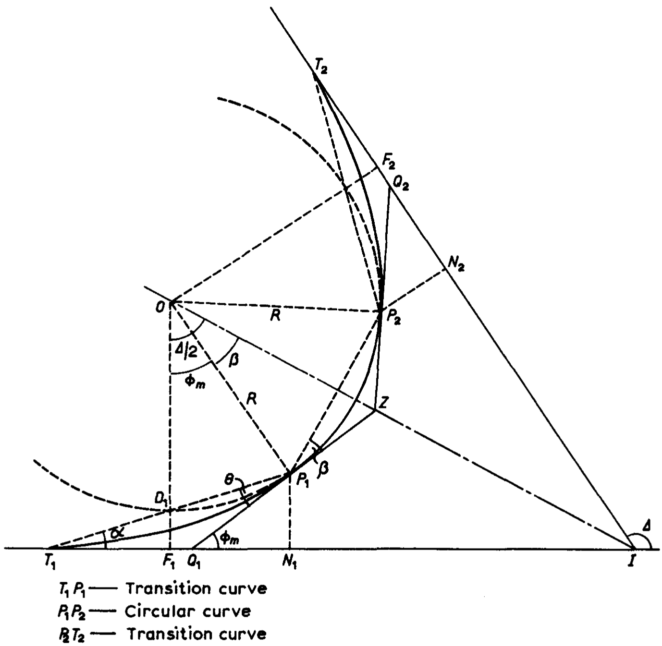


Fig. 12.27 Setting-out processes

By Eq. (12.96),

$$FD = \text{shift}\ (s) = \frac{L^2}{24R}$$

$$FI = (R+s)\tan\tfrac{1}{2}\Delta$$

$$FT \simeq \tfrac{1}{2}L$$

$$\text{Tangent length } T_1I = T_2I = \tfrac{1}{2}L + (R+s)\tan\tfrac{1}{2}\Delta \quad (12.98)$$

Setting out (Fig. 12.27)

(1) Produce straights (if possible) to meet at I. Measure Δ.

(2) Measure tangent lengths $IT_1 = IT_2$ to locate T_1 and T_2.

(3) Set out transition from T_1. Two methods are possible, (a) by offsets from the tangent, (b) by tangential angles. For either method, accuracy is reduced when

$$l > 0{\cdot}4R \qquad \text{i.e.} \qquad \phi > 12°$$

Method (b) is more accurate, even assuming chord = arc.

(4) *Offsets from the tangent* (Fig. 12.28)

From Eq. (12.88),

$$y_1 = \frac{x_1^3}{6RL}$$

If $x_2 = 2x_1$, $x_3 = 3x_1$

then $y_2 = 2^3 y_1 = 8y_1$

$y_3 = 3^3 y_1 = 27y_1$

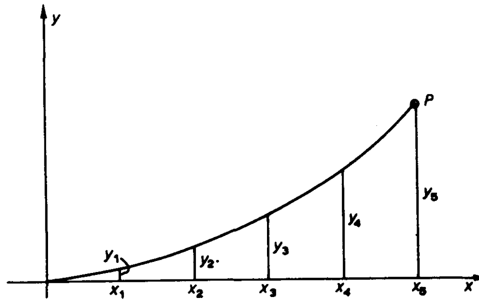


Fig. 12.28 Offsets from the tangent

(5) *Tangential angles* α (Fig. 12.29)

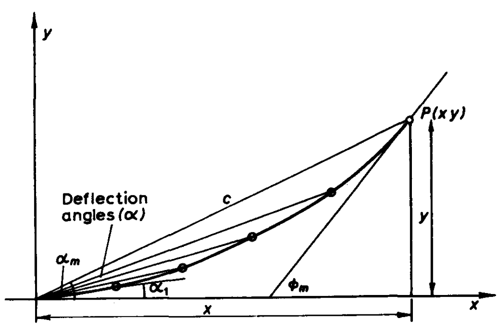


Fig. 12.29 Deflection angles from the tangent

From Eq. (12.89)

$$\tan\alpha = \frac{y}{x}$$

$$= \frac{x^2}{6RL}$$

If α is small, then

$$\alpha'' = \frac{206\,265\,x^2}{6RL} \tag{12.99}$$

if $x \simeq c$

$$\simeq \frac{206\,265\,c^2}{6RL} \tag{12.100}$$

$$\alpha' \simeq 573\,c^2/RL \tag{12.101}$$

For α''_{max} $c \simeq L$

$$\therefore \quad \alpha''_m = \frac{206\,265\,c}{6R} \tag{12.102}$$

$$\alpha'_m = 573\,c/R$$

$$\phi''_{\max} = 3\alpha_m = 206\,265c/2R \tag{12.103}$$

$$\phi'_m = 1719\ c/R \tag{12.104}$$

(6) *Check on* P_1 (Join of transition to circular curve)

$$N_1P_1 = N_2P_2 = \frac{L^2}{6R} \tag{12.105}$$

(7) Move theodolite to P_1.

Set out circular curve by offsets or deflection angles from the tangent QPZ.

N.B. Angle $T_1P_1Q_1 = \theta = 2\alpha = \frac{2}{3}\phi_m$ (12.106)

Check $ZP_1P_2 = \beta = \frac{1}{2}\Delta - \phi_m$ (12.107)

$P_1P_2 = 2R\sin\beta$ (12.108)

Check $N_2P_2 = N_1P_1 = \dfrac{L^2}{6R}$ Eq. (12.105)

(8) Move theodolite to T_2 and set out the transition backwards towards P_2 as in (3).

N.B. The use of metric units does not make any difference to the solution, providing these are compatible, i.e.

v – m/s

L and R – m

a – ft/s³ converted to m/s³

(e.g. 1 ft/s³ = 0·305 m/s³)

Example 12.6. A circular curve of 2000 ft radius deflects through an angle of 40° 30′. This curve is to be replaced by one of smaller radius so as to admit transitions 350 ft long at each end. The deviation of the new curve from the old at their midpoint is 1·5 ft towards the intersect tion point.

Determine the amended radius assuming the shift can be calculated with sufficient accuracy on the old radius. Calculate the length of the track to be lifted and the new track to be laid.

(L.U.)

By (12.94) Shift $s_1 = \dfrac{L^2}{24R} = \dfrac{350^2}{24 \times 2000} = 2{\cdot}55$ ft.

Tangent length of circular curve $(T_1 I) = 2000 \tan 40°\ 30\frac{1}{2} = 737{\cdot}84$ ft.

In triangle O_1IT_1 $\quad O_1I = \dfrac{2000}{\cos 20°\ 15'} = 2131{\cdot}8$ ft

$X_1I = 2131{\cdot}8 - 2000 = 131{\cdot}8$ ft

new value $\quad X_2I = 131{\cdot}8 - 1{\cdot}5 = 130{\cdot}3$ ft

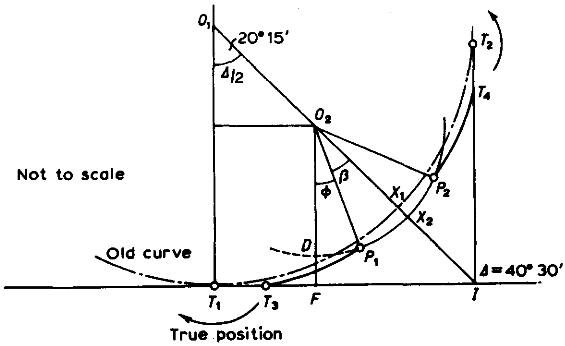


Fig. 12.30

In triangle O_2IF $\quad \dfrac{R+s_1}{R+130{\cdot}3} = \cos 20°\,15' = 0{\cdot}93\,819$

i.e. $\quad R + 2{\cdot}55 = (R + 130{\cdot}3)\,0{\cdot}93\,819$

$$\therefore \quad R = \frac{122{\cdot}3 - 0{\cdot}6}{0{\cdot}06\,181} = \underline{1936{\cdot}5\text{ ft}}$$

By (12.103)
$$\phi_{max} = 3\alpha = \frac{206\,265 \times 350}{2 \times 1936{\cdot}5} = 18\,639''$$
$$= 5°\,10'\,39''$$

By (12.107)
$$\beta = \tfrac{1}{2}\Delta - \phi_{max}$$
$$= 20°\,15' - 5°\,10'\,39'' = 15°\,04'\,21''$$

To find length of track to be lifted ($T_1 T_2$)

Length of circular curve $= 2000 \times 40°\,30'_{rad} = 1413{\cdot}72$ ft

Tangent length $\quad T_3 I = \dfrac{L}{2} + (R + s)\tan\dfrac{\Delta}{2}$

new shift $\quad s_2 = s_1 \times \dfrac{2000}{1936{\cdot}5}$

$$= 2{\cdot}55 \times \frac{2000}{1936{\cdot}5} = 2{\cdot}64$$

$$T_3 I = 175{\cdot}0 + (1936{\cdot}5 + 2{\cdot}64)\tan 20°\,15'$$
$$= 175{\cdot}0 + 715{\cdot}39 = 890{\cdot}39\text{ ft}$$

but $\quad T_1 I = 737{\cdot}84$ ft

$\therefore \quad T_1 T_3 = 152{\cdot}55$ ft

therefore total length of track to be lifted

$$= \text{length of arc} + 2 \times T_1 T_3$$
$$= 1413{\cdot}72 + 305{\cdot}10 = \underline{1718{\cdot}8\text{ ft}}$$

To find the length of track to be laid

$$= 2 \times \text{transition curve } (T_3 P_1) + \text{circular arc } (P_1 P_2)$$
$$= 2 \times 350 + 2 \times 1936{\cdot}5\beta$$
$$= 700 + 3873{\cdot}0 \times 0{\cdot}26306$$
$$= 700 + 1018{\cdot}83 \qquad = \underline{1718{\cdot}8 \text{ ft.}}$$

12.13 Transition Curves Applied to Compound Curves

In this case, the transition would be applied at the entry and exit (i.e. at T_1 and T_2 (Fig. 12.31)

The amount of superelevation cannot be designed to conform to both circular arcs and the design speed must relate to the smaller radius.

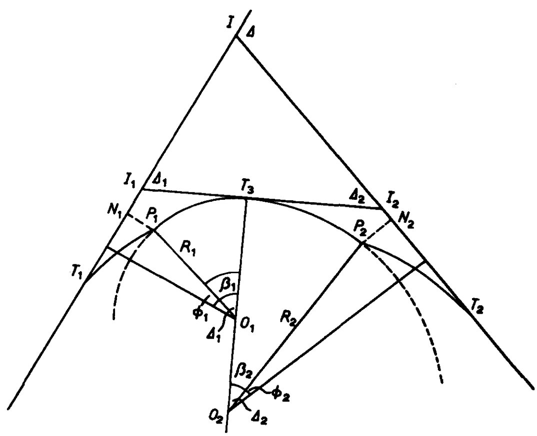


Fig. 12.31 Transition curves applied to compound curves

If the two curves are to be connected by a transition curve of length l, the shortest distance c (Fig. 12.32) between the two circular curves is given by Glover* as

$$c = \frac{L^2}{24}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (12.109)$$

The length of the transition L, is bisected at Q by this shift c and the shift is bisected by the transition.

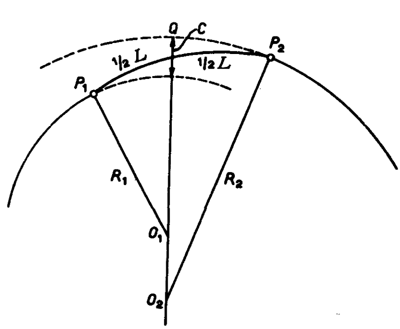


Fig. 12.32

*Transition curves for railways, *Proc.Inst.Civ.Eng.*, Vol. 140.

If transitions are applied to reverse curves, the radii must be reduced to allow the transition curves to be introduced, Fig. 12.33.

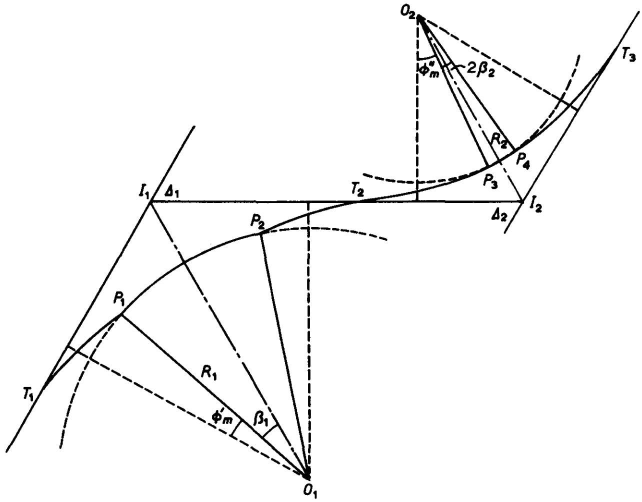


Fig. 12.33 Transitions applied to reverse curves

$$I_1I_2 = I_1T_2 + T_2I_2$$
$$= (R_1 + s_1)\tan\tfrac{1}{2}\Delta_1 + \tfrac{1}{2}L_1 + \tfrac{1}{2}L_2 + (R_2 + s_2)\tan\tfrac{1}{2}\Delta_2 \qquad (12.110)$$

If $L = \sqrt{R}$ (based on Gunter chains),

$$\text{Shift } s_1 = s_2 = \frac{L^2}{24R} = \frac{R}{24R} = \frac{1}{24} \text{ chains} \qquad (12.111)$$

$$\therefore \quad I_1I_2 = (R_1 + \tfrac{1}{24})\tan\tfrac{1}{2}\Delta_1 + \tfrac{1}{2}\sqrt{R_1} + \tfrac{1}{2}\sqrt{R_2} + (R_2 + \tfrac{1}{24})\tan\tfrac{1}{2}\Delta_2 \qquad (12.112)$$

If $R_1 = R_2$, then

$$I_1I_2 = (R + \tfrac{1}{24})(\tan\tfrac{1}{2}\Delta_1 + \tan\tfrac{1}{2}\Delta_2) + \sqrt{R} \qquad (12.113)$$

This may be solved as a quadratic in $\sqrt{R}$

Example 12.7. Two railway lines have straights which are deflected through 70°. The circular radius is to be 1500 feet with a maximum superelevation of 5 in. The gradient of the line is to be 1 in. in 1 chain (Gunter).

Calculate the distance from the beginning of the transition to the intersection point (i.e. tangent length), the lengths of the separate portions of the curve and sufficient data for setting out the curve by

offsets from the tangent and by the method of tangential angles.

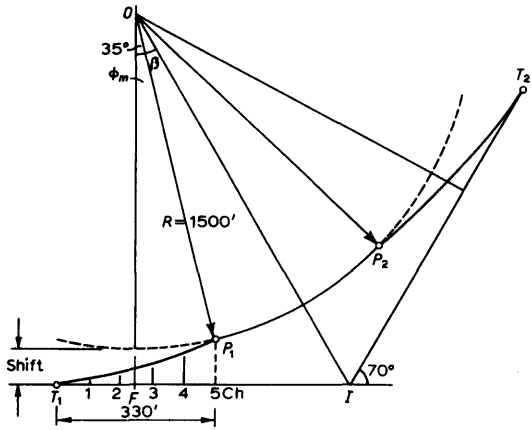


Fig. 12.34

Gradient is 1 in. in 66 ft, therefore for the total superelevation it is 5in.

$$L = 5 \times 66\text{ ft} = 330\text{ ft}$$

$$\text{Shift} = \frac{L^2}{24R} = \frac{330^2}{24 \times 1500}$$

$$= \underline{3{\cdot}03\text{ ft}}$$

$$T_1I = 1503{\cdot}03\tan 35° + \frac{330}{2}$$

$$= \underline{1217{\cdot}43\text{ ft}}$$

$$\phi_{max} = \tan^{-1}\frac{L}{2R} = \tan^{-1}\frac{330}{3000} = 6°\ 16'\ 37''$$

$$\beta = 35° - 6°\,16'\,37'' = 28°\,43'\,23''$$

$$\text{Length of circular curve} = 1500 \times 57°\,26'\,46''_{rad} = 1503{\cdot}93\text{ ft}$$

Offsets from tangents $\qquad y = \dfrac{x^3}{6RL}$

Let the curve be subdivided into 5 equal parts. (1 chain each.)

$$y_1 = \frac{66^3}{6 \times 1500 \times 330} = 0{\cdot}0968\text{ ft}$$

$$y_2 = 0{\cdot}0968 \times 2^3 = 0{\cdot}77\text{ ft}$$

$$y_3 = 0{\cdot}0968 \times 3^3 = 2{\cdot}61\text{ ft}$$

$$y_4 = 0{\cdot}0968 \times 4^3 = 6{\cdot}19\text{ ft}$$

$$y_5 = 0{\cdot}0968 \times 5^3 = 12{\cdot}10\text{ ft}$$

Tangential angles, based on 1 chain chords.

$$\alpha_1 = \frac{206\,265\,c^2}{6RL} = \frac{206\,265 \times 66^2}{6 \times 1500 \times 330} = 3025'' \; (05'\,03'')$$

$$\alpha_2 = 4\alpha_1 = 1210'' = 20'\,10''$$

$$\alpha_3 = 9\alpha_1 = 2722{\cdot}5'' = 45'\,23''$$

$$\alpha_4 = 16\alpha_1 = 4840'' = 80'\,40''$$

$$\alpha_5 = 25\alpha_1 = 7562{\cdot}5 = 126'\,03''$$

$$= \tfrac{1}{3}\phi = 2^\circ\,05'\,32'' = \underline{125'\,32''}$$

$$\text{error} = 31''$$

Example 12.8. An existing circular curve of 1500 ft radius is to be improved by sharpening the ends to 1300 ft radius and inserting a transition curve 400 ft long at each straight.

Using the cubic parabola type, calculate:

(a) the length of curve to be taken up,

(b) the movement of the tangent points,

(c) the offsets for the quarter points of the transition curves.

(L.U.)

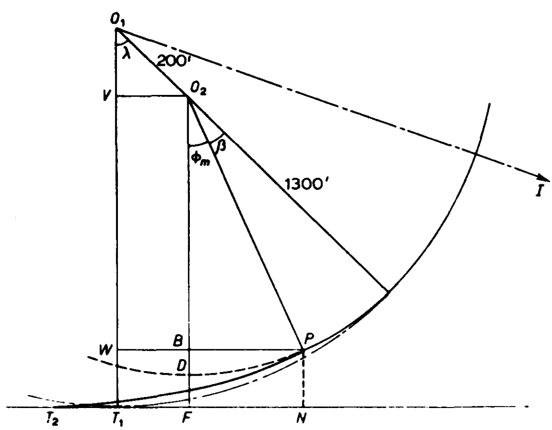


Fig. 12.35

This is Case (4) in Section 12.11.

In Fig. 12.35,

$$PN = y = \frac{L^2}{6R}$$

$$= \frac{400^2}{6 \times 1300} = 20{\cdot}51\text{ ft}$$

$$\phi_m = \tan^{-1}\frac{L}{2R}$$

$$= \tan^{-1}\frac{400}{2 \times 1300} = 8^\circ 48'$$

In triangle $O_2 BP$,

$$O_2B = 1300 \cos 8^\circ 48' = 1284{\cdot}66 \text{ ft}$$

$$BP = FN = 1300 \sin 8^\circ 48' = 198{\cdot}90 \text{ ft}$$

$$O_1V = O_1T_1 - WT_1 - WV = O_1T_1 - PN - O_2B$$

$$= 1500 - 20{\cdot}51 - 1284{\cdot}66 = 194{\cdot}83 \text{ ft}$$

In triangle O_1O_2V,

$$\lambda = \cos^{-1}\frac{194{\cdot}83}{200{\cdot}0} = 13^\circ 05'$$

$$VO_2 = 200 \sin 13^\circ 05' = 45{\cdot}28 \text{ ft}$$

(a) Length of curve taken up $= 1500 \times 13^\circ 05'_{\text{rad}} = 342{\cdot}5$ ft.

(b) Movement of the tangent points, i.e. distance T_1T_2:

$$FN = T_2F = 198{\cdot}90 \text{ ft}$$

$$T_1F = VO_2 = 45{\cdot}28 \text{ ft}$$

$$\therefore \quad T_1T_2 = T_2F - T_1F = 153{\cdot}62 \text{ ft}$$

i.e. along straight from T_1.

(c) Offsets to transition:

$$y = \frac{l^3}{6RL}$$

$$\therefore \quad y_1 = \frac{100^3}{6 \times 1300 \times 400} = 0{\cdot}3205 \text{ ft} = 0{\cdot}32 \text{ ft}$$

$$y_2 = 2^3y_1 = 8y_1 = 2{\cdot}56 \text{ ft}$$

$$y_3 = 3^3y_1 = 27y_1 = 8{\cdot}66 \text{ ft}$$

$$y_4 = 4^3y_1 = 64y_1 = 20{\cdot}51 \text{ ft} \quad \textit{check}$$

Example 12.9. *AB*, *BC* and *CD* are three straights. The length of *BC* is 40 Gunter chains. *BC* deflects 60° right from *AB* and *CD* 45° left from *BC*. Find the radius r for two equal circular curves, each with transition curves of length $\sqrt{r}$ at both ends to connect *AB* and *CD*. *BC* is to be the common tangent without intermediate straight. Find also the total length of curve. (L.U.)

By Eq. (12.112),

$$40{\cdot}0 = (r + \tfrac{1}{24})(\tan 30^\circ + \tan 22^\circ 30') + \sqrt{r}$$

$$= (r + \tfrac{1}{24})(0{\cdot}577\,4 + 0{\cdot}414\,2) + \sqrt{r}$$

i.e.

$$0{\cdot}991\,6r + \sqrt{r} - 39{\cdot}96 = 0$$

Solving this quadratic equation in $\sqrt{r}$, i.e. let $l^2 = r$,

$$l = \sqrt{r} = \frac{-1 \pm \sqrt{(1 + 4 \times 0{\cdot}991\,6 \times 39{\cdot}96)}}{2 \times 0{\cdot}991\,6}$$

$$l = 5{\cdot}864 \text{ chains}$$

$$r = \underline{34{\cdot}39} \text{ chains}$$

$$\phi'_{max} = \frac{206\,265\,L}{2R} = \frac{206\,265\sqrt{r}}{2r} = \frac{103\,132 \times 5{\cdot}864}{68{\cdot}78}$$

$$= 8793''$$

$$= 2^\circ 26' 33''$$

$$\beta_1 = 30 - 2^\circ 26' 33'' = 27^\circ 33' 27''$$

$$\text{Length of circular arc, } A_1 = 2 \times 34{\cdot}39 \times 27^\circ 33' 27''_{rad}$$

$$= 68{\cdot}78 \times 0{\cdot}480\,96 = \underline{33{\cdot}08} \text{ ch}$$

$$\beta_2 = 22^\circ 30' - 2^\circ 26' 33'' = 19^\circ 33' 27''$$

$$A_2 = 2 \times 34{\cdot}39 \times 19^\circ 33' 27''$$

$$= 68{\cdot}78 \times 0{\cdot}341\,33 = 23{\cdot}48 \text{ ch}$$

$$\text{Total length} = A_1 + A_2 + 2L$$

$$= 33{\cdot}08 + 23{\cdot}48 + 11{\cdot}73$$

$$= \underline{68{\cdot}29 \text{ chains}}$$

Exercises 12(b)

14. A road curve of 2000 ft radius is to be connected to two straights by means of transition curves of the cubic parabola type at each end. The maximum speed on this part of the road is to be 70 mile/h and the rate of change of radial acceleration is 1 ft/s^3. The angle of intersection of the two straights is 50° and the chainage of the intersection point is 5872·84 ft.

Calculate:

(a) the length of each transition curve,

(b) the shift of the circular arc,

(c) the chainage at the beginning and the end of the composite curve,

(d) the value of the first two deflection angles for setting out the

first two pegs of the transition curve from the first tangent point assuming that the pegs are set out at 50 ft intervals.

(I.C.E. Ans. (a) 542·1 ft (b) 6·10 ft
(c) 4666·33 ft; 6955·93 ft
(d) 44″, 3′ 39″)

15. Two tangents which intersect at an angle of $41° 40'$ are to be connected by a circular curve of 3000 ft radius with a transition curve at each end. The chainage of the intersection point is 2784 + 26. The transition curves are to be of the cubic parabolic type, designed for a maximum speed of 60 mile/h and a rate of change of radial acceleration is not to exceed 1 ft/s^3.

Find the chainage of the beginning and end of the first transition curve and draw up a table of deflection angles for setting out the curve in 50 ft chord lengths, chainage running continuously through the tangent point (I.C.E. Ans. 2771 + 70·5; 2773 + 97·7; 40″; 5′ 08″; 13′ 54″; 26′ 42″; 43′ 18″)

16. The limiting speed around a circular curve of 2000 ft radius calls for a superelevation of 1/24 across the 30 ft carriageway. Adopting the Ministry of Transport's recommendation of a rate of 1 in 200 for the application of superelevation along the transition curve leading from the straight to the circular curve, calculate the tangential angles for setting out of the transition curve with pegs at 50 ft intervals from the tangent with the straight.

(I.C.E. Ans. 02′ 52″; 11′ 28″; 25′ 48″; 45′ 52″; 1° 11′ 40″)

17. Two straights of a proposed length of railway track are to be joined by a circular curve of 2200 ft radius with cubic parabolic transitions 220 ft long at entry, and exit. The deflection angle between the two straights is $22° 38'$ and the chainage of the intersection point on the first straight produced is 2553·0 ft. Determine the chainages at the ends of both transitions and the information required in the field for setting out the midpoint and end of the first transition curve.

If the transition curve is designed to give a rate of change of radial acceleration of 1 ft/s^3, what will be the superelevation of the outer rail at the midpoint of the transition, if the distance between the centres of the rails is 4 ft 11 in.?

(I.C.E. Ans. 2002·6 ft; 2222·6 ft; 2871·6 ft; 3091·6 ft;
Offsets 0·46 ft and 3·67 ft; 2·2 in.)

18. A transition curve of the cubic parabola type is to be set out from a straight centre line. It must pass through a point which is 20 ft away from the straight, measured at right-angles from a point on the straight produced, 200 ft from the start of the curve.

Tabulate the data for setting out a 400 ft length of curve at 50 ft intervals.

Calculate the rate of change of radial acceleration for a speed of 30 mile/h.

(L.U. Ans. $0°20'20''$, $1°26'00''$, $3°13'20''$, $5°42'40''$, $8°52'50''$, $12°40'50''$, $17°01'40''$, $21°48'05''$; $1{\cdot}28\ \text{ft/s}^3$)

19. Two straight portions of a railway line, intersecting at an angle of 155°, are to be connected by two cubic parabolic transition curves, each 250 ft long and a circular arc of 1000 ft radius.

Calculate the necessary data for setting out the curve using chords 50 ft long.

(R.I.C.S./M Ans. shift 2·61 ft; tangent length 4647·9 ft; $0°05'40''$, $0°23'00''$, $0°51'30''$, $1°31'40''$, $2°23'10''$)

20. Two straights of a railway with $4'8\frac{1}{2}'$ gauge, intersect at an angle of 135°. They are to be connected by a curve of 12 chains radius with cubic parabolic transitions at either end.

The curve is to be designed for a maximum speed of 35 mile/h with a rate of gain of radial acceleration of 1 ft/s^3.

Calculate (a) the required length of transition,
(b) the maximum super elevation of the outer rail,
(c) the amount of shift required for the transition, and
(d) the lengths of the tangent points from the intersection of the straights.

(R.I.C.S./M Ans. 170·7 ft, 5·8 in., 1·53 ft, 2001·1 ft)

21. Two railway straights, having an intersection (deviation) angle of $14°02'40''$, are to be connected by a circular curve of radius 2000 ft with spiral transitions at each end.

(a) Calculate the superelevation for equilibrium on the circular arc, if the design speed is 45 mile/h, $g = 32{\cdot}2\ \text{ft/s}^2$ and the effective gauge between rails = 5 ft and thence,

(b) if this super elevation is introduced with a gradient of 1 in 600, what is the length of each transition and of the circular curve.

(c) Hence, given the point of intersection of the straights, compute all data required for setting out one of the spirals by means of deflection angles and 50 ft chords.

(R.I.C.S./L Ans. 4 in., 200 ft, $3'35''$, $14'19''$, $32'13''$, $57'17''$)

22. (a) Calculate the setting out data for a circular curve, radius 500 ft joining two straights with a deviation angle of $30°00'00''$.

(b) Show that a curve having a polar deflection angle equal to one third of its tangent deflection angle is a lemniscate.

For the lemniscate, the ideal transition curve relationship between length of curve and radius of curvature does not hold. Show why this is not usually important. (N.U.)

23. The curve connecting two straights is to be wholly transitional without intermediate circular arc, and the junction of the two transitions is to be 16 ft from the intersection point of the straights which deflects through an angle of 18°.

Calculate the tangent distances and the minimum radius of curvature. If the superelevation is limited to 1 vertical to 16 horizontal, determine the correct velocity for the curve and the rate of gain of radial acceleration.

(L.U. Ans. 304·2 ft; 958·3 ft; 30 mile/h; 0·292 ft/s³)

24. The superelevation on a road 50 ft wide is to be 3 ft. Calculate the radius for a design speed of 40 mile/h and then give the data for setting out the curve if the two straights have a deflection angle of 30°. Transition curves 300 ft long will be applied at each end, but the data for setting out of these is not required.

(L.U. Ans. Rad 1781·5 ft; tangent length 627·9 ft; shift 2·10 ft; Circular arc 632·8 ft)

25. Two straights of a road 20 ft wide intersect at a through chainage of 8765·9 ft, the deflection angle being 44° 24′. The straights are to be connected by a circular arc of radius 900 ft with cubic parabolic transitions at entry and exit. The curve is to be designed for a speed of 45 mile/h, with a rate of gain of radial acceleration of 2 ft/s³. Determine the required length of the transition and the maximum superelevation of the outer kerb. Tabulate all the necessary data for setting out the first transition with pegs at every 50 ft of through chainage.

(L.U. Ans. L = 159·7 ft; c = 3·0 ft; Chainage T_1 = 8318·3 ft; Offsets 0·04, 0·63, 2·65, 4·72 ft)

26. Assuming an equation $\lambda = f(\phi)$ where λ and ϕ are the intrinsic co-ordinates of any point on a transition spiral, prove that $\lambda^2 = 2RL\phi$, where R = minimum radius of curvature; L = length of spiral, ϕ = the spiral angle.

A curve on a trunk road is to be transitional throughout with a total deviation of 52° 24′. The design speed is to be 60 mile/h, the maximum centripetal ratio 0·25, and the rate of change of radial acceleration 1 ft/s³.

Calculate (1) the length of each spiral,
(2) the minimum radius of curvature,
(3) the tangent distance.

(L.U. Ans. L = 879·8 ft; R = 962 ft; T_1I = 926·6 ft)

27. A suburban road 30 ft wide, designed for a maximum speed of 40 mile/h is to deflect through 38° 14′ with a radius of 1600 ft. A cubic parabola transition is required, with a rate of gain of radial acceleration of 1 ft/s³.

Calculate (a) the maximum superelevation of the outer kerb,

(b) the length of the transition,

(c) the chainage of the tangent points if the forward chainage of the intersection point is 5829·60 ft,

(d) the chainages of the junctions of the transition and circular arcs. (N.R.C.T. Ans. (a) 2·0 ft (b) 126·22 ft (c) 5211·77; 6405·69 ft (d) 5337·99; 6279·47 ft)

28. The co-ordinates of three points, K, L and M are as follows:

Point	North (ft)	East (ft)
K	700	867
L	700	1856
M	1672	2031

These points define the direction of two railway straights $JK(M)$ and LM, which are to be connected by a reverse curve formed by circular arcs of equal radius.

The circular arcs are to be linked together and to the straights by easement curves of length (in Gunter's chains) equal to $\sqrt{R}$ where R is the radius of the circular arcs in chains. Calculate the radius of the circular arcs. (L.U. Ans. 9·87 chains)

29. A road 30 ft wide is to turn through an angle of 26° 24′ with a centre line radius of 600 ft, the forward chainage of the intersection point being 3640·6 ft. A transition curve is to be used at each end of the circular curve of such a length that the rate of gain of radial acceleration is 1 ft/s³ when the speed is 30 mile/h. Find the length of the transition curve, the banking of the road for this speed, the chainage of the beginning of the combined curve, and the angle to turn off these for the peg at 3500 ft.

(L.U. Ans. 142 ft; 2·99 ft; 3428·6 ft; 0° 34′20″)

30. A road curve of 2000 ft radius is to be connected by two straights by means of transition curves of the cubic parabola type at each end. The maximum speed on this part of the road is to be 70 mile/h and the rate of change of radial acceleration is 1 ft/s³. The angle of intersection of the two straights is 50° and the chainage at the intersection point is 5872·84 ft.

Calculate:

(a) the length of each transition curve,

(b) the shift of the circular arc,

(c) the chainage at the beginning and end of the composite curve,

(d) the value of the first two deflection angles for setting out the first two pegs of the transition curve from the first tangent point, assuming that the pegs are set out at 50 ft intervals.

(I.C.E. Ans. (a) 541 ft, (b) 6·10 ft (c) 4666·95 ft; 6953·25 ft (d) 35″; 3′ 40″)

31. A circular curve of radius 700 ft and length 410·70 ft connects two straights of railway track. In order that the track may be modernised to allow for the passage of faster traffic and induce less track wear, the whole curve and certain lengths of the connecting straights are to be removed and replaced by a new circular curve of radius 2500 ft, with transitions of the cubic parabola type at entry and exit.

Given that the maximum speed of the traffic on the new curve is to be 60 mile/h, and the rate of change of radial acceleration is not to exceed 0·90 ft/s^3, determine:

(a) the length of the new composite curve,

(b) the length of the straight track to be removed

(c) the necessary superelevation of the track on the circular curve, the gauge of the track being 4 ft 8½ in.

(I.C.E. Ans. (a) 1769·7 ft; (b) 2 – 695·7 ft; (c) 5·42in)

Bibliography

CLARK, D., *Plane and Geodetic Surveying,* Vol. 1 (Constable)
BANISTER, A. and RAYMOND, S., *Surveying* (Pitman)
JAMESON, A.H., *Advanced Surveying* (Pitman)
JENKINS, R.B.M., *Curve Surveying* (Macmillan)
SEARLES, W.H. and IVES, H.C., ed. by P. KIRSAM, *Field Engineering,* 22nd Ed. (John Wiley)
HIGGINS, A.L., *Higher Surveying* (Macmillan)
SHORTT, W.H., A Practical Method for Improvement of Existing Railway Curves, *Proc. Inst. Civ. Eng.,* Vol. 176
ROYAL-DAWSON, F.G., *Elements of Curve Design for Road, Railway and Racing Track* (Spon, 1932)
ROYAL-DAWSON, F.G., *Road Curves for Safe Modem Traffic* (Spon, 1936)
PERROTT, S.W. and BADGER, F.E.G., *The Practice of Railway Surveying and Permanent Way Work* (Edward Arnold, 1920)
SHARMA, R.C. and SHARMA, S.K., *Principles and Practice of Highway Engineering* (Asia Publishing House)

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